An ideal Waring problem with restricted summands

by

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1. Introduction. If we define \( g(k) \) to be the order of the set \( \{1^k, 2^k, \ldots\} \) as an additive basis for the positive integers, then the ideal Waring problem is to show that

\[
g(k) = 2^k + \lfloor (3/2)^k \rfloor - 2
\]

for all \( k \in \mathbb{N} \) ([\( x \)] is the integer part of \( x \)). By work of Mahler [9], this holds for all but finitely many \( k \), but the result is ineffective and does not yield a bound upon these exceptional values. Computations by Kubina and Wunderlich [8], however, have shown (1) to obtain for all \( k \leq 471 \, 600 \, 000. \)

We consider representations of positive integers as sums of elements of

\[
S_N^{(k)} = \{1^k, N^k, (N+1)^k, \ldots\}
\]

where \( N \geq 2 \) is an integer. A theorem of Rieger [10] gives that \( S_N^{(k)} \) forms an additive basis for \( \mathbb{N} \) for any natural number \( k \). If we let \( g_N(k) \) denote the order of this basis (so that \( g_2(k) = g(k) \)), then the aim of this paper is to prove an analog of (1). To be precise, we have

**Theorem 1.1.** If \( 4 \leq N \leq (k+1)^{(k-1)/k} - 1 \), then

\[
g_N(k) = N^k + \left\lfloor \left( \frac{N+1}{N} \right)^k \right\rfloor - 2.
\]

This follows from two results of the author, namely

**Theorem 1.2 (Bennett [2]).** Suppose \( k \geq 6 \) and \( M \geq e^{446k^6} \) are positive integers. Then there exist \( s \) integers \( x_1, x_2, \ldots, x_s \), where \( s < 6k \log k + (3 \log 6 + 4)k \), such that \( x_i \geq M^{1/(8k^3)} \) for \( i = 1, 2, \ldots, s \) and

\[
M = x_1^k + x_2^k + \ldots + x_s^k.
\]
1.3 (Bennett [3]). Define \( \|x\| = \min_{M \in \mathbb{Z}} |x - M| \). If \( 4 \leq N \leq k \cdot 3^k \), then

\[
\left\| \left( \frac{N + 1}{N} \right)^k \right\| > 3^{-k}.
\]

The first of these is essentially a slight generalization of Vinogradov’s earliest upper bound for \( G(k) \) in the standard Waring problem (see [11]). Since its proof entails making only minor modifications to a well known argument (to compensate for the restriction to \( k \)th powers of integers \( \geq N \)), we will not duplicate it here. We use this rather old fashioned approach instead of later versions of order \( 3k \log k \) or \( 2k \log k \) because these induce a lower bound for \( M \) which is too large to be practical for our purposes (though they increase the bound for \( x_i \)). The difficulty chiefly arises from the size of the implied constant in

\[
\eta(a) \ll q^\varepsilon
\]

where \( \eta(a) \) is the number of solutions to the congruence

\[
v^k \equiv a \pmod{q}
\]

for \( v \) and \( a \) integers in \([0, q-1] \).

The second theorem we use is an effective sharpening of a result of Beukers [4] on fractional parts of powers of rationals. It utilizes Padé approximation to the polynomial \((1 - z)^k\) and some estimates on primes dividing binomial coefficients.

2. Dickson’s ascent argument. We adopt the notation

\[
\alpha = \left[ \left( \frac{N + 1}{N} \right)^k \right] \quad \text{and} \quad \beta = (N + 1)^k - N^k \cdot \left[ \left( \frac{N + 1}{N} \right)^k \right].
\]

Suppose \( N \leq (k + 1)^{(k-1)/k} - 1 \) and write \([a, b] \in S_N^{(k)}(m)\) (or \( (a, b) \in S_N^{(k)}(m) \)) if every integer in \([a, b]\) (respectively \((a, b)\)) can be written as a sum of at most \( m \) elements of \( S_N^{(k)} \) (where we allow repetitions). Following Dickson [6], we count the number of elements of \( S_N^{(k)} \) required for representations of “small” integers before applying an ascent argument to enable the use of Theorem 1.2.

Before we begin, we need a pair of preliminary lemmas.

**Lemma 2.1.** If \( N, k \geq 2 \) and \( M \) are integers then

\[
(N + 1)^k - MN^k = 1
\]

has only the solutions \( N = 2 \) and \( k = 2 \) or \( 4 \).

**Proof.** Suppose that

\[
(N + 1)^k = MN^k + 1
\]
where $N \geq 2$ and $k \geq 2$ (but not $N = k = 2$). If $k$ is even, then we may write

\[(3) \quad ((N + 1)^{k/2} - 1)((N + 1)^{k/2} + 1) = MN^k\]

and so conclude if $N$ is odd that $N^k$ divides $(N+1)^{k/2} - 1$. Since this implies $N^2 < N + 1$, it contradicts $N \geq 2$. If, however, $N$ is even, then we have

\[(4) \quad N^k | 2((N + 1)^{k/2} - 1) \quad \text{if } N \equiv 0 \mod 4\]

or

\[(5) \quad N^k | 2^k((N + 1)^{k/2} - 1) \quad \text{if } N \equiv 2 \mod 4.\]

From (4), we have $N^2 < 2(N + 1)$, which contradicts $N \equiv 0 \mod 4$ while (5) implies that $N = 2$. Since 3 belongs to the exponent $2^{k-2}$ modulo $2^k$, we must have $2^{k-2}$ dividing $k$, so that $k \leq 4$.

It remains only to consider odd $k$. We can write, from (2),

\[(6) \quad \sum_{i=1}^{k} \binom{k}{i} N^i = MN^k\]

and proceed via induction, proving that $\text{ord}_N(k) \to \infty$, thus contradicting any a priori upper bound for $k$. From (6), we clearly have $N \mid k$ and if we suppose that $N^a \mid k$, then since

\[\text{ord}_p \left( \binom{k}{i} \right) \geq \text{ord}_p k - \text{ord}_p i \quad (p \text{ prime})\]

we have

\[\text{ord}_N \left( \binom{k}{i} \right) \geq a - \max_{p | i} \text{ord}_p i .\]

It follows that

\[\text{ord}_N \left( \binom{k}{i} N^i \right) \geq a - \max_{p \text{ odd}} \text{ord}_p i + i\]

and so if $i \geq 2$,

\[\text{ord}_N \left( \binom{k}{i} N^i \right) \geq a + 2 .\]

We conclude, then, that $N^{a+1} \mid k$ as required and hence (6) has no solutions for $k$ odd.

We will also use

\textbf{Lemma 2.2.} If $n$ and $l$ are integers with $n > l \geq (N + 1)^k$, then there is an element of $S_N^{(k)}$, say $i^k$, such that

\[(7) \quad l \leq n - i^k < l + kn^{(k-1)/k}.\]
Proof. Suppose first that \( n \geq l + N^k \) and choose \( i \) such that \( i^k \leq n - l < (i + 1)^k \). Then \( i^k \in S_N^{(k)} \) and since, by calculus,
\[
n - l - i^k \leq k(n - l)^{(k-1)/k} < k(n+1)^{(k-1)/k},
\]
we have (7). If, however, \( n < l + N^k \), take \( i = 1 \) and write \( n = l + m \) (so that \( 1 \leq m < N^k \)). We conclude
\[
k(l + m)^{(k-1)/k} > k(N + 1)^{k-1} = \frac{k}{N + 1}(N + 1)^k.
\]
Since \( k \geq N + 1 \), this is at least \((N + 1)^k\) and hence greater than \( m \), as desired.

Let us now begin to consider representations of comparatively small integers as sums of elements of \( S_N^{(k)} \). We have

**Lemma 2.3.** \([1, \alpha N^k] \in S_N^{(k)}(I_{N^k}^{(k)}) \) where \( I_{N^k}^{(k)} = N^k + \alpha - 2 \).

**Proof.** If \( M \leq \alpha N^k - 1 \), then we can write \( M = N^k x + y \) with \( 0 \leq y \leq N^k - 1 \) and \( x < \alpha \). It follows that \( M \) is a sum of \( x + y \leq N^k + \alpha - 2 \) elements of \( S_N^{(k)} \). If, however, \( M = \alpha N^k \), clearly \( M \in S_N^{(k)}(\alpha) \).

**Lemma 2.4.** \((\alpha N^k, (\alpha + 1)N^k) \in S_N^{(k)}(E) \) where \( E = \max\{\alpha + \beta - 1, N^k - \beta\} \).

**Proof.** The integers \( \alpha N^k, \alpha N^k + 1, \ldots, \alpha N^k + \beta - 1 \) are in \( S_N^{(k)}(\alpha + \beta - 1) \) while \( \alpha N^k + \beta = (N+1)^k, \ldots, \alpha N^k + N^k - 1 = (N+1)^k - \beta + N^k - 1 \) belong to \( S_N^{(k)}(N^k - \beta) \). Since \((\alpha + 1)N^k \in S_N^{(k)}(\alpha + 1) \) and \( \beta \geq 2 \) via Lemma 2.1, we are done.

The beginning of our ascent argument, following Dickson [6], lies in

**Lemma 2.5.** If \( p \) and \( L \) are positive integers with \( p \geq N \) and \((L, L + p^k) \in S_N^{(k)}(m) \), then \((L, L + 2p^k) \in S_N^{(k)}(m + 1) \).

**Proof.** Let \( M \) be an integer satisfying
\[
L + p^k \leq M < L + 2p^k.
\]
Then \( M - p^k \in S_N^{(k)}(m) \) and so \( M \in S_N^{(k)}(m + 1) \). If \( M \in (L, L + p^k) \), the result is trivial.

By induction on \( n \), we readily obtain

**Lemma 2.6.** If \( p, n \) and \( L \) are positive integers with \( p \geq N \) and \((L, L + p^k) \in S_N^{(k)}(m) \), then \((L, L + p^k(n + 1)) \in S_N^{(k)}(m + n) \).

Taking \( L = \alpha N^k, p = N, n = \alpha + 1 \) and applying Lemmas 2.4 and 2.6 we conclude, from \( nN^k > (N + 1)^k \),

**Lemma 2.7.** \((\alpha N^k, \alpha N^k + (N + 1)^k) \in S_N^{(k)}(E + \alpha) \).
If we now successively apply Lemma 2.7 and Lemma 2.6 with \( p = N + 1, N + 2, \ldots, k \) and
\[
n = \left[ \left( \frac{N + 2}{N + 1} \right)^k \right], \left[ \left( \frac{N + 3}{N + 2} \right)^k \right], \ldots, \left[ \left( \frac{k + 1}{k} \right)^k \right],
\]
it follows that

**Lemma 2.8.**
\[
(\alpha N^k, \alpha N^k + (k + 1)^k) \in S_N^{(k)} \left( E + \alpha + \left[ \left( \frac{N + 2}{N + 1} \right)^k \right] + \ldots + \left[ \left( \frac{k + 1}{k} \right)^k \right] \right).
\]

Our main ascent relies upon the following result, which is essentially a variant of a theorem of Dickson [5, Theorem 12].

**Proposition 2.9.** Let \( l \) and \( L_0 \) be integers with
\[
L_0 > l \geq (N + 1)^k, \quad v = (1 - l/L_0)/k \quad \text{and} \quad v^k L_0 \geq 1.
\]
If for \( t \in \mathbb{N} \) we define \( L_t \) by
\[
\log L_t = \left( \frac{k}{k - 1} \right)^t (\log L_0 + k \log v) - k \log v
\]
and if \( (l, L_0) \in S_N^{(k)}(m) \), then \( (l, L_t) \in S_N^{(k)}(m + t) \).

**Proof.** We suppose \( (l, L_0) \in S_N^{(k)}(m) \) and that \( n \in (l, L_1) \). Now for \( t = 1, (8) \) is equivalent to
\[
L_1 = (vL_0)^{k/(k-1)}
\]
and hence we may use Lemma 2.2 to find \( i^k \in S_N^{(k)} \) such that
\[
l \leq n - i^k < l + kn^{(k-1)/k} < l + kvL_0.
\]
Since \( v = (1 - l/L_0)/k \), we have \( l \leq n - i^k < L_0 \), whence \( (l, L_1) \in S_N^{(k)}(m + 1) \). In general, (8) yields
\[
L_{t+1} = (vL_t)^{k/(k-1)}
\]
and the result obtains by induction upon \( t \).

**3. Proof of Theorem 1.1.** Assume \( N \geq 4 \). To apply the preceding proposition, we let \( l = (N + 1)^k \) and \( L_0 = (k + 1)^k \). The condition that \( v^k L_0 \geq 1 \) is then equivalent to
\[
N \leq (k + 1)^{(k-1)/k} - 1.
\]
If we choose \( t \) large enough that
\[
L_t > \max \{ N^{8k^3}, e^{446k^6} \} = e^{446k^6}
\]
then Theorem 1.2 gives $[L_t, \infty) \in S_N^{(k)}(6k \log k + (3 \log 6 + 4)k)$. Now from $v = (1 - l/L_0)/k$, we may write

$$\log L_t = \left(\frac{k}{k-1}\right)^t (k \log(k+1) - k \log v) - k \log v$$

$$> \left(\frac{k}{k-1}\right)^t \left(k \log \left(\frac{k+1}{k}\right)\right).$$

Since

$$\log \left(\frac{k+1}{k}\right) > \frac{1}{k} - \frac{1}{2k^2} \geq \frac{11}{12k} \quad \text{for } k \geq 6,$$

this implies

$$\log L_t > \frac{11}{12} \left(\frac{k}{k-1}\right)^t.$$  

If we note that

$$\log \left(\frac{k}{k-1}\right) > \frac{1}{k-1} - \frac{1}{2(k-1)^2} > \frac{1}{k},$$

we obtain (9) provided

$$t > k \left(6 \log k + \log \left(\frac{5352}{11}\right)\right).$$

Taking $t = [6k \log k + 7k]$, then, yields the desired conclusion. By Lemma 2.3, it remains to show for this choice of $t$ that $(\alpha N^k, L_t) \in S_N^{(k)}(I_N^{(k)})$ (we have $[L_t, \infty) \in S_N^{(k)}(I_N^{(k)})$ because $6k \log k + (3 \log 6 + 4)k < I_N^{(k)}$ for $4 \leq N \leq (k + 1)(k-1)/k - 1$).

By Lemma 2.8 and Proposition 2.9, we have

$$(\alpha N^k, L_t) \in S_N^{(k)}\left(E + \alpha + t + (k - N)\left[\left(\frac{N + 2}{N + 1}\right)^k\right]\right)$$

and this follows from

$$E + \alpha + t + (k - N)\left[\left(\frac{N + 2}{N + 1}\right)^k\right] \leq I_N^{(k)} = N^k + \alpha - 2.$$  

If $E = \alpha + \beta - 1$, then (10) becomes

$$\alpha + \beta + t + (k - N)\left[\left(\frac{N + 2}{N + 1}\right)^k\right] - N^k \leq -1$$

while $E = N^k - \beta$ implies the inequality

$$t + (k - N)\left[\left(\frac{N + 2}{N + 1}\right)^k\right] - \beta \leq -2.$$
To prove that (11) and (12) obtain for all \( N \) and \( k \) satisfying
\[
4 \leq N \leq (k + 1)^{(k-1)/k} - 1
\]
we employ Theorem 1.3 to deduce
\[
3^{-k} < \beta/N^k < 1 - 3^{-k}.
\]
The left hand side of (11) is then bounded above by
\[
\left( \frac{N+1}{N} \right)^k - \left( \frac{N}{3} \right)^k + 6k \log k + 7k + (k - N) \left( \frac{N+2}{N+1} \right)^k
\]
and hence is \( \leq -1 \) for \( N \) and \( k \) unless
(i) \( N = 4, 6 \leq k \leq 34 \), or
(ii) \( N = 5, 8 \leq k \leq 11 \).

Additionally, we bound the left hand side of (12) by
\[
6k \log k + 7k + (k - N) \left( \frac{N+2}{N+1} \right)^k - \left( \frac{N}{3} \right)^k,
\]
which is \( \leq -2 \) for all values of \( N \) and \( k \) under consideration except
(iii) \( N = 4, 6 \leq k \leq 32 \), and
(iv) \( N = 5, 8 \leq k \leq 11 \).

Checking that (11) and (12) hold for the cases (i), (ii) and (iii), (iv) respectively, we conclude the proof of the theorem by noting that \( M = \alpha N^k - 1 \not\in S_N^{(k)} (N^k + \alpha - 3) \) and thus
\[
N^k + \left\lfloor \left( \frac{N+1}{N} \right)^k \right\rfloor - 2 \leq g_N(k) \leq N^k + \left\lfloor \left( \frac{N+1}{N} \right)^k \right\rfloor - 2.
\]

4. Concluding remarks. If \( N = 3 \) and \( k \geq 6 \), we can show that
\[
g_3(k) = 3^k + \lfloor (4/3)^k \rfloor - 2
\]
provided
\[
\left\| (4/3)^k \right\| > (9/4)^{-k}
\]
(in general, we require only
\[
\left\| \left( \frac{N+1}{N} \right)^k \right\| > \left( \frac{N^2}{N+1} \right)^{-k},
\]
which is rather weaker than Theorem 1.3). Though we have (13) for all but finitely many \( k \) by Mahler’s result, it seems difficult to prove effective bounds approaching the above in strength (see Baker and Coates [1] for the only known nontrivial bound in this situation). As mentioned previously,
the case \( N = 2 \) (the ideal Waring problem) also remains open. The best effective result for \( \|(3/2)^k\| \) is due to Dubitskas, who proved

**Theorem 4.1 (Dubitskas [7]).** There is an effectively computable \( k_0 \) such that if \( k \geq k_0 \), then

\[
\|(3/2)^k\| > (1.734)^{-k}.
\]

Unfortunately, this falls rather short of the desired lower bound of \( (4/3)^{-k} \).

**References**


