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An ideal Waring problem with restricted summands

by

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1. Introduction. If we define $g(k)$ to be the order of the set $\{1^k, 2^k, \ldots\}$ as an additive basis for the positive integers, then the ideal Waring problem is to show that

$$g(k) = 2^k + [(3/2)^k] - 2 \tag{1}$$

for all $k \in N$ ($[x]$ is the integer part of x). By work of Mahler [9], this holds for all but finitely many k, but the result is ineffective and does not yield a bound upon these exceptional values. Computations by Kubina and Wunderlich [8], however, have shown (1) to obtain for all $k \leq 471\,600\,000$.

We consider representations of positive integers as sums of elements of

$$S_N^{(k)} = \{1^k, N^k, (N+1)^k, \ldots\}$$

where $N \geq 2$ is an integer. A theorem of Rieger [10] gives that $S_N^{(k)}$ forms an additive basis for $\mathbb{N}$ for any natural number k. If we let $g_N(k)$ denote the order of this basis (so that $g_2(k) = g(k)$), then the aim of this paper is to prove an analog of (1). To be precise, we have

THEOREM 1.1. *If* $4 \leq N \leq (k+1)^{(k-1)/k} - 1$, *then*

$$g_N(k) = N^k + \left[\left(\frac{N+1}{N}\right)^k\right] - 2\,.$$

This follows from two results of the author, namely

THEOREM 1.2 (Bennett [2]). *Suppose* $k \geq 6$ *and* $M \geq e^{446k^6}$ *are positive integers. Then there exist* s *integers* $x_1, x_2, \ldots, x_s$, *where* $s < 6k\log k + (3\log 6 + 4)k$, *such that* $x_i \geq M^{1/(8k^3)}$ *for* $i = 1, 2, \ldots, s$ *and*

$$M = x_1^k + x_2^k + \ldots + x_s^k\,.$$

THEOREM 1.3 (Bennett [3]). *Define* $\|x\| = \min_{M\in\mathbb{Z}} |x - M|$. *If* $4 \le N \le k \cdot 3^k$, *then*

$$\left\| \left(\frac{N+1}{N}\right)^k \right\| > 3^{-k}.$$

The first of these is essentially a slight generalization of Vinogradov's earliest upper bound for $G(k)$ in the standard Waring problem (see [11]). Since its proof entails making only minor modifications to a well known argument (to compensate for the restriction to kth powers of integers $\ge N$), we will not duplicate it here. We use this rather old fashioned approach instead of later versions of order $3k\log k$ or $2k\log k$ because these induce a lower bound for M which is too large to be practical for our purposes (though they increase the bound for x_i). The difficulty chiefly arises from the size of the implied constant in

$$\eta(a) \ll q^{\varepsilon}$$

where $\eta(a)$ is the number of solutions to the congruence

$$v^k \equiv a \bmod q$$

for v and a integers in $[0, q-1]$.

The second theorem we use is an effective sharpening of a result of Beukers [4] on fractional parts of powers of rationals. It utilizes Padé approximation to the polynomial $(1-z)^k$ and some estimates on primes dividing binomial coefficients.

2. Dickson's ascent argument. We adopt the notation

$$\alpha = \left[\left(\frac{N+1}{N}\right)^k\right] \quad \text{and} \quad \beta = (N+1)^k - N^k \cdot \left[\left(\frac{N+1}{N}\right)^k\right].$$

Suppose $N \le (k+1)^{(k-1)/k} - 1$ and write $[a,b] \in S_N^{(k)}(m)$ (or $(a,b) \in S_N^{(k)}(m)$) if every integer in $[a,b]$ (respectively (a,b)) can be written as a sum of at most m elements of $S_N^{(k)}$ (where we allow repetitions). Following Dickson [6], we count the number of elements of $S_N(k)$ required for representations of "small" integers before applying an ascent argument to enable the use of Theorem 1.2.

Before we begin, we need a pair of preliminary lemmas.

LEMMA 2.1. *If* N, $k \ge 2$ *and* M *are integers then*

$$(N+1)^k - MN^k = 1$$

has only the solutions $N = 2$ *and* $k = 2$ *or* 4.

P r o o f. Suppose that

$$(N+1)^k = MN^k + 1 \tag{2}$$

where $N \geq 2$ and $k \geq 2$ (but not $N = k = 2$). If k is even, then we may write

$$((N+1)^{k/2} - 1)((N+1)^{k/2} + 1) = MN^k \tag{3}$$

and so conclude if N is odd that N^k divides $(N+1)^{k/2} - 1$. Since this implies $N^2 < N + 1$, it contradicts $N \geq 2$. If, however, N is even, then we have

$$N^k \mid 2((N+1)^{k/2} - 1) \quad \text{ if } N \equiv 0 \bmod 4 \tag{4}$$

or

$$N^k \mid 2^k((N+1)^{k/2} - 1) \quad \text{ if } N \equiv 2 \bmod 4\,. \tag{5}$$

From (4), we have $N^2 < 2(N + 1)$, which contradicts $N \equiv 0 \bmod 4$ while (5) implies that $N = 2$. Since 3 belongs to the exponent 2^{k-2} modulo 2^k, we must have 2^{k-2} dividing k, so that $k \leq 4$.

It remains only to consider odd k. We can write, from (2),

$$\sum_{i=1}^{k} \binom{k}{i} N^i = MN^k \tag{6}$$

and proceed via induction, proving that $\operatorname{ord}_N(k) \to \infty$, thus contradicting any a priori upper bound for k. From (6), we clearly have $N \mid k$ and if we suppose that $N^a \mid k$, then since

$$\operatorname{ord}_p \binom{k}{i} \geq \operatorname{ord}_p k - \operatorname{ord}_p i \qquad (p \text{ prime})$$

we have

$$\operatorname{ord}_N \binom{k}{i} \geq a - \max_{\substack{p \mid i \\ p \text{ odd}}} (\operatorname{ord}_p i)\,.$$

It follows that

$$\operatorname{ord}_N \left(\binom{k}{i} N^i \right) \geq a - \max_{\substack{p \mid i \\ p \text{ odd}}} (\operatorname{ord}_p i) + i$$

and so if $i \geq 2$,

$$\operatorname{ord}_N \left(\binom{k}{i} N^i \right) \geq a + 2\,.$$

We conclude, then, that $N^{a+1} \mid k$ as required and hence (6) has no solutions for k odd. ∎

We will also use

LEMMA 2.2. *If n and l are integers with $n > l \geq (N + 1)^k$, then there is an element of $S_N^{(k)}$, say i^k, such that*

$$l \leq n - i^k < l + kn^{(k-1)/k}\,. \tag{7}$$

Proof. Suppose first that $n \ge l+N^k$ and choose i such that $i^k \le n-l < (i+1)^k$. Then $i^k \in S_N^{(k)}$ and since, by calculus,

$$n - l - i^k \le k(n-l)^{(k-1)/k} < kn^{(k-1)/k},$$

we have (7). If, however, $n < l + N^k$, take $i = 1$ and write $n = l + m$ (so that $1 \le m < N^k$). We conclude

$$k(l+m)^{(k-1)/k} > k(N+1)^{k-1} = \frac{k}{N+1}(N+1)^k.$$

Since $k \ge N+1$, this is at least $(N+1)^k$ and hence greater than m, as desired. ∎

Let us now begin to consider representations of comparatively small integers as sums of elements of $S_N^{(k)}$. We have

Lemma 2.3. $[1, \alpha N^k] \in S_N^{(k)}(I_N^{(k)})$ *where* $I_N^{(k)} = N^k + \alpha - 2$.

Proof. If $M \le \alpha N^k - 1$, then we can write $M = N^k x + y$ with $0 \le y \le N^k - 1$ and $x < \alpha$. It follows that M is a sum of $x + y \le N^k + \alpha - 2$ elements of $S_N^{(k)}$. If, however, $M = \alpha N^k$, clearly $M \in S_N^{(k)}(\alpha)$. ∎

Lemma 2.4. $(\alpha N^k, (\alpha+1)N^k) \in S_N^{(k)}(E)$ *where* $E = \max\{\alpha + \beta - 1, N^k - \beta\}$.

Proof. The integers $\alpha N^k, \alpha N^k+1, \ldots, \alpha N^k+\beta-1$ are in $S_N^{(k)}(\alpha+\beta-1)$ while $\alpha N^k+\beta = (N+1)^k, \ldots, \alpha N^k+N^k-1 = (N+1)^k-\beta+N^k-1$ belong to $S_N^{(k)}(N^k-\beta)$. Since $(\alpha+1)N^k \in S_N^{(k)}(\alpha+1)$ and $\beta \ge 2$ via Lemma 2.1, we are done. ∎

The beginning of our ascent argument, following Dickson [6], lies in

Lemma 2.5. *If p and L are positive integers with $p \ge N$ and $(L, L+p^k) \in S_N^{(k)}(m)$, then $(L, L+2p^k) \in S_N^{(k)}(m+1)$.*

Proof. Let M be an integer satisfying

$$L + p^k \le M < L + 2p^k.$$

Then $M - p^k \in S_N^{(k)}(m)$ and so $M \in S_N^{(k)}(m+1)$. If $M \in (L, L+p^k)$, the result is trivial. ∎

By induction on n, we readily obtain

Lemma 2.6. *If p, n and L are positive integers with $p \ge N$ and $(L, L+p^k) \in S_N^{(k)}(m)$, then $(L, L+p^k(n+1)) \in S_N^{(k)}(m+n)$.*

Taking $L = \alpha N^k$, $p = N$, $n = \alpha + 1$ and applying Lemmas 2.4 and 2.6 we conclude, from $nN^k > (N+1)^k$,

Lemma 2.7. $(\alpha N^k, \alpha N^k + (N+1)^k) \in S_N^{(k)}(E + \alpha)$.

If we now successively apply Lemma 2.7 and Lemma 2.6 with $p = N+1$, $N+2, \ldots, k$ and

$$n = \left[\left(\frac{N+2}{N+1}\right)^k\right], \left[\left(\frac{N+3}{N+2}\right)^k\right], \ldots, \left[\left(\frac{k+1}{k}\right)^k\right],$$

it follows that

LEMMA 2.8.

$$(\alpha N^k, \alpha N^k + (k+1)^k) \in S_N^{(k)}\left(E + \alpha + \left[\left(\frac{N+2}{N+1}\right)^k\right] + \ldots + \left[\left(\frac{k+1}{k}\right)^k\right]\right).$$

Our main ascent relies upon the following result, which is essentially a variant of a theorem of Dickson [5, Theorem 12].

PROPOSITION 2.9. *Let l and L_0 be integers with*

$$L_0 > l \geq (N+1)^k, \quad v = (1 - l/L_0)/k \quad \textit{and} \quad v^k L_0 \geq 1\,.$$

If for $t \in \mathbb{N}$ we define L_t by

$$\log L_t = \left(\frac{k}{k-1}\right)^t (\log L_0 + k \log v) - k \log v \tag{8}$$

and if $(l, L_0) \in S_N^{(k)}(m)$, then $(l, L_t) \in S_N^{(k)}(m+t)$.

Proof. We suppose $(l, L_0) \in S_N^{(k)}(m)$ and that $n \in (l, L_1)$. Now for $t = 1$, (8) is equivalent to

$$L_1 = (vL_0)^{k/(k-1)}$$

and hence we may use Lemma 2.2 to find $i^k \in S_N^{(k)}$ such that

$$l \leq n - i^k < l + kn^{(k-1)/k} < l + kvL_0\,.$$

Since $v = (1-l/L_0)/k$, we have $l \leq n - i^k < L_0$, whence $(l, L_1) \in S_N^{(k)}(m+1)$. In general, (8) yields

$$L_{t+1} = (vL_t)^{k/(k-1)}$$

and the result obtains by induction upon t. ∎

3. Proof of Theorem 1.1. Assume $N \geq 4$. To apply the preceding proposition, we let $l = (N+1)^k$ and $L_0 = (k+1)^k$. The condition that $v^k L_0 \geq 1$ is then equivalent to

$$N \leq (k+1)^{(k-1)/k} - 1\,.$$

If we choose t large enough that

$$L_t > \max\{N^{8k^3}, e^{446k^6}\} = e^{446k^6} \tag{9}$$

then Theorem 1.2 gives $[L_t,\infty) \in S_N^{(k)}(6k\log k + (3\log 6+4)k)$. Now from $v = (1-l/L_0)/k$, we may write

$$\log L_t = \left(\frac{k}{k-1}\right)^t (k\log(k+1) - k\log v) - k\log v$$
$$> \left(\frac{k}{k-1}\right)^t \left(k\log\left(\frac{k+1}{k}\right)\right).$$

Since

$$\log\left(\frac{k+1}{k}\right) > \frac{1}{k} - \frac{1}{2k^2} \ge \frac{11}{12k} \quad \text{for } k \ge 6\,,$$

this implies

$$\log L_t > \frac{11}{12}\left(\frac{k}{k-1}\right)^t.$$

If we note that

$$\log\left(\frac{k}{k-1}\right) > \frac{1}{k-1} - \frac{1}{2(k-1)^2} > \frac{1}{k}\,,$$

we obtain (9) provided

$$t > k\left(6\log k + \log\left(\frac{5352}{11}\right)\right).$$

Taking $t = [6k\log k + 7k]$, then, yields the desired conclusion. By Lemma 2.3, it remains to show for this choice of t that $(\alpha N^k, L_t) \in S_N^{(k)}(I_N^{(k)})$ (we have $[L_t,\infty) \in S_N^{(k)}(I_N^{(k)})$ because $6k\log k + (3\log 6+4)k < I_N^{(k)}$ for $4 \le N \le (k+1)^{(k-1)/k} - 1$).

By Lemma 2.8 and Proposition 2.9, we have

$$(\alpha N^k, L_t) \in S_N^{(k)}\left(E + \alpha + t + (k-N)\left[\left(\frac{N+2}{N+1}\right)^k\right]\right)$$

and this follows from

$$E + \alpha + t + (k-N)\left[\left(\frac{N+2}{N+1}\right)^k\right] \le I_N^{(k)} = N^k + \alpha - 2\,. \tag{10}$$

If $E = \alpha + \beta - 1$, then (10) becomes

$$\alpha + \beta + t + (k-N)\left[\left(\frac{N+2}{N+1}\right)^k\right] - N^k \le -1 \tag{11}$$

while $E = N^k - \beta$ implies the inequality

$$t + (k-N)\left[\left(\frac{N+2}{N+1}\right)^k\right] - \beta \le -2\,. \tag{12}$$

To prove that (11) and (12) obtain for all N and k satisfying

$$4 \le N \le (k+1)^{(k-1)/k} - 1$$

we employ Theorem 1.3 to deduce

$$3^{-k} < \beta/N^k < 1 - 3^{-k}\,.$$

The left hand side of (11) is then bounded above by

$$\left(\frac{N+1}{N}\right)^k - \left(\frac{N}{3}\right)^k + 6k\log k + 7k + (k-N)\left(\frac{N+2}{N+1}\right)^k$$

and hence is ≤ -1 for N and k unless

(i) $N = 4$, $6 \le k \le 34$, or
(ii) $N = 5$, $8 \le k \le 11$.

Additionally, we bound the left hand side of (12) by

$$6k\log k + 7k + (k-N)\left(\frac{N+2}{N+1}\right)^k - \left(\frac{N}{3}\right)^k,$$

which is ≤ -2 for all values of N and k under consideration except

(iii) $N = 4$, $6 \le k \le 32$, and
(iv) $N = 5$, $8 \le k \le 11$.

Checking that (11) and (12) hold for the cases (i), (ii) and (iii), (iv) respectively, we conclude the proof of the theorem by noting that $M = \alpha N^k - 1 \notin S_N^{(k)}(N^k + \alpha - 3)$ and thus

$$N^k + \left[\left(\frac{N+1}{N}\right)^k\right] - 2 \le g_N(k) \le N^k + \left[\left(\frac{N+1}{N}\right)^k\right] - 2\,.$$

4. Concluding remarks. If $N = 3$ and $k \ge 6$, we can show that

$$g_3(k) = 3^k + [(4/3)^k] - 2$$

provided

$$\|(4/3)^k\| > (9/4)^{-k} \qquad (13)$$

(in general, we require only

$$\left\|\left(\frac{N+1}{N}\right)^k\right\| > \left(\frac{N^2}{N+1}\right)^{-k},$$

which is rather weaker than Theorem 1.3). Though we have (13) for all but finitely many k by Mahler's result, it seems difficult to prove effective bounds approaching the above in strength (see Baker and Coates [1] for the only known nontrivial bound in this situation). As mentioned previously, the case $N = 2$ (the ideal Waring problem) also remains open. The best effective result for $\|(3/2)^k\|$ is due to Dubitskas, who proved

THEOREM 4.1 (Dubitskas [7]). *There is an effectively computable k_0 such that if $k \geq k_0$, then*

$$\|(3/2)^k\| > (1.734)^{-k}.$$

Unfortunately, this falls rather short of the desired lower bound of $(4/3)^{-k}$.

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