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On Franel–Kluyver integrals of order three

by

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1. Introduction. In 1924 Franel [2] proved the formula

$$\frac{1}{ab}\int_0^{ab}\left(\left\{\frac{x}{a}\right\}-\frac{1}{2}\right)\left(\left\{\frac{x}{b}\right\}-\frac{1}{2}\right)dx=\frac{(a,b)}{12[a,b]} \tag{1}$$

(where $\{x\}$ denotes the fractional part of x), which he used to establish a connection between the Riemann hypothesis and the distribution of Farey sequences.

In Greaves, Hall, Huxley and Wilson [3] we defined the Franel integral of order n by

$$J(a_1,\ldots,a_n)=\frac{1}{a_1\ldots a_n}\int_0^{a_1\ldots a_n}\varrho\left(\frac{x}{a_1}\right)\ldots\varrho\left(\frac{x}{a_n}\right)dx\,,$$

where $a_1,\ldots,a_n$ are positive integers and $\varrho(x)=[x]-x+1/2$. In particular, for $n=4$, we evaluated certain cases in terms of elementary functions of the h.c.f.'s and l.c.m.'s of $a_1,\ldots,a_n$: others involved generalized Dedekind sums and related cotangent sums.

In fact, twenty years before Franel proved equation (1), Kluyver [6] had implicitly proved the more general result

$$\frac{1}{ab}\int_0^{ab}\overline{B}_m\left(\frac{x}{a}\right)\overline{B}_n\left(\frac{x}{b}\right)dx=(-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n}\frac{(a,b)^{m+n}}{a^mb^n} \tag{2}$$

for all positive integers m,n,a,b. Here $\overline{B}_r(x)$ is the periodic extension into $\mathbb{R}$ of the Bernoulli polynomial $B_r(x)$ on $[0,1)$ given by the relation

$$\frac{ze^{xz}}{e^z-1}=\sum_{r=0}^{\infty}B_r(x)\frac{z^r}{r!}\qquad(|z|<2\pi)\,. \tag{3}$$

In this paper we generalize the Franel integral of order 3 in two ways. Firstly, following Kluyver, we replace the function ϱ by higher order Bernoulli functions and define the Franel–Kluyver integral of order 3 by

$$J_{l,m,n}(a_1,a_2,a_3)=\frac{1}{a_1a_2a_3}\int\limits_0^{a_1a_2a_3}\overline{B}_l\Big(\frac{x}{a_1}\Big)\overline{B}_m\Big(\frac{x}{a_2}\Big)\overline{B}_n\Big(\frac{x}{a_3}\Big)\,dx \tag{4}$$

(where $l+m+n\equiv 0 \pmod 2$: if $l+m+n$ is odd then the integrand is an odd, periodic function and the integral is zero). We show that this integral can be evaluated as a linear combination of the generalized Dedekind sums

$$S_{m,n}(h,k)=\sum_{r=0}^{k-1}\overline{B}_m\Big(\frac{r}{k}\Big)\overline{B}_n\Big(\frac{rh}{k}\Big). \tag{5}$$

Secondly, we define

$$J(a_1,a_2,a_3;\theta)=\frac{1}{a_1a_2a_3}\int\limits_0^{a_1a_2a_3}\overline{B}_1\Big(\frac{x}{a_1}\Big)\overline{B}_1\Big(\frac{x}{a_2}\Big)\overline{B}_1\Big(\frac{x}{a_3}+\theta\Big)\,dx. \tag{6}$$

The evaluation of this integral involves the further generalized Dedekind–Rademacher sum

$$S_{m,n}(h,k;x)=\sum_{r=0}^{k-1}\overline{B}_m\Big(\frac{r}{k}\Big)\overline{B}_n\Big(\frac{rh}{k}+x\Big). \tag{7}$$

(Carlitz [1] has defined $\phi_{m,n}(h,k;x,y)$ where $S_{m,n}(h,k;x)=\phi_{n,m}(h,k;x,0)$ and proved reciprocity formulae for these sums.)

In Section 2 we show how both (4) and (6) can be reduced to integrals involving the functions $\overline{B}_r(a_ix)$, in which we need only consider pairwise coprime variables. We work out the Fourier series for $\overline{B}_l(ax)\overline{B}_m(bx)$ in Section 3 which we then use to evaluate integrals equivalent to (4) and (6) in Theorems 1 and 2 respectively.

We shall make use of the following alternative expression for the generalized Dedekind sum (5):

$$S_{m,n}(h,k)=\frac{i^{n-m}}{(2\pi)^{m+n}}\frac{mn}{k^{m+n-1}}\sum_{r=0}^{k-1}C^{(m)}\Big(\frac{rh}{k}\Big)C^{(n)}\Big(\frac{r}{k}\Big), \tag{8}$$

where we have defined

$$C^{(m)}(z)=\frac{d^m}{dz^m}\log(\sin\pi z)=-(m-1)!\sum_{t=-\infty}^{\infty}\frac{1}{(t-z)^m} \tag{9}$$

for $z\notin\mathbb{Z}$, and

$$C^{(m)}(0)=\begin{cases}-2\zeta(m)(m-1)! & \text{if } m \text{ is even},\\ 0 & \text{if } m \text{ is odd}.\end{cases} \tag{10}$$

Eisenstein proved that

$$\overline{B}_1\Big(\frac{r}{k}\Big) = \frac{i}{2\pi k}\sum_{a=0}^{k-1} C^{(1)}\Big(\frac{a}{k}\Big)e\Big(\frac{ra}{k}\Big)$$

($e(x) := \exp(2\pi ix)$), so that the ordinary Dedekind sum may be expressed in terms of cotangents. Analogously, using the generalization

$$\overline{B}_m\Big(\frac{r}{k}\Big) = \frac{m}{k^m}\Big(\frac{i}{2\pi}\Big)^m\sum_{a=0}^{k-1} C^{(m)}\Big(\frac{a}{k}\Big)e\Big(\frac{ra}{k}\Big), \tag{11}$$

we see that

$$S_{m,n}(h,k) = \frac{mn}{k^{m+n}}\Big(\frac{i}{2\pi}\Big)^{m+n}\sum_{r=0}^{k-1}\sum_{a=0}^{k-1}\sum_{b=0}^{k-1} C^{(m)}\Big(-\frac{a}{k}\Big)C^{(n)}\Big(\frac{b}{k}\Big)e\Big(\frac{rhb-ra}{k}\Big)$$

which, since

$$\sum_{r=0}^{k-1} e\Big(\frac{rhb-ra}{k}\Big) = \begin{cases} k & \text{if } a\equiv hb \pmod{k},\\ 0 & \text{else,}\end{cases}$$

gives (8).

2. Reduction steps and related integrals. We let

$$J_{l,m,n}(a_1,a_2,a_3;\theta) = \frac{1}{a_1a_2a_3}\int_0^{a_1a_2a_3} \overline{B}_l\Big(\frac{x}{a_1}\Big)\overline{B}_m\Big(\frac{x}{a_2}\Big)\overline{B}_n\Big(\frac{x}{a_3}+\theta\Big)\,dx \tag{12}$$

and define the related integral $I_{l,m,n}(a_1,a_2,a_3;\theta)$ by

$$I_{l,m,n}(a_1,a_2,a_3;\theta) = \int_0^1 \overline{B}_l(a_1x)\overline{B}_m(a_2x)\overline{B}_n(a_3x+\theta)\,dx\,. \tag{13}$$

Since the integrand in (12) has period $[a_1,a_2,a_3]$ we may write (12) as

$$J_{l,m,n}(a_1,a_2,a_3;\theta) = \frac{1}{[a_1,a_2,a_3]}\int_0^{[a_1,a_2,a_3]} \overline{B}_l\Big(\frac{x}{a_1}\Big)\overline{B}_m\Big(\frac{x}{a_2}\Big)\overline{B}_n\Big(\frac{x}{a_3}+\theta\Big)\,dx\,. \tag{14}$$

Then, substituting $x=[a_1,a_2,a_3]y$, we find that

$$J_{l,m,n}(a_1,a_2,a_3;\theta) = I_{l,m,n}(A_1,A_2,A_3;\theta) \tag{15}$$

where

$$A_i = \frac{[a_1,a_2,a_3]}{a_i}\qquad (1\le i\le 3)\,.$$

There is an analogous transformation in the opposite direction.

We now show that the integral in (12) can be reduced to an I integral in which the variables are pairwise coprime. Firstly we may assume the a_i in (12) have no common divisor, since putting $x = ky$ gives

$$J_{l,m,n}(ka_1, ka_2, ka_3; \theta) = \frac{1}{k^3 a_1 a_2 a_3} \int_0^{k^2 a_1 a_2 a_3} \overline{B}_l\left(\frac{y}{a_1}\right) \overline{B}_m\left(\frac{y}{a_2}\right) \overline{B}_n\left(\frac{y}{a_3} + \theta\right) k\,dy = J_{l,m,n}(a_1, a_2, a_3; \theta)$$

by periodicity.

We can also write $J_{l,m,n}(a_1, a_2, a_3; \theta)$ as

$$J_{l,m,n}(a_1, a_2, a_3; \theta) = \frac{1}{a_1 a_2 a_3} \sum_{s=0}^{a_1 a_2 a_3 - 1} \int_s^{s+1} \overline{B}_l\left(\frac{x}{a_1}\right) \overline{B}_m\left(\frac{x}{a_2}\right) \overline{B}_n\left(\frac{x}{a_3} + \theta\right) dx$$
$$= \frac{1}{a_1 a_2 a_3} \int_0^1 \sum_{s=0}^{a_1 a_2 a_3 - 1} \overline{B}_l\left(\frac{s+y}{a_1}\right) \overline{B}_m\left(\frac{s+y}{a_2}\right) \overline{B}_n\left(\frac{s+y}{a_3} + \theta\right) dy\,. \tag{16}$$

If we now let $K = [a_1, a_2]$ and $k = (a_3, K)$, where $a_3 = hk$, then $hK = [a_1, a_2, a_3]$ and, by periodicity,

$$J_{l,m,n}(a_1, a_2, a_3; \theta) = \frac{1}{hK} \int_0^1 \sum_{s=0}^{hK-1} \overline{B}_l\left(\frac{s+y}{a_1}\right) \overline{B}_m\left(\frac{s+y}{a_2}\right) \overline{B}_n\left(\frac{s+y}{a_3} + \theta\right) dy\,.$$

We write $s = tK + u$, for $0 \le u \le K - 1$ and $0 \le t \le h - 1$; then

$$J_{l,m,n}(a_1, a_2, a_3; \theta) = \frac{1}{hK} \int_0^1 \sum_{t=0}^{h-1} \sum_{u=0}^{K-1} \overline{B}_l\left(\frac{u+y}{a_1}\right) \overline{B}_m\left(\frac{u+y}{a_2}\right) \overline{B}_n\left(\frac{tK+u+y}{hk} + \theta\right) dy\,.$$

Now $(h, K/k) = 1$, so that if t runs through all residue classes modulo h, then so does tK/k and

$$\sum_{t=0}^{h-1} \overline{B}_n\left(\frac{tK+u+y}{hk} + \theta\right) = \sum_{t=0}^{h-1} \overline{B}_n\left(\frac{t}{h} + \frac{u+y+hk\theta}{hk}\right) = \frac{1}{h^{n-1}} \overline{B}_n\left(\frac{u+y}{k} + h\theta\right).$$

Thus

$$\begin{aligned}(17)\quad & J_{l,m,n}(a_1,a_2,a_3;\theta)\\ &= \frac{1}{h^n K}\int\limits_0^1 \sum_{u=0}^{K-1} \overline{B}_l\left(\frac{u+y}{a_1}\right)\overline{B}_m\left(\frac{u+y}{a_2}\right)\overline{B}_n\left(\frac{u+y}{k}+h\theta\right)dy\\ &= \frac{1}{h^n} J_{l,m,n}(a_1,a_2,k;h\theta)\,.\end{aligned}$$

It is convenient at this stage to introduce the idea of total decomposition sets, which we now define.

With any positive integers $a_1,\ldots,a_n$ we associate 2^n-1 further positive integers $d(S)$, where S runs through the non-empty subsets of $\{1,\ldots,n\}$, having the properties:

(i) For any non-empty $T\subseteq\{1,\ldots,n\}$ we have

$$\text{h.c.f.}\,(a_i : i\in T) = \prod\{d(S) : T\subseteq S\}\,.$$

(ii) For any non-empty T we have

$$\text{l.c.m.}\,[a_i : i\in T] = \prod\{d(S) : S\cap T\neq\emptyset\}\,.$$

We refer to $\{d(S)\}$ as the *total decomposition set* of $\{a_1,\ldots,a_n\}$. Its existence and uniqueness were established in [4].

We shall also make use of the following lemma, which was proved by Hall in [5].

LEMMA 1. *Let 2^n-1 positive integers $e(S)$ be given, where S runs through the non-empty subsets of $\{1,\ldots,n\}$. Then $\{e(S)\}$ is a total decomposition set if, and only if, for every pair of subsets R, S neither of which contains the other, we have $(e(R),e(S))=1$.*

Using this notation we can write

$$a_1 = d_1d_{12}d_{13}d_{123}\,,\quad a_2 = d_2d_{12}d_{23}d_{123}\,,\quad a_3 = d_3d_{13}d_{23}d_{123}\,,$$

and it follows from (16) and (17) that

$$J_{l,m,n}(a_1,a_2,a_3;\theta) = \frac{1}{d_1^l d_2^m d_3^n} J_{l,m,n}(d_{12}d_{13}, d_{12}d_{23}, d_{13}d_{23}; d_3\theta)\,.$$

Now, by Lemma 1,

$$[d_{12}d_{13}, d_{12}d_{23}, d_{13}d_{23}] = d_{12}d_{13}d_{23}$$

so that, from (15),

$$J_{l,m,n}(a_1,a_2,a_3;\theta) = \frac{1}{d_1^l d_2^m d_3^n} I_{l,m,n}(d_{23}, d_{13}, d_{12}; d_3\theta)$$

and we notice that the variables d_{23}, d_{13} and d_{12} are pairwise relatively prime.

3. The Fourier series for $\overline{B}_l(ax)\overline{B}_m(bx)$. Let a and b be coprime positive integers. The function $\overline{B}_l(ax)\overline{B}_m(bx)$ has period 1 and so has an expansion in complex Fourier series

$$\overline{B}_l(ax)\overline{B}_m(bx) \sim \sum_{k=-\infty}^{\infty} c_k(a,b)e(kx)$$

where

$$c_k(a,b) = \int_0^1 \overline{B}_l(ax)\overline{B}_m(bx)e(-kx)\,dx\,.$$

We apply Parseval's theorem to the functions $\overline{B}_l(ax)$ and $\overline{B}_m(bx)e(-kx)$. (See Whittaker and Watson [8], §9.5.) Since

$$\overline{B}_r(x) \sim -r! \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{e(nx)}{(2\pi in)^r}$$

(where the dash denotes throughout that undefined terms are excluded from the sum) this gives

$$\begin{aligned} c_k(a,b) &= \frac{l!m!}{(2\pi i)^{l+m}} \sum\sum_{ga+hb-k=0} \frac{1}{g^l h^m} \\ &= \frac{l!m!}{(2\pi i)^{l+m}} \sideset{}{'}\sum_{d=-\infty}^{\infty} \frac{1}{(k\overline{a}+db)^l(k\overline{b}-da)^m}\,, \end{aligned} \tag{18}$$

where $\overline{a}a + \overline{b}b = 1$. Now

$$\sideset{}{'}\sum_{d=-\infty}^{\infty} \frac{1}{d^r} = \begin{cases} 2\zeta(r) = -\dfrac{(2\pi i)^r B_r}{r!} & \text{if } r \text{ is even}\,, \\ 0 & \text{if } r \text{ is odd}\,. \end{cases}$$

Therefore, since $B_r = 0$ for odd $r > 1$, we may write

$$\sideset{}{'}\sum_{d=-\infty}^{\infty} \frac{1}{d^r} = -\frac{(2\pi i)^r B_r}{r!} \tag{19}$$

for any $r > 1$. Hence

$$c_0(a,b) = \frac{(-1)^{m-1}l!m!B_{l+m}}{a^m b^l (l+m)!}\,. \tag{20}$$

To find $c_k(a,b)$ for $k \neq 0$, we consider the integral

$$\frac{1}{2\pi i}\int_{Q_M} f(z)\,dz$$

where

$$f(z) = \frac{\pi \cot(\pi z)}{(k\overline{a} + zb)^l (k\overline{b} - za)^m}$$

and Q_M is the square with corners $(M+1/2)(\pm 1 \pm i)$. We note that $k\overline{b}/a \neq -k\overline{a}/b$ for $k \neq 0$ and let R_1 and R_2 be the residues at $z = -k\overline{a}/b$ and $z = k\overline{b}/a$ respectively. Then, since the integral round Q_M tends to zero as $M \to \infty$, we have

$$0 = \lim_{M\to\infty} \sideset{}{'}\sum_{l=-M}^{M} \frac{1}{(k\overline{a} + db)^l (k\overline{b} - da)^m} + R_1 + R_2$$

and so, for $k \neq 0$,

$$c_k(a,b) = -\frac{l!m!}{(2\pi i)^{l+m}} (R_1 + R_2)\,. \tag{21}$$

We consider the different cases separately.

Case 1: $a \nmid k$, $b \nmid k$. Since $f(z)$ has a pole of order l at $z = -k\overline{a}/b$, we have

$$\begin{aligned} R_1 &= \frac{1}{(l-1)!} \frac{d^{l-1}}{dz^{l-1}} \bigg\{ \frac{\pi \cot(\pi z)}{b^l (k\overline{b} - za)^m} \bigg\}_{z \to -k\overline{a}/b} \\ &= \frac{1}{a^m b^l (l-1)!} \\ &\quad \times \bigg\{ \sum_{s=0}^{l-1} \binom{l-1}{s} \frac{d^{l-1-s}}{dz^{l-1-s}} C^{(1)}(z) \frac{d^s}{dz^s} \bigg(\frac{k\overline{b}}{a} - z\bigg)^{-m} \bigg\}_{z \to -k\overline{a}/b} \\ &= \frac{1}{a^m b^l (l-1)!} \\ &\quad \times \bigg\{ \sum_{s=0}^{l-1} \binom{l-1}{s} C^{(l-s)}(z) \frac{(m+s-1)!}{(m-1)!} \bigg(\frac{k\overline{b}}{a} - z\bigg)^{-(m+s)} \bigg\}_{z \to -k\overline{a}/b} \\ &= \frac{(-1)^l b^{m-l}}{k^m (l-1)!(m-1)!} \\ &\quad \times \sum_{s=0}^{l-1} \binom{l-1}{s} (-1)^s (m+s-1)! \bigg(\frac{ab}{k}\bigg)^s C^{(l-s)} \bigg(\frac{k\overline{a}}{b}\bigg)\,. \end{aligned}$$

Similarly,

$$R_2 = \frac{(-1)^m a^{l-m}}{k^l (l-1)!(m-1)!} \sum_{s=0}^{m-1} \binom{m-1}{s} (-1)^s (l+s-1)! \bigg(\frac{ab}{k}\bigg)^s C^{(m-s)} \bigg(\frac{k\overline{b}}{a}\bigg)$$

and so, when $a\nmid k$, $b\nmid k$, we have

$$(22)\quad c_k(a,b)=\frac{lm}{(2\pi i)^{l+m}}$$
$$\times\biggl\{\frac{b^{m-l}}{k^m}\sum_{s=0}^{l-1}\binom{l-1}{s}(-1)^{l+s-1}(m+s-1)!\Bigl(\frac{ab}{k}\Bigr)^s C^{(l-s)}\Bigl(\frac{k\overline{a}}{b}\Bigr)$$
$$+\frac{a^{l-m}}{k^l}\sum_{s=0}^{m-1}\binom{m-1}{s}(-1)^{m+s-1}(l+s-1)!\Bigl(\frac{ab}{k}\Bigr)^s C^{(m-s)}\Bigl(\frac{k\overline{b}}{a}\Bigr)\biggr\}.$$

Case 2: $a\mid k$ but $b\nmid k$. At $z=-k\overline{a}/b$, $f(z)$ has a pole of order l, with residue

$$R_1=\frac{(-1)^l b^{m-l}}{k^m(l-1)!(m-1)!}$$
$$\times\sum_{s=0}^{l-1}\binom{l-1}{s}(-1)^s(m+s-1)!\Bigl(\frac{ab}{k}\Bigr)^s C^{(l-s)}\Bigl(\frac{k\overline{a}}{b}\Bigr).$$

Also $f(z)$ has a pole of order $m+1$ at $z=k\overline{b}/a$. We put $z=w+k\overline{b}/a$; then the Laurent expansion becomes

$$\frac{c_{-(m+1)}}{w^{m+1}}+\ldots+\frac{c_{-1}}{w}+\ldots=\frac{\pi\cot(\pi w+\pi k\overline{b}/a)}{(k\overline{a}+wb+k\overline{b}b/a)^l(-wa)^m}$$
$$=\frac{\pi\cot(\pi w)}{(-w)^m k^l a^{m-l}(1+abw/k)^l}.$$

We use the expansion

$$\pi\cot(\pi w)=\sum_{h=0}^{\infty}\frac{(2\pi i)^{2h}B_{2h}w^{2h-1}}{(2h)!}$$

so that we require the coefficient of w^{m-1} in

$$\frac{(-1)^m}{k^l a^{m-l}}\sum_{h=0}^{\infty}\frac{(2\pi i)^{2h}B_{2h}w^{2h-1}}{(2h)!}\sum_{g=0}^{\infty}\binom{g+n-1}{g}\Bigl(-\frac{abw}{k}\Bigr)^g.$$

This is

$$\frac{1}{k^l a^{m-l}}\sum_{r=0}^{m/2}\frac{(2\pi i)^{m-2r}B_{m-2r}}{(m-2r)!}\binom{l+2r-1}{2r}\Bigl(\frac{ab}{k}\Bigr)^{2r}$$

when m is even, and

$$\frac{1}{k^l a^{m-l}}\sum_{r=0}^{(m-1)/2}\frac{(2\pi i)^{m-1-2r}B_{m-1-2r}}{(m-1-2r)!}\binom{l+2r}{2r+1}\Bigl(\frac{ab}{k}\Bigr)^{2r+1}$$

when m is odd. Again, since $B_r = 0$ for odd $r > 1$, we can write

$$R_2 = \frac{1}{k^l a^{m-l}} \sum_{s=0}^{m-2} \frac{(2\pi i)^{m-s} B_{m-s}}{(m-s)!} \binom{l+s-1}{s} \left(\frac{ab}{k}\right)^s + \frac{a^l b^m}{k^{l+m}} \binom{l+m-1}{m}$$

for any m. Hence, when $a \mid k$ but $b \nmid k$, we have

(23) $c_k(a,b)$

$$= \frac{a^{l-m} lm}{k^l (2\pi i)^{l+m}} \sum_{s=0}^{m-1} \binom{m-1}{s} (-1)^{m+s-1} (l+s-1)! \left(\frac{ab}{k}\right)^s C^{(m-s)}\left(\frac{k\overline{b}}{a}\right)$$
$$- \frac{l}{a^{m-l}} \sum_{s=0}^{m-2} \binom{m}{s} \frac{(ab)^s (l+s-1)! B_{m-s}}{(2\pi ik)^{l+s}} - \frac{l(l+m-1)! a^l b^m}{(2\pi ik)^{l+m}}.$$

Case 3: $a \nmid k$ but $b \mid k$. Since the formula for $c_k(a,b)$ is symmetric in a and b, we have

(24) $c_k(a,b)$

$$= \frac{a^{l-m} lm}{k^l (2\pi i)^{l+m}} \sum_{s=0}^{m-1} \binom{m-1}{s} (-1)^{m+s-1} (l+s-1)! \left(\frac{ab}{k}\right)^s C^{(m-s)}\left(\frac{k\overline{b}}{a}\right)$$
$$- \frac{m}{b^{l-m}} \sum_{s=0}^{l-2} \binom{l}{s} \frac{(ab)^s (m+s-1)! B_{l-s}}{(2\pi ik)^{m+s}} - \frac{m(l+m-1)! a^l b^m}{(2\pi ik)^{l+m}}$$

for this case.

Case 4: $a \mid k$ and $b \mid k$. In this case $f(z)$ has poles of order $l+1$ at $z = -k\overline{a}/b$ and $m+1$ at $z = k\overline{b}/a$, so that

(25) $c_k(a,b)$

$$= -\frac{l}{a^{m-l}} \sum_{s=0}^{m-2} \binom{m}{s} \frac{(ab)^s (l+s-1)! B_{m-s}}{(2\pi ik)^{l+s}} - \frac{l(l+m-1)! a^l b^m}{(2\pi ik)^{l+m}}$$
$$- \frac{m}{b^{l-m}} \sum_{s=0}^{l-2} \binom{l}{s} \frac{(ab)^s (m+s-1)! B_{l-s}}{(2\pi ik)^{m+s}} - \frac{m(l+m-1)! a^l b^m}{(2\pi ik)^{l+m}}.$$

Thus, after a slight rearrangement, (20), (22), (23), (24) and (25) give

(26)
$$\overline{B}_l(ax)\overline{B}_m(bx) \sim \frac{(-1)^{m-1} l! m! B_{l+m}}{a^m b^l (l+m)!}$$
$$- \sideset{}{'}\sum_{\substack{k=-\infty \\ a \mid k}}^{\infty} \left\{ \frac{l}{a^{m-l}} \sum_{s=0}^{m-2} \binom{m}{s} \frac{(ab)^s (l+s-1)! B_{m-s}}{(2\pi ik)^{l+s}} \right.$$
$$\left. + \frac{l(l+m-1)! a^l b^m}{(2\pi ik)^{l+m}} \right\} e(kx)$$

$$- \sideset{}{'}\sum_{\substack{k=-\infty \\ b|k}}^{\infty} \left\{ \frac{m}{b^{l-m}} \sum_{s=0}^{l-2} \binom{l}{s} \frac{(ab)^s (m+s-1)! B_{l-s}}{(2\pi i k)^{m+s}} + \frac{m(l+m-1)! a^l b^m}{(2\pi i k)^{l+m}} \right\} e(kx)$$

$$- \sideset{}{'}\sum_{\substack{k=-\infty \\ a\nmid k}}^{\infty} \frac{a^{l-m} lm}{k^l (2\pi i)^{l+m}} \times \sum_{s=0}^{m-1} \binom{m-1}{s} (-1)^{m+s} (l+s-1)! \left(\frac{ab}{k}\right)^s C^{(m-s)}\left(\frac{k\overline{b}}{a}\right) e(kx)$$

$$- \sideset{}{'}\sum_{\substack{k=-\infty \\ b\nmid k}}^{\infty} \frac{b^{m-l} lm}{k^m (2\pi i)^{l+m}} \times \sum_{s=0}^{l-1} \binom{l-1}{s} (-1)^{l+s} (m+s-1)! \left(\frac{ab}{k}\right)^s C^{(l-s)}\left(\frac{k\overline{a}}{b}\right) e(kx)\,.$$

4. The triple Franel–Kluyver integral. Let

$$I_{l,m,n}(a,b,c) = \int_0^1 \overline{B}_l(ax)\overline{B}_m(bx)\overline{B}_n(cx)\,dx$$

where we may assume a, b and c are pairwise coprime and that $l+m+n$ is even since the integral is zero otherwise. Then we have

THEOREM 1.

$$I_{l,m,n}(a,b,c) \tag{27}$$
$$= \frac{(-1)^{l+n} l! m! n! a^{l-1}}{c^l} \sum_{s=0}^{m} \binom{l+s-1}{s} \left(\frac{b}{c}\right)^s (-1)^{s+1} \frac{S_{m-s,l+n+s}(c\overline{b}, a)}{(m-s)!(l+n+s)!}$$
$$+ \frac{(-1)^{m+n} l! m! n! b^{m-1}}{c^m} \sum_{s=0}^{l} \binom{m+s-1}{s} \left(\frac{a}{c}\right)^s (-1)^{s+1} \frac{S_{l-s,m+n+s}(c\overline{a}, b)}{(l-s)!(m+n+s)!}\,.$$

Proof. We apply Parseval's formula to the functions $\overline{B}_l(ax)\overline{B}_m(bx)$ and $\overline{B}_n(cx)$ to obtain

$$I_{l,m,n}(a,b,c) \tag{28}$$
$$= \sideset{}{'}\sum_{a|k,\,k=hc} \left\{ \frac{ln!}{a^{m-l}} \sum_{s=0}^{m-2} \binom{m}{s} \frac{(ab)^s (l+s-1)! B_{m-s}}{(2\pi i)^{l+n+s} h^n k^{l+s}} + \frac{l(l+m-1)! n! a^l b^m}{(2\pi i)^{l+m+n} h^n k^{l+m}} \right\}$$

$$+\sum_{b\mid k,k=hc}{}' \left\{ \frac{mn!}{b^{l-m}} \sum_{s=0}^{l-2} \binom{l}{s} \frac{(ab)^s (m+s-1)! B_{l-s}}{(2\pi i)^{m+n+s} h^n k^{m+s}} + \frac{m(l+m-1)!n!a^l b^m}{(2\pi i)^{l+m+n} h^n k^{l+m}} \right\}$$

$$+\sum_{a\nmid k,k=hc}{}' \left\{ \frac{a^{l-m} lmn!}{(2\pi i)^{l+m+n}} \times \sum_{s=0}^{m-1} \binom{m-1}{s} \frac{(-1)^{m+s}(l+s-1)!(ab)^s}{h^n k^{l+s}} C^{(m-s)}\left(\frac{k\overline{b}}{a}\right) \right\}$$

$$+\sum_{b\nmid k,k=hc}{}' \left\{ \frac{b^{m-l} lmn!}{(2\pi i)^{l+m+n}} \times \sum_{s=0}^{l-1} \binom{l-1}{s} \frac{(-1)^{l+s}(m+s-1)!(ab)^s}{h^n k^{m+s}} C^{(l-s)}\left(\frac{k\overline{a}}{b}\right) \right\}.$$

In the first sum here, since $a \mid k$ and $(a,c)=1$, we must have $a \mid h$ so we put $h = da$; then $k = cda$ and the first sum is

$$\frac{ln!}{a^{m+n}c^l} \sum_{s=0}^{m-2} \binom{m}{s} \left(\frac{b}{c}\right)^s \frac{(l+s-1)! B_{m-s}}{(2\pi i)^{l+n+s}} \sum_{d=-\infty}^{\infty}{}' \frac{1}{d^{l+n+s}} + \frac{l(l+m-1)!n!b^m}{(2\pi i)^{l+m+n} a^{m+n} c^{l+m}} \sum_{d=-\infty}^{\infty}{}' \frac{1}{d^{l+m+n}}.$$

Using (19) we can write this as

$$(29)\qquad -\frac{ln!}{a^{m+n}c^l} \sum_{s=0}^{m} \binom{m}{s} \left(\frac{b}{c}\right)^s \frac{(l+s-1)! B_{m-s} B_{n+l+s}}{(n+l+s)!} = -\frac{l!m!n!}{a^{m+n}c^l} \sum_{s=0}^{m} \binom{l+s-1}{s} \left(\frac{b}{c}\right)^s \frac{B_{m-s} B_{n+l+s}}{(m-s)!(n+l+s)!}.$$

Similarly, the second sum in (28) gives

$$(30)\qquad -\frac{l!m!n!}{b^{l+n}c^m} \sum_{s=0}^{l} \binom{m+s-1}{s} \left(\frac{a}{c}\right)^s \frac{B_{l-s} B_{n+m+s}}{(l-s)!(n+m+s)!}.$$

For the third sum in (28) we have $a \nmid k$ and $k = hc$ so we put $h = da + r$, $0 < r < a$; then this sum is

$$\frac{a^{l-m} lmn!}{(2\pi i)^{l+m+n} c^l} \sum_{s=0}^{m-1} \left(\frac{ab}{c}\right)^s \binom{m-1}{s} (-1)^{m+s} (l+s-1)! \times \sum_{r=1}^{a-1} C^{(m-s)}\left(\frac{rc\overline{b}}{a}\right) \sum_{d=-\infty}^{\infty} \frac{1}{(da+r)^{l+n+s}}.$$

Now, from (9),

$$\sum_{d=-\infty}^{\infty} \frac{1}{(da+r)^{l+n+s}} = \frac{(-1)^{l+n+s-1}}{a^{l+n+s}(l+n+s-1)!} C^{(l+n+s)}\left(\frac{r}{a}\right)$$

so the third sum is

$$\begin{aligned} &-\frac{lmn!}{(2\pi i)^{l+m+n}a^{m+n}c^l} \\ &\quad\times \sum_{s=0}^{m-1}\binom{m-1}{s}\left(\frac{b}{c}\right)^s \frac{(l+s-1)!}{(l+n+s-1)!}\sum_{r=0}^{a-1} C^{(m-s)}\left(\frac{rc\bar{b}}{a}\right)C^{(l+n+s)}\left(\frac{r}{a}\right) \\ &+\frac{lmn!}{(2\pi i)^{l+m+n}a^{m+n}c^l} \\ &\quad\times \sum_{s=0}^{m-1}\binom{m-1}{s}\frac{(l+s-1)!}{(l+n+s-1)!}\left(\frac{b}{c}\right)^s C^{(m-s)}(0)C^{(l+n+s)}(0)\,, \end{aligned}$$

which, from (8) and (10), is

$$\begin{aligned} (31)\quad &\frac{(-1)^{l+n}l!m!n!a^{l-1}}{c^l} \\ &\quad\times \sum_{s=0}^{m-1}\binom{l+s-1}{s}\left(\frac{b}{c}\right)^s (-1)^{s+1}\frac{S_{m-s,l+n+s}(c\bar{b},a)}{(m-s)!(l+n+s)!} \\ &\quad+\frac{l!m!n!}{a^{m+n}c^l}\sum_{s=0}^{m-1}\binom{l+s-1}{s}\left(\frac{b}{c}\right)^s \frac{B_{m-s}B_{l+n+s}}{(m-s)!(l+n+s)!}\,. \end{aligned}$$

Similarly the fourth sum in (28) is

$$\begin{aligned} (32)\quad &\frac{(-1)^{m+n}l!m!n!b^{m-1}}{c^m} \\ &\quad\times \sum_{s=0}^{l-1}\binom{m+s-1}{s}\left(\frac{a}{c}\right)^s (-1)^{s+1}\frac{S_{l-s,m+n+s}(c\bar{a},b)}{(l-s)!(m+n+s)!} \\ &\quad+\frac{l!m!n!}{b^{l+n}c^m}\sum_{s=0}^{l-1}\binom{m+s-1}{s}\left(\frac{a}{c}\right)^s \frac{B_{l-s}B_{m+n+s}}{(l-s)!(m+n+s)!}\,. \end{aligned}$$

Thus, (29)–(32) give

$$\begin{aligned} &I_{l,m,n}(a,b,c) \\ &= \frac{(-1)^{l+n}l!m!n!a^{l-1}}{c^l}\sum_{s=0}^{m-1}\binom{l+s-1}{s}\left(\frac{b}{c}\right)^s (-1)^{s+1}\frac{S_{m-s,l+n+s}(c\bar{b},a)}{(m-s)!(l+n+s)!} \end{aligned}$$

$$+\frac{(-1)^{m+n}l!m!n!b^{m-1}}{c^m}\sum_{s=0}^{l-1}\binom{m+s-1}{s}\left(\frac{a}{c}\right)^s(-1)^{s+1}\frac{S_{l-s,m+n+s}(c\overline{a},b)}{(l-s)!(m+n+s)!}$$
$$-\frac{l!m!n!}{(l+m+n)!}\frac{b^m}{a^{m+n}c^{l+m}}\binom{l+m-1}{m}B_{l+m+n}$$
$$-\frac{l!m!n!}{(l+m+n)!}\frac{a^l}{b^{l+n}c^{l+m}}\binom{l+m-1}{l}B_{l+m+n}\,,$$

which is equivalent to (27).

5. Shifted triple integrals. Let

$$I(a,b,c;\theta)=\int_0^1\overline{B}_1(ax)\overline{B}_1(bx)\overline{B}_1(cx+\theta)\,dx$$

where we may assume a,b and c are pairwise coprime and, since the integrand has period 1 in θ, that $0<\theta<1$. In order to evaluate this integral we shall require the following lemma.

LEMMA 2. *For $\alpha\notin\mathbb{Z}$ and all real x we have*

$$\sum_{n=-\infty}^{\infty}\frac{e((n+\alpha)x)}{(n+\alpha)^k}=\delta_k(x)+\frac{e([x]\alpha)2^{k-1}(i\pi)^k\{x\}^{k-1}}{(k-1)!}+\frac{e([x]\alpha)}{(k-1)!}\sum_{s=0}^{k-1}\binom{k-1}{s}(-1)^sC^{(s+1)}(\alpha)(2\pi i\{x\})^{k-1-s}\tag{33}$$

where $\{x\}$ denotes the fractional part of x and

$$\delta_k(x)=\begin{cases}1 & \text{if } k=1 \text{ and } x\in\mathbb{Z},\\ 0 & \text{else}.\end{cases}$$

Proof. We begin with the Fourier series

$$e(\alpha(1/2-\{x\}))\sim\frac{\sin\pi\alpha}{\pi\alpha}+\frac{2\sin\pi\alpha}{\pi}\sum_{n=1}^{\infty}\frac{\alpha\cos2\pi nx-in\sin2\pi nx}{\alpha^2-n^2}\,.\tag{34}$$

The function on the left has bounded variation on any interval $[a,b]$ and is continuous for all $x\in\mathbb{R}\setminus\mathbb{Z}$. It follows that we may replace $\sim$ by $=$ above provided we add the term $\delta_1(x)i\sin\pi\alpha$ on the right to take care of integral x. We write, formally,

$$e(\alpha(1/2-\{x\}))=\delta_1(x)i\sin\pi\alpha+\sum_{n=-\infty}^{\infty}\frac{\sin\pi\alpha}{\pi(n+\alpha)}e(nx)$$

with the understanding that the terms involving $\pm n$ are bracketed together. We multiply through by $\pi e(\alpha x)/\sin\pi\alpha$ to obtain (33) in the case $k=1$.

We may write the summand on the right-hand side of (34) in the form

$$\frac{(\alpha-n)e(nx)+(\alpha+n)e(-nx)}{2(\alpha^2-n^2)}.$$

Let $x\in[0,1)$. We multiply (34) by $\pi e(\alpha x)/\sin\pi\alpha$ and integrate term-by-term from 0 to y. This step is justified by §13.53 of Titchmarsh [7], and we obtain, for $y\in[0,1)$,

$$\begin{aligned}\frac{\pi e(\alpha/2)}{\sin\pi\alpha}y &= \frac{e(\alpha y)-1}{2\pi i\alpha^2}+\sum_{n=1}^{\infty}\left(\frac{e((\alpha+n)y)-1}{2\pi i(\alpha+n)^2}+\frac{e((\alpha-n)y)-1}{2\pi i(\alpha-n)^2}\right)\\ &= \frac{1}{2\pi i}\left(\sum_{n=-\infty}^{\infty}\frac{e((\alpha+n)y)}{(\alpha+n)^2}-\pi^2\operatorname{cosec}^2\pi\alpha\right).\end{aligned}$$

We extend this periodically to obtain (33) in the case $k=2$. We now proceed by induction on k, and from this point the term-by-term integrations may be justified by uniform convergence. Accordingly, assume that $k\ge 2$, $x\in[0,1)$ and (33) holds. Integrating from 0 to y gives

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{e((n+\alpha)y)-1}{(n+\alpha)^{k+1}} &= 2\pi i\biggl\{\frac{2^{k-1}(i\pi)^k y^k}{k!}\\ &\quad+\frac{1}{(k-1)!}\sum_{s=0}^{k-1}\binom{k-1}{s}\frac{(-1)^s C^{(s+1)}(\alpha)(2\pi i)^{k-1-s}y^{k-s}}{k-s}\biggr\}\end{aligned}$$

so that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{e((n+\alpha)y)}{(n+\alpha)^{k+1}} &= \frac{2^k(i\pi)^{k+1}y^k}{k!}+\sum_{n=-\infty}^{\infty}\frac{1}{(n+\alpha)^{k+1}}\\ &\quad+\frac{1}{k!}\sum_{s=0}^{k-1}\binom{k}{s}(-1)^s C^{(s+1)}(\alpha)(2\pi iy)^{k-s}.\end{aligned}$$

Therefore, since

$$C^{(k+1)}(\alpha)=(-1)^k k!\sum_{n=-\infty}^{\infty}\frac{1}{(n+\alpha)^{k+1}}$$

equation (33) holds for $k+1$ and hence for all $k\ge 1$.

We see from (9) that (33) also holds when $x=0$ and therefore, since

$$\sum_{n=-\infty}^{\infty}\frac{e((n+\alpha)x)}{(n+\alpha)^k}=\sum_{n=-\infty}^{\infty}\frac{e((n+\alpha)\{x\})e([x]\alpha)}{(n+\alpha)^k},$$

(33) follows for $x\in\mathbb{R}$.

THEOREM 2. *For a, b, c pairwise coprime integers and $0 < \theta < 1$,*

$$
\begin{aligned}
(35)\qquad I(a,b,c;\theta) = {} & \frac{\{a\theta\}}{ac} S_{1,1}(\bar{b}c, a; [a\theta]/a) + \frac{\{b\theta\}}{bc} S_{1,1}(\bar{a}c, b; [b\theta]/b) \\
& + \frac{1}{2c}\{S_{1,2}(\bar{b}c, a; [a\theta]/a) + S_{1,2}(\bar{a}c, b; [b\theta]/b)\} \\
& + \frac{\{a\theta\}}{2ac}\overline{B}_1\left(\frac{[a\theta]b\bar{c}}{a}\right) + \frac{\{b\theta\}}{2bc}\overline{B}_1\left(\frac{[b\theta]a\bar{c}}{b}\right) \\
& - \frac{1}{6a^2b^2c^2}\{b^3\overline{B}_3(a\theta) + a^3\overline{B}_3(b\theta)\}\,.
\end{aligned}
$$

P r o o f. Since

$$\overline{B}_1(cx+\theta) \sim -\sum_{g=-\infty}^{\infty}{}' \frac{e(g\theta)e(gcx)}{2\pi i g}$$

and, from (26),

$$
\begin{aligned}
\overline{B}_1(ax)\overline{B}_1(bx) \sim {} & \frac{1}{12ab} + \sum_{\substack{h=-\infty\\ a\mid h}}^{\infty}{}' \frac{ab}{4\pi^2h^2}e(-hx) + \sum_{\substack{h=-\infty\\ b\mid h}}^{\infty}{}' \frac{ab}{4\pi^2h^2}e(-hx) \\
& - \sum_{\substack{h=-\infty\\ a\nmid h}}^{\infty} \frac{1}{4\pi h}\cot\left(\frac{\pi h\bar{b}}{a}\right)e(-hx) \\
& - \sum_{\substack{h=-\infty\\ b\nmid h}}^{\infty} \frac{1}{4\pi h}\cot\left(\frac{\pi h\bar{a}}{b}\right)e(-hx)\,,
\end{aligned}
$$

Parseval's formula gives

$$
\begin{aligned}
(36)\qquad I(a,b,c;\theta) = {} & \sum{}'\left\{-\frac{e(g\theta)ab}{8\pi^3igh^2};\ a\mid h,\ h=gc\right\} \\
& + \sum{}'\left\{-\frac{e(g\theta)ab}{8\pi^3igh^2};\ b\mid h,\ h=gc\right\} \\
& + \sum{}'\left\{-\frac{e(g\theta)}{8\pi^2igh}\cot\left(\frac{\bar{b}h\pi}{a}\right);\ a\nmid h,\ h=gc\right\} \\
& + \sum{}'\left\{-\frac{e(g\theta)}{8\pi^2igh}\cot\left(\frac{\bar{a}h\pi}{b}\right);\ b\nmid h,\ h=gc\right\}.
\end{aligned}
$$

For the first sum we let $g = la$, then $h = cla$ and the sum is

$$(37)\qquad -\sum_{l=-\infty}^{\infty}{}' \frac{e(la\theta)b}{8\pi^3ia^2c^2l^3} = -\frac{b}{8\pi^3ia^2c^2}\sum_{l=-\infty}^{\infty}{}' \frac{e(la\theta)}{l^3} = -\frac{b}{6a^2c^2}\overline{B}_3(a\theta)\,.$$

Similarly, the second sum is

$$-\frac{a}{b^2c^2}\overline{B}_3(b\theta)\,. \tag{38}$$

In the third sum in (36) we must have $a \nmid g$. Put $g = la + r$, $0 < r < a$; then the sum is

$$\frac{1}{8\pi^2 ic}\sum_{r=1}^{a-1}\cot\left(\frac{cr\overline{b}\pi}{a}\right)\sum_{l=-\infty}^{\infty}\frac{e(la\theta + r\theta)}{(la+r)^2}\,. \tag{39}$$

Now, by Lemma 2,

$$\sum_{l=-\infty}^{\infty}\frac{e(la\theta + r\theta)}{(la+r)^2}$$
$$= \frac{e([a\theta]r/a)}{a^2}\left\{2(i\pi)^2\{a\theta\} + 2\pi i\{a\theta\}C^{(1)}\left(\frac{r}{a}\right) - C^{(2)}\left(\frac{r}{a}\right)\right\},$$

so that (39) is

$$\begin{aligned}\frac{i\{a\theta\}}{4\pi ca^2}\sum_{r=1}^{a-1}C^{(1)}\left(\frac{cr\overline{b}}{a}\right)e\left(\frac{[a\theta]r}{a}\right)\\ + \frac{\{a\theta\}}{4\pi^2ca^2}\sum_{r=1}^{a-1}C^{(1)}\left(\frac{cr\overline{b}}{a}\right)C^{(1)}\left(\frac{r}{a}\right)e\left(\frac{[a\theta]r}{a}\right)\\ + \frac{i}{8\pi^3ca^2}\sum_{r=1}^{a-1}C^{(1)}\left(\frac{cr\overline{b}}{a}\right)C^{(2)}\left(\frac{r}{a}\right)e\left(\frac{[a\theta]r}{a}\right).\end{aligned} \tag{40}$$

From (11) we have

$$\overline{B}_n\left(\frac{h\mu}{k} + x\right) = \frac{n}{k^n}\left(\frac{i}{2\pi}\right)^n\sum_{\lambda=0}^{k-1}C^{(n)}\left(\frac{\lambda}{k}\right)e\left(\frac{\lambda(h\mu + kx)}{k}\right)$$

so that we can also write

$$S_{m,n}(h,k;x) = \frac{(-1)^{(n-m)/2}mn}{(2\pi)^{m+n}k^{m+n-1}}\sum_{\lambda=0}^{k-1}e(\lambda x)C^{(m)}\left(\frac{h\lambda}{k}\right)C^{(n)}\left(\frac{\lambda}{k}\right)$$

and the third sum in (36) is

$$\frac{\{a\theta\}}{2ac}\overline{B}_1\left(\frac{[a\theta]b\overline{c}}{a}\right) + \frac{\{a\theta\}}{ac}S_{1,1}(\overline{b}c, a; [a\theta]/a) + \frac{1}{2c}S_{1,2}(\overline{b}c, a; [a\theta]/a)\,. \tag{41}$$

Similarly, the fourth sum is

$$\frac{\{b\theta\}}{2bc}\overline{B}_1\left(\frac{[b\theta]a\overline{c}}{b}\right) + \frac{\{b\theta\}}{bc}S_{1,1}(\overline{a}c, b; [b\theta]/b) + \frac{1}{2c}S_{1,2}(\overline{a}c, b; [b\theta]/b)\,, \tag{42}$$

so that, from (37), (38), (41) and (42) we obtain (35) as required.

Notice that the integral on the left-hand side of (35) is bounded in absolute value by $1/32$ for any a, b and c (using Hölder's inequality), whereas there are terms on the right-hand side which can be very large in certain cases. For example, if a is very large in comparison with b and c then there must be cancellation between the terms

$$\frac{1}{2c}S_{1,2}(\bar{b}c, a; [a\theta]/a) - \frac{a}{6b^2c^2}\overline{B}_3(b\theta)$$

which suggests the existence of reciprocity relations for the homogeneous Dedekind–Rademacher sums

$$S_{m,n}(c, b, a; x) = S_{m,n}(\bar{b}c, a; x)\,.$$

In fact such relations do exist, but the appropriate choice in a particular problem depends on the relative magnitudes of a, b and c: we do not have a single formula for $I(a, b, c; \theta)$ in terms of Dedekind sums in which all the terms are uniformly bounded. I hope to consider this matter further in a later paper.

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