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On oscillations in the additive divisor problem, 1

by

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1. Introduction. In its simplest form, the additive divisor problem is to determine the asymptotic behaviour of the sum

$$S_k(x)=\sum_{n\le x} d(n)d(n+k) \quad (x>0)\,,$$

where $d(n)$ stands for the number of positive divisors of n and k is a positive integer.

On the assumption that the shift k is fixed, the best result was obtained by J.-M. Deshouillers and H. Iwaniec [3], who proved that for every $\varepsilon>0$ we have

$$S_k(x)=xP_k(\log x)+E_k(x)$$

with

$$E_k(x)\ll_{k,\varepsilon} x^{2/3+\varepsilon} \quad (x\to\infty)\,, \tag{1.1}$$

where P_k is a quadratic polynomial.

On the other hand, confirming a conjecture by A. Ivić, Y. Motohashi [10] has recently proved that for each fixed k we have

$$E_k(x)=\Omega(x^{1/2}) \quad (x\to\infty)\,. \tag{1.2}$$

In this note we shall prove a slight improvement of this result.

Theorem. *For fixed $k\ge 1$, we have*

$$E_k(x)=\Omega_{\pm}(x^{1/2}) \quad (x\to\infty)\,.$$

The proof of (1.2) [10] (and of (1.1) [3]) proceeds via Kloosterman sums and Kuznetsov's trace formulas (cf. [2], [7], [8] and [12]). But it is perhaps easy to conceive that there should be a more direct approach avoiding these

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tools, namely the one using the zeta-function of our problem,

$$\zeta_k(s) := \sum_{n=1}^{\infty} \frac{d(n)d(n+k)}{n^s} \qquad (\operatorname{Re} s > 1)\,.$$

Moreover, within this approach it would then be natural to try to apply a certain general result of Landau [9] (cf. Lemma 0 in Section 2). Actually, we choose this line of argument.

The function $\zeta_k(s)$ was analyzed earlier by L. A. Takhtajan and A. I. Vinogradov [14]; see also [5] for some revision of [14]. They applied the spectral theory of the hyperbolic Laplacian (cf. [6]) directly to a modification of the Eisenstein series.

Needed facts from [14] (and [5]) will be given below in Lemmas 1 and 2 (Section 4). Lemma 3 in Section 4 (non-vanishing lemma) is not new. It is stated in [10] as a fact needed for completing the proof of (1.2). It is also remarked there that this fact is a consequence of a lemma of [11] which in turn is proved via Kloosterman sums and Kuznetsov's trace formulas. We shall prove Lemma 3 in another way.

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2. Consequence of a theorem of Landau. The following lemma is a corollary of a classical result of Landau [9] (cf. e.g. [1]).

LEMMA 0. *Suppose $g(x)$ is a piecewise continuous function bounded on finite intervals such that*

$$G(s) := \int_1^{\infty} g(x)x^{-s-1}\,dx$$

converges absolutely for $\operatorname{Re} s > \sigma_a$. *Suppose $G(s)$ analytically continues into a region including the reals $s \ge \sigma_0$ (with no singularity at σ_0) while $G(s)$ has a simple pole at $\sigma_0 + it_0$, $t_0 \ne 0$ with residue r. Then*

$$\limsup_{x\to\infty} g(x)x^{-\sigma_0} \ge |r|\,, \qquad \liminf_{x\to\infty} g(x)x^{-\sigma_0} \le -|r|\,.$$

3. Notations and auxiliary facts. The following notations will be used (cf. [5], [7] and [14]):

$$K_\nu(v) := \int_0^{\infty} e^{-v\cosh t}\cosh(\nu t)\,dt \qquad (v>0,\ \nu\in\mathbb{C}) \tag{3.1}$$

(the K-Bessel function);

$$\sigma_s(k) := \sum_{d\mid k} d^s;$$

$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$;

$d\mu(z) := y^{-2}\,dx\,dy$ (the invariant hyperbolic measure in the upper half-plane $z = x + iy,\ y > 0$);

$\kappa_j := \sqrt{\lambda_j - 1/4}$, where λ_j is the jth (non-zero) eigenvalue of the hyperbolic Laplacian (it is well known that $\lambda_j > 1/4$);

$z_j := 1/2 + i\kappa_j$;

$\varrho_j(1)$ — the first Fourier coefficient of the Maass wave form attached to λ_j;

$H_j(s) := \sum_{n=1}^{\infty} t_j(n)/n^s$ ($\operatorname{Re} s > 1$) (the Maass L-function attached to λ_j);

$$E^*(z) := \sqrt{y}(\log y - c) + 2\sqrt{y} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} d(|n|)K_0(2\pi|n|y)e(nx) \quad (y > 0), \tag{3.2}$$

where $e(\alpha) := \exp(2\pi i\alpha)$ and $c := \log(4\pi) - \gamma$ with Euler's constant γ;

$$I_k(s) := \int_{\Pi} |E^*(z)|^2 e(kz) y^s\, d\mu(z) \qquad (\operatorname{Re} s > 1), \tag{3.3}$$

where Π is the strip $|x| \le 1/2$, $y > 0$;

$$I_k(w,s) := \frac{2^{2-2s}\pi^{1-s}k^{w-s}\sigma_{1-2w}(k)\Gamma(s-w)\Gamma(s-1+w)}{\xi(2w)\Gamma(s)}. \tag{3.4}$$

We will also use the following facts about the K-Bessel function (3.1):

$$K_0(v) > 0 \qquad (v > 0)\,, \tag{3.5}$$

$$K_0(v) \asymp v^{-1/2}e^{-v} \qquad (v \to \infty)\,, \tag{3.6}$$

$$\left(\frac{\partial}{\partial\nu}\right)^n K_\nu(v)\bigg|_{\nu=1/2} \ll v^{-1/2}e^{-v} \qquad (v \to \infty\,;\ n = 0,1,2)\,, \tag{3.7}$$

$$|K_{iu}(v)| \le K_0(v) \qquad (u \ge 0\,,\ v > 0)\,, \tag{3.8}$$

$$K_{iu}(v) \ll e^{(-3/2)u} \qquad (u \ge 1\,,\ v \ge 1) \tag{3.9}$$

and

$$\int_0^\infty K_\nu(t)e^{-t}t^{s-1}\,dt = \sqrt{\pi}\,2^{-s}\frac{\Gamma(s+\nu)\Gamma(s-\nu)}{\Gamma(s+1/2)} \qquad (\operatorname{Re}(s \pm \nu) > 0)\,. \tag{3.10}$$

The facts (3.5) and (3.8) follow directly from (3.1). The fact (3.6) is, of course, a corollary of the asymptotic formula for $K_\nu(v)$ (see [4], p. 86, (7)). The estimate (3.7) can be derived from (3.1), and (3.9) from a suitable integral representation for $K_{iu}(v)$ (see for example [13], (8.8)). Finally, (3.10) is a particular case of the formula (26) on p. 50 of [4].

We will use the following simple estimates from the theory of the Riemann zeta-function:

$$\zeta(1/2+it) \ll t \quad (t\to\infty)\,, \tag{3.11}$$

$$\zeta(1+it)^{-1} \ll \log^7 t \quad (t\to\infty) \tag{3.12}$$

and

$$\int_0^T |\zeta(1/2+it)|^4\,dt \ll T\log^4 T \quad (T\to\infty)\,; \tag{3.13}$$

see [15], (2.12.2), (3.6.3) and (7.6.1).

4. Analytic properties of $\zeta_k(s)$. All needed facts from [5] and [14] are stated in the following two lemmas.

Lemma 1. *The function $\zeta_k(s)$ can be meromorphically continued onto the whole complex plane. The only singularities of $\zeta_k(s)$ in the half-plane* $\operatorname{Re} s \ge 1/2$ *are: a triple pole at $s=1$ and simple poles at $s=z_j$, $\overline{z}_j$ $(j=1,2,\ldots)$. For $z=1/2+i\kappa\in\{z_1,z_2,\ldots\}$ we have*

$$\operatorname*{Res}_{s=z}\zeta_k(s) = \frac{\sqrt{k}\,|\Gamma(z/2)|^4\Gamma(2\kappa i)}{(4k)^z\Gamma(z/2)^4}\sum_{z_j=z}|\varrho_j(1)|^2 t_j(k)H_j^2(1/2)\,. \tag{4.1}$$

Lemma 2. *For* $\operatorname{Re} s > 1/2$ *we have*

$$I_k(s) = B(s)+C(s)+D(s)\,, \tag{4.2}$$

where

$$B(s) = \left(\frac{\partial}{\partial w}-c\right)^2 I_k(w,s)\bigg|_{w=1}\,, \tag{4.3}$$

$$C(s) = \frac{\pi\sqrt{k}}{(4\pi k)^s\Gamma(s)} \times \int_{-\infty}^{\infty}\frac{k^{-iu}\sigma_{2iu}(k)|\xi(1/2+iu)|^4\Gamma(s-1/2-iu)\Gamma(s-1/2+iu)}{|\Gamma(1/2+iu)|^2|\zeta(1+2iu)|^2}\,du \tag{4.4}$$

and

$$D(s) = \frac{\sqrt{k}}{(4\pi k)^s\Gamma(s)} \times\sum_{j=1}^{\infty}|\varrho_j(1)|^2 t_j(k)H_j^2(1/2)|\Gamma(z_j/2)|^4\Gamma(s-z_j)\Gamma(s-\overline{z}_j)\,. \tag{4.5}$$

The above series converges absolutely.

LEMMA 3. *There is a $\kappa > 0$ such that*
$$\sum_{\kappa_j=\kappa} |\varrho_j(1)|^2 t_j(k) H_j^2(1/2) \neq 0\,.$$

P r o o f. Suppose the contrary. Then by (4.2) and (4.5) we have
$$I_k(s) = B(s) + C(s)\,.$$

For $y > 0$ consider

$$\begin{aligned}(4.6)\qquad m(y) :&= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} I_k(s) y^{-s}\,ds\\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} B(s) y^{-s}\,ds + \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} C(s) y^{-s}\,ds\\ &=: b(y) + c(y)\,.\end{aligned}$$

From (4.3), (3.4) and (3.10) we obtain

$$\begin{aligned}b(y) &= \left(\frac{\partial}{\partial w} - c\right)^2 \left[\frac{4\pi k^w \sigma_{1-2w}(k)}{\xi(2w)}\right.\\ &\qquad\left.\times \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s-w)\Gamma(s+w-1)}{\Gamma(s)}(4\pi k y)^{-s}\,ds\right]\bigg|_{w=1}\\ &= \left(\frac{\partial}{\partial w} - c\right)^2 \left[\begin{pmatrix}\text{a factor which depends}\\ \text{only on } k \text{ and } w\end{pmatrix} \cdot \frac{K_{w-1/2}(2\pi k y) e^{-2\pi k y}}{\sqrt{y}}\right]\bigg|_{w=1}.\end{aligned}$$

From this and (3.7) it follows that

$$(4.7)\qquad b(y) \ll y^{-1} e^{-4\pi k y} \qquad (y \to \infty)\,.$$

Next, using (4.4) and (3.10), we first obtain

$$c(y) = \frac{e^{-2\pi k y}}{2\sqrt{y}} \int_{-\infty}^{\infty} \frac{k^{-iu}\sigma_{2iu}(k)|\xi(1/2+iu)|^4 K_{iu}(2\pi k y)}{|\Gamma(1/2+iu)|^2|\zeta(1+2iu)|^2}\,du\,.$$

Let $T \geq 2$. From Stirling's formula, (3.11)–(3.13), (3.8) and (3.9) it follows that

$$\begin{aligned}c(y) &\ll \frac{e^{-2\pi k y}}{\sqrt{y}} \left[K_0(2\pi k y)\int_1^T |\zeta(1/2+iu)|^4 \frac{\log^{14} u}{u}\,du\right.\\ &\qquad\qquad \left. + \int_T^{\infty} u^3 \log^{14} u \cdot e^{(-3/2)u}\,du\right]\\ &\ll \frac{e^{-2\pi k y}}{\sqrt{y}}[K_0(2\pi k y)\log^{18} T + e^{-T}]\,.\end{aligned}$$

Putting $T := 3\pi ky$, we obtain by (3.6),

$$c(y) \ll y^{-2/3} e^{-4\pi ky} \quad (y \to \infty)\,. \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$m(y) \ll y^{-2/3} e^{-4\pi ky} \quad (y \to \infty)\,. \tag{4.9}$$

On the other hand, by (3.2), (3.3) and (4.6), we have

$$\begin{aligned} m(y) = \exp(-2\pi ky)\Big[&4d(k)K_0(2\pi ky)(\log y - c) \\ &+ 4\sum_{n=1}^{k-1} d(n)d(k-n)K_0(2\pi ny)K_0(2\pi(k-n)y) \\ &+ 8\sum_{n=1}^{\infty} d(n)d(n+k)K_0(2\pi ny)K_0(2\pi(n+k)y)\Big]\,. \end{aligned}$$

Thus, by (3.5) and (4.9), we conclude that

$$K_0(2\pi ky) \ll y^{-2/3} e^{-2\pi ky} \quad (y \to \infty)\,.$$

Comparison with (3.6) gives the desired contradiction.

5. Proof of the theorem. We are going to check whether the assumptions of Lemma 0 (Section 2) will be satisfied if we put there

$$g(x) := E_k(x) \quad (x \ge 1)\,.$$

We have of course

$$G(s) := \int_1^{\infty} g(x)x^{-s-1}\,dx = \frac{\zeta_k(s)}{s} - \sum_{\nu=1}^{3} \frac{a_\nu}{(s-1)^\nu} \quad (\mathrm{Re}\, s > 1) \tag{5.1}$$

with some constants a_ν ($\nu = 1, 2, 3$). By (1.1) and Lemma 1 (Section 4) we conclude that $G(s)$ is regular in the half-plane $\mathrm{Re}\, s > 1/2$ and that $G(s)$ has no singularity at $s = 1/2$. Also, by (4.1), Lemma 3 (Section 4) and (5.1), $G(s)$ has a simple pole at some $s = z \neq 1/2$ with $\mathrm{Re}\, z = 1/2$ such that

$$r := \operatorname*{Res}_{s=z} G(s) = \frac{1}{z} \operatorname*{Res}_{s=z} \zeta_k(s) \neq 0\,.$$

The theorem follows now immediately from Lemma 0.

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