The greatest prime factor of the integers in an interval

by

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Introduction. Let $P(x)$ denote the greatest prime factor of
\[ \prod_{n \leq x \leq x + x^{1/2}} n, \]
where $x$ is a large positive number. To estimate the lower bound of $P(x)$ is one of the “greatest prime factor” problems to which the Chebyshev–Hooley’s machinery can be applied. Chronologically Ramachandra [11], [12], Graham [4], Baker [1], and Jia [7] have contributed to the topic. In particular, Baker [1] proved
\[ P(x) > x^{0.7}. \]
Jia [7] improved the above result. But as [7] contained a substantial mistake concerning a multiple exponential sum (which was already picked out by the present author in 1987), the announced estimate $P(x) > x^{0.71}$ was actually not attainable there.

We prove the following better estimate.

THEOREM.
\[ P(x) > x^{0.723}. \]

We begin with
\[ x^{1/2} \ln x + O(x^{1/2}) = \sum_{p \leq P(x)} N(p) \ln p, \quad p \text{ primes}, \]
where
\[ N(p) = \sum_{x \leq pn \leq x + x^{1/2}} 1. \]

Thus to prove the Theorem we need to show
\[ \sum_{p \leq x^{0.723}} N(p) \ln p < (1 - \varepsilon)x^{1/2} \ln x, \quad \varepsilon > 0.\]
For a suitable number $\sigma$, $1/2 < \sigma < 0.723$, we write
\[
\sum_1 = \sum_{p \leq x^\sigma} N(p) \ln p, \quad \sum_2 = \sum_{x^\sigma \leq p \leq x^{0.723}} N(p) \ln p.
\]
We need to establish an asymptotic formula for $\sum_1$ with $\sigma$ as large as possible, and to estimate $\sum_2$ via the sieve.

The proof, as usual, depends mainly on treating some exponential sums, in particular we can prove an asymptotic formula for $\sum_1$ with $\sigma = 0.6 - \varepsilon$. We also make an innovation in the sieve part. To save space we do not analyze the method term by term; the interested reader can find the advantages by himself.

Notations. $e(\xi) = \exp(2\pi i \xi)$. $[\xi]$ is the integer part of $\xi$, and $\{\xi\} = \xi - [\xi]$. $\|\xi\| = \min(\{\xi\}, 1 - \{\xi\})$.

\[
\sum_{(x_1, x_2, \ldots) \in D} f(x, y) g(x, y)
\]
means summation over all lattice points $(x_1, x_2, \ldots)$ inside a domain $D$. $f(x, y) \tilde{\Delta} g(x, y)$ means that
\[
f_{x', y'}(x, y) = g_{x', y'}(x, y) + O(\Delta g_{x', y'}(x, y))
\]
whenever both sides make sense. $x \sim X$ means $X \leq x < 2X$, $x \asymp X$ means $C_1 X \leq x \leq C_2 X$ for some constants $C_1$ and $C_2$. The meaning of $x = O(X)$ or $x \ll X$ is as usual. $\varepsilon$, of course, is a sufficiently small number.

1. An asymptotic formula. In this section, we prove

**Proposition 1.** For $\sigma = 0.6 - \varepsilon$, we have
\[
(2) \quad \sum_1 = (0.6 - \varepsilon) x^{1/2} \ln x + O(x^{1/2}).
\]

As
\[
N(n) = \frac{x^{1/2}}{n} + \psi\left(\frac{x + x^{1/2}}{n}\right) - \psi\left(\frac{x}{n}\right), \quad \psi(\xi) = \frac{1}{2} - \{\xi\},
\]
in view of the Prime Number Theorem, to prove (2) it remains to show that (with $A(n)$ being the von Mangoldt function)
\[
\sum_{x^{1/2} \leq n \leq x^{0.6-\varepsilon}} A(n) \left(\psi\left(\frac{x + x^{1/2}}{n}\right) - \psi\left(\frac{x}{n}\right)\right) \ll x^{1/2}.
\]
We split the range $x^{1/2} \leq n \leq x^{0.6-\varepsilon}$ into ranges of the form $v \leq n \leq v'$, $v' \leq 2v$. Then we need to show that
\[
(\ast) \quad \sum_{v \leq n \leq v'} A(n) f(n) \ll x^{1/2} (\ln x)^{-1}, \quad f(n) = \psi\left(\frac{x + x^{1/2}}{n}\right) - \psi\left(\frac{x}{n}\right).
\]
In view of [1], (*) holds for \( x^{1/2} \ll v \ll x^{13/22-\varepsilon} \), thus we only need to consider the range \( x^{0.59} \leq v \leq x^{0.6-\varepsilon} \). We appeal to the following decomposition.

**Lemma 1.1.** Let \( 3 \leq u < w < z \leq 2v \), suppose that \( z - 1/2 \) is an integer, and \( z \geq 4u^2 \), \( v \geq 32z^2u \), \( w^3 \geq 64v \). Set
\[
U = \max_B \sum_{m=1}^{\infty} d_3(m) \sum_{\substack{z < n < B \\ v \leq mn \leq v'}} f(mn),
\]
\[
W = \sup_g \sum_{m=1}^{\infty} d_4(m) \sum_{\substack{u \leq n \leq w \\ v \leq mn \leq v'}} g(n)f(mn),
\]
where the supremum is taken over all arithmetic functions \( g(n) \) such that \( |g(n)| \leq d_3(n) \). Then
\[
(3) \quad U \ll x^{1/2}(\ln x)^{-10} \quad \text{and} \quad W \ll x^{1/2}(\ln x)^{-10}
\]
imply the estimate
\[
\sum_{v \leq n \leq v'} A(n)f(n) \ll x^{1/2}(\ln x)^{-1}.
\]

**Proof.** See [5].

Choose
\[
u = vx^{-1/2+15\eta}, \quad w = x^2v^{-3}x^{-10\eta}, \quad z = [x^{1/4-10\eta}] + \frac{1}{2}
\]
in Lemma 1.1, where \( \eta = \varepsilon^2 \), \( x^{0.59} \ll v \ll x^{0.6-\varepsilon} \). Then it is easy to verify that Lemma 1.1 is applicable. As in [1], by a reduction using the Fourier expansion of the function \( \psi(\xi) \), in order to show (3) it suffices to demonstrate, with \( H = vx^{-1/2+\eta} \), the following estimates:
\[
(4) \quad \sum_{0 < h \leq H} \left| \sum_{m \sim M} a(m) \sum_{n \sim N} e\left( \frac{hx'}{mn} \right) \right| \ll vx^{-\eta}
\]
whenever \( v \gg MN \), \( N \gg z \), \( |a(m)| \leq x^\eta \), \( x \ll x' \); and
\[
(5) \quad \sum_{0 < h \leq H} \left| \sum_{m \sim M} a(m) \sum_{n \sim N} b(n)e\left( \frac{hx'}{mn} \right) \right| \ll vx^{-\eta}
\]
whenever \( v \gg MN \), \( u \ll N \ll w \), \( |a(m)|, |b(n)| \leq x^\eta \), \( x \ll x' \).

**Lemma 1.2.** (5) holds for \( v \) in the \((\text{larger})\) range \( v \leq x^{5/8-\varepsilon} \).

**Proof.** This is Lemma 9 of [1].

Now it remains to show

**Lemma 1.** (4) holds.
We cite three lemmata of basic importance.

**Lemma 1.3.** Let $I$ be a subinterval of $[X, 2X]$, $Q$ be a positive number, and $z_n (X \leq n \leq 2X)$ be complex numbers. Then

$$\left| \sum_{n \in I} z_n \right|^2 \leq (1 + XQ^{-1}) \sum_{|q| \leq Q} (1 - |q|Q^{-1}) \sum_{n, n+q \in I} z_n z_{n+q}. $$

**Proof.** This is Weyl’s inequality. Cf. [1].

**Lemma 1.4.** Let $f(x)$ and $g(x)$ be algebraic functions in the interval $[a, b]$, and

$$|f''(x)| \equiv R^{-1}, \quad |f'''(x)| \ll (RU)^{-1},$$

$$|g(x)| \leq H, \quad |g'(x)| \ll HU_1^{-1}, \quad U, U_1 \geq 1.$$ Then

$$\sum_{\alpha \leq n \leq \beta} g(n)e(f(n)) = \sum_{\alpha \leq n \leq \beta} b_u \frac{g(n(u))}{\sqrt{f''(n(u))}} e(f(n(u)) - u n + 1/8)$$

$$+ O(H \min(\beta - \alpha + 2, b - a + R)(U^{-1} + U_1^{-1}))$$

$$+ O(H \min(R^{1/2}, \max(1/\langle \alpha \rangle, 1/\langle \beta \rangle))) ,$$

where $[\alpha, \beta]$ is the image of $[a, b]$ under the mapping $y = f'(x)$, $n(u)$ is determined by the equation $f'(n(u)) = u$, $b_u = 1/2$ or 1 according as $u$ is one of $\alpha$ and $\beta$ or not, and $\langle x \rangle$ is defined as follows:

$$\langle x \rangle = \begin{cases} \|x\| & \text{if } x \text{ is not an integer,} \\ \beta - \alpha & \text{if } x \text{ is an integer.} \end{cases}$$

Moreover, $\sqrt{f''} > 0$ if $f'' > 0$, $\sqrt{f''} = i \sqrt{|f''|}$ if $f'' < 0$.

**Proof.** This is Theorem 2.2 of S. H. Min [10].

**Lemma 1.5.** Let $f(x, y)$ be an algebraic function in the rectangle $D_0 = \{(x, y) | x \sim X, y \sim Y\}$, $f(x, y) \Delta x^\alpha y^\beta$ hold throughout $D_0$, and $D$ be a subdomain of $D_0$ bounded by $O(1)$ algebraic curves. Suppose that $X \geq Y, N = XY, A > 0, F = AX^\alpha Y^\beta, \alpha(\alpha + \beta - 1)(\alpha + \beta - 2) \neq 0, 0 < \Delta < \varepsilon_0$, where $\varepsilon_0$ is a small number depending at most on $\alpha$ and $\beta$. Then

$$\sum_{(x,y) \in D} e(f(x, y)) \ll_{\varepsilon, \alpha, \beta} (\sqrt{F^2 N^3} + N^{5/6} + \sqrt{\Delta Y^4} F^2 N^5)^6$$

$$+ \sqrt{F^{-1} X^{-1} N^8} + NF^{-1/4}$$

$$+ \sqrt{\Delta X^{-1} N^4} + NY^{-1/2})(NF)^9.$$ 

**Proof.** This is Lemma 1 of [9].
We also need the following auxiliary lemma.

**Lemma 1.6.** Let $M \leq N < N_1 \leq M_1$, and $a_n$ $(M \leq n \leq M_1)$ be complex numbers. Then

$$
\left| \sum_{N < n \leq N_1} a_n \right| \leq \int_{-\infty}^{\infty} K(\theta) \left| \sum_{M < m \leq M_1} a_m e(\theta m) \right| d\theta,
$$

with $K(\theta) = \min(M_1 - M + 1, (\pi |\theta|)^{-1}, (\pi \theta)^{-2})$, and

$$
\int_{-\infty}^{\infty} K(\theta) d\theta \leq 3 \ln(2 + M_1 - M).
$$

**Proof.** This is Lemma 2.2 of [2].

**Proof of Lemma 1.** For notational simplicity, we let $x' = x$ in (4). For $1 \leq h \leq H$, let

$$
S(h) = \sum_{m \sim M} a(m) \sum_{n \sim N} e\left(\frac{hx}{mn}\right).
$$

By Lemma 1.4, after a partial summation, we obtain

$$
(6) \quad x^{-\eta} S(h) \ll \left(\frac{vN^2}{hx}\right)^{1/2} \sum_{m \sim M} \left| \sum_{u \in I(m)} e(F(u, m)) \right| + \left(\frac{v^3}{hx}\right)^{1/2} + \frac{v^2}{hx} + M,
$$

where $I(m)$ is a subinterval of $[C_1 U, C_2 U]$, $U = h x v^{-1} N^{-1}$, $F(u, m) = C_3(h x u m^{-1})^{1/2}$ (for $i \geq 1$, $C_i$ denotes a constant). By Lemma 1.6 we get

$$
x^{-\eta} \sum_{m \sim M} \left| \sum_{u \in I(m)} e(F(u, m)) \right| \ll \sum_{m \sim M} \left| \sum_{u \in U} e(G(u, m)) \right| = S'(h), \quad \text{say},
$$

where $G(u, m) = F(u, m) + \theta u$, $\theta$ is independent of $u$ and $m$, and $0 \leq \theta < 1$.

Let

$$
Q = \min\left(\sqrt[5]{hxM^5 N^{-5}}, hxM^{-1} N^{-2}\right)x^{-\eta}.
$$

If $Q \leq 100$, then we get, trivially,

$$
\left(\frac{vN^2}{hx}\right)^{1/2} S'(h) \ll (hxMN^{-1} Q^{-1})^{1/2} \ll (MN^{1/2} + \sqrt[5]{hx^7 M^3 N^{-3}}) x^{-\eta}
$$

$$
\ll (v x^{-1/8} + (v^5 x)^{1/8}) x^{10\eta} \ll x^{1/2 - 10\eta}.
$$

If $Q > 100$, by Cauchy’s inequality, Lemma 1.3, and partial summation, we get, with some $Q_1, 1 \leq Q_1 \leq Q$, the inequality

$$
(7) \quad |S'(h)|^2 \ll (MQ)^2 Q^{-1} + M^{3/2} U Q^{-1} \left| \sum_{(u, q) \in D} \sum_{m \sim M} m^{-1/2} e(f(u, q, m)) \right|,
$$

where $D = \{(u, q) \mid q \sim Q_1, u \gtrsim U, u + q \gtrsim U\}$ and $f(u, q, m) = C_3(hxm^{-1})^{1/2}(u^{1/2} - (u + q)^{1/2}) - q\theta$. We apply Lemma 1.4 to transform
the sum over \( m \) to a sum over \( w \), where \( w \equiv W = NQ_1M^{-1} \). Then we change the order of summation, and estimate the sum over \( w \) trivially, to get, with some \( w \), the estimate

\[
(8) \quad M^{3/2}UQ^{-1} \left| \sum_{(u,q) \in D} \sum_{m \sim M} m^{-1/2} e(f(u,q,m)) \right| \\
\ll hxN^{-3/2}Q^{-1/2} \left| \sum_{(u,q) \in D_1} e(g(u,q)) \right| \\
+ (hx)^2Q^{-1/2}N^{-9/2} + (hx)^2Q^{-1}N^{-5} + (hx)^2M^{-1}N^{-4},
\]

where

\[
D_1 = D \cap \{ (u,q) \mid 1 \leq |C_3(2w)^{-1}(hx)^{1/2}M^{-3/2}(u^{1/2} - (u + q)^{1/2})| \leq \sqrt[3]{4} \},
\]

\[
g(u,q) = C_4(hxw(u^{1/2} - (u + q)^{1/2}))^{1/3} - q \theta.
\]

It is easy to verify that

\[
g(u,q) \Delta C_5(hxw)^{1/3}u^{-1/3}q^{2/3}, \quad \Delta = Q_1U^{-1}.
\]

Choosing \( X = U, Y = Q_1, F \equiv NQ_1, \Delta = Q_1U^{-1} \) in Lemma 1.5, we find that

\[
(9) \quad hxN^{-3/2}Q^{-1/2} \left| \sum_{(u,q) \in D_1} e(g(u,q)) \right| \\
\ll \left( \sqrt{(hx)^9Q^2M^{-3}N^{-13}} + \sqrt{(hx)^{11}Q^2M^{-5}N^{-15}} \right) \\
+ 10\sqrt{(hx)^{11}Q^{10}M^{-1}N^{-15}} + 8\sqrt{(hx)^{15}Q^4M^{-7}N^{-27}} \\
+ 4\sqrt{(hx)^8QM^{-4}N^{-15}} + 4\sqrt{(hx)^6Q^3M^{-2}N^{-16}} \\
+ \sqrt{(hx)^4M^{-2}N^{-7}}x^{2\eta}.
\]

From (6)–(9), we deduce that

\[
(10) \quad \left( \frac{vN^2}{hx} \right)^{1/2} S'(h) \ll \left( \frac{12\sqrt{(hx)^3M^3N^5Q^2} + 2\sqrt{(hx)^5MN^{-1}Q^2}}{vN^2} \right) \\
+ \frac{2\sqrt{(hx)MN^9Q^{10}} + 16\sqrt{(hx)^7MN^{-5}Q^3}}{vN^2} \\
+ 4\sqrt{(hx)^4N^{-3}Q} + 4\sqrt{(hx)^2v^2Q^3} + \sqrt{(hx)^2N^{-1}} \\
+ 4\sqrt{(hx)^2M^2N^{-3}Q^{-1}} + \sqrt{hxMN^{-1}Q^{-1}}x^{10\eta}.
\]

As we have already seen,

\[
\sqrt{hxMN^{-1}Q^{-1}} \ll x^{1/2 - 20\eta},
\]

thus
From (9)–(11), we get
\[
\begin{align*}
&\left(\frac{vN^2}{hx}\right)^{1/2} S'(h) \ll \left(\frac{48}{\sqrt{hx}}\right)^{13} M^{17} N^{15} + \frac{48}{\sqrt{hx}} 21 M^9 N^{-9} \\
&\quad + \frac{96}{\sqrt{hx}^3} M^{33} N^{-29} + \frac{96}{\sqrt{hx}^3} M^{23} N^{-39} \\
&\quad + \frac{48}{\sqrt{hx}^2} M^{5} N^{-29} + \frac{48}{\sqrt{hx}^2} M^{31} N^{-61} \\
&\quad + \frac{48}{\sqrt{hx}} M^{21} N^{-11} \right)^{10\eta} + x^{1/2-10\eta} \\
&\ll \left( \frac{96}{\sqrt{vx}^4} N^{-4} + \frac{96}{\sqrt{vx}^2} N^{-36} + \frac{160}{\sqrt{v}140} N^{-32} \\
&\quad + \frac{256}{\sqrt{v}76} x^{33} N^{-68} \\
&\quad + \frac{128}{\sqrt{v}60} x^{19} N^{-60} + \frac{128}{\sqrt{v}50} x^{30}\eta + x^{1/2-10\eta} \\
&\ll \left( \frac{96}{\sqrt{vx}^4} x^{12} + \frac{160}{\sqrt{v}140} x^{-4} + \frac{128}{\sqrt{v}82} x^{14} + \frac{64}{\sqrt{v}38} x^{8} \\
&\quad + \frac{256}{\sqrt{v}50} x^{2} + \frac{160}{\sqrt{v}33} x^{30}\eta + x^{1/2-10\eta} \right)^{1/2-10\eta}.
\end{align*}
\]

From (6), (7) and the above estimate, we get
\[
S(h) \ll \left( \frac{v^3}{hx} \right)^{1/2} + \frac{v^2}{hx} + \frac{vx^{-4}}{1/4} \right) x^{20\eta} + x^{1/2-8\eta},
\]
hence
\[
\sum_{1 \leq h \leq H} |S(h)| \ll (v^2 x^{-3/4}) x^{30\eta} + vx^{-\eta} \ll vx^{-\eta}.
\]
The proof of Lemma 1 is finished.

Remark. The proof of Lemma 6 of [7] is false. On p. 191 we find the equality $E_1 = E_2$, where
\[
E_1 = \sum_{q=1}^{Q} \sum_{(h,m) \in \mathcal{P}_q} \sum_{(h',m') \in \mathcal{P}_q} \varepsilon_h \varepsilon_{h'} \sum_{n \sim N} e \left( \frac{x}{n} \left( \frac{h}{m} - \frac{h'}{m'} \right) \right),
\]
\[
E_2 = \sum_{h \sim J} \sum_{h' \sim J} \sum_{0 \leq k \leq M hh' (JQ)^{-1}} \sum_{m \sim M} \sum_{m' \sim M} \sum_{n \sim N} e \left( \frac{hx}{mm' n} \right),
\]
and
\[
\mathcal{P}_q = \left\{ (h, m) \mid h \sim J, m \sim M, \frac{M(q-1)}{JQ} \leq \frac{m}{h} \leq \frac{Mq}{JQ} \right\}.
\]
But we observe that
\[
0 \leq k \leq M hh' (JQ)^{-1} \quad \text{and} \quad m'h - mh' = k.
\]
cannot imply that there is a \( q, 1 \leq q \leq Q, \) such that both \((h, m) \in P_q\) and \((h', m') \in P_q\); this means that there is no one-to-one correspondence between the summation variables of \( E_1 \) and those of \( E_2 \), thus in general \( E_1 \neq E_2 \).

2. Multiple exponential sums. Here we give several results on exponential sums, thus preparing an application of the sieve.

**Lemma 2.** For \( v \approx MN, H = vx^{-1/2+\eta}, x^{0.6} \ll vx^\epsilon \ll x^{2/3}, |a(n)|, |b(m)| \leq x^\eta, \) and \( v x^{-1/2+30\eta} \ll N \ll (vx^{1/2})^{1/7} x^{-30\eta} \), we have

\[
\sum_{1 \leq h \leq H} \sum_{n \sim N} \sum_{m \sim M} a(n)b(m)e \left( \frac{hx}{mn} \right) \ll vx^{-\eta}.
\]

To prove Lemma 2, we need two lemmata.

**Lemma 2.1.** Let \( H_1 \geq H_1' \geq 1, H_2 \geq H_2' \geq 1, n_1 \) and \( n_2 \) be positive integers with \((n_1, n_2) = 1\). Then

\[
\omega(n_1, n_2; r) := \sum_{h_1 n_1 - h_2 n_2 = r}^1 \hat{\omega}(n_1, n_2; \theta)e(\theta r) \, d\theta
\]

and

\[
\int_0^1 |\hat{\omega}(n_1, n_2; \theta)| \, d\theta \ll (1 + H_1 H_2 n_1^{-1} n_2^{-1})^{1/2}(\ln(2H_1 H_2))^2,
\]

where \(*\) means the condition \( H_1' \leq h_1 \leq H_1, H_2' \leq h_2 \leq H_2.\)

**Proof.** This is Lemma 8 of [3].

**Lemma 2.2.** Suppose \( \alpha = a/q + \theta/q^2 \), \((a, q) = 1, q \geq 1, |\theta| \leq 1. \) Then for any \( \beta, U > 0, P \geq 1, \) we have

\[
\sum_{x=1}^P \min \left( U, \frac{1}{\|\alpha x + \beta\|} \right) \leq 6 \left( \frac{P}{q} + 1 \right) (U + q \ln q).
\]

**Proof.** This is Lemma 6 of Chapter 5 of [8].

**Proof of Lemma 2.** We assume that \( x \) is irrational. Pick an integer \( j \) such that \( M \sim M_1 = 2^j \). We denote the triple exponential sum of Lemma 2 by \( S(M, N) \). By Cauchy’s inequality, we have \((m \sim M_1 \text{ means } M_1 \leq m \leq 4M_1)\)

\[
x^{-2\eta}S(M, N)^2 \ll M^{3/2} \sum_{m \leq M_1} m^{-1/2} \left| \sum_{h=1}^H \sum_{n \sim N} a(n)e \left( \frac{hx}{mn} \right) \right|^2
\]

\[
= M^{3/2} \sum_{h, h_1, n, n_1} a(n_1 \overline{a(n)}) \sum_{m \sim M_1} m^{-1/2}e \left( \left( \frac{h_1}{n_1} - \frac{h}{n} \right) \frac{x}{m} \right).
\]
The terms with $h_1n = hn_1$ contribute at most $O(M^2HNx^n)$. We classify the remaining terms according to the value of $(n, n_1)$. After a familiar argument, we get

\begin{equation}
\sum_{d \sim D} \sum_{n, n_1 \sim N_1} \sum_{r \sim R} \omega(n, n_1; r) \sum_{m \geq M_1} e\left(\frac{-r\chi}{dnn_1}\right) m^{-1/2} + M^2HN,
\end{equation}

for some $1 \leq D \leq N$, $N_1 = N/d$, and some $R$, $1 \leq R \leq HN/D$. By Lemma 1.4, the innermost sum is equal to

\begin{equation}
\sum_{\alpha \leq u \leq \beta} u^{-1/2} e\left(C_6\left(\frac{ux}{dnn_1}\right)^{1/2}\right) C_7
\end{equation}

\begin{align*}
+ O\left(M^{-1/2} \min \left(\left(\frac{M^3N^2}{RDx}\right)^{1/2}, \max \left(\frac{1}{\|\alpha\|}, \frac{1}{\|\beta\|}\right)\right)\right) \\
+ O\left(M^{-1/2} N^2 x^n\right) + O(M^{-1/2} x^n),
\end{align*}

where

\begin{align*}
\alpha &= \frac{rx}{4M_1^2dnn_1}, \\
\beta &= \frac{rx}{M^2dnn_1}.
\end{align*}

We consider the sum

\begin{equation}
S^\ast(D, R) = M \sum_{d \sim D} \sum_{n, n_1 \sim N_1} \sum_{r \sim R} \omega(n, n_1; r) \min \left(\left(\frac{M^3N^2}{RDx}\right)^{1/2}, \frac{1}{\|\beta\|}\right);
\end{equation}

a similar sum with $\beta$ being replaced by $\alpha$ can be treated analogously.

We shall prove the following estimate for $S^\ast(D, R)$:

\begin{equation}
S^\ast(D, R) \ll v^2x^{-10\eta} \quad \text{if } N \leq x^{1/4-20\eta}.
\end{equation}

If $D \geq (MN)^2x^{-1-\eta}$, then we trivially get

\begin{equation}
S^\ast(D, R) \ll M \left(\frac{M^3N^2}{RDx}\right)^{1/2} D \sum_{|nh_1-nh_1| \leq 2R} 1 \\
\ll (MN)^{5/2} H^{3/2} x^{-1/2} D^{-1} x^n \\
\ll x^{1/2} (MN)^{1/2} H^{3/2} x^{2\eta} \ll v^2x^{-1/4+20\eta}.
\end{equation}

If $D \leq (MN)^2x^{-1-\eta}$, then we see that

\[ \frac{\beta}{r} = \frac{1}{q} + \frac{\theta}{q^2}, \quad \text{with } q = [dnn_1M_1^2x^{-1}] \geq 1 \text{ and } |\theta| \leq 1, \]
hence by Lemmata 2.1 and 2.2, we have the estimate
\[ S^*(D,R) \ll MD \frac{N^2}{D^2} \left(1 + \frac{DH}{N}\right) \max_{d,n,n_1} \sum_{r \sim R} \min \left( \left( \frac{M^3N^2}{RDx} \right)^{1/2}, 1 \right) \]
\[ \ll MN^2 \left( \frac{RDx}{M^2N^2} + 1 \right) \left( \frac{N^2M^3}{RDx} \right)^{1/2} + \frac{N^2M^2}{Dx} x^\eta \]
\[ \ll (vNx^{1/4} + v^2x^{-1/2}N^2 + v^{5/2}N^{1/2}x^{-1/2} + v^3Nx^{-1})x^{10\eta} \]
\[ \ll v^2x^{-10\eta} \]
p provided only \( N \leq x^{1/4-20\eta} \). Thus anyhow (2.3) holds. By inserting (2.2) in (2.1), and taking into account (2.3), we get, after changing the order of summation, the following estimate (with \( U = RDxv^{-2} \)):
\[ x^{-3\eta}|S(M,N)|^2 \]
\[ \ll \frac{NM^5/2}{(RDx)^{1/2}} \sum_{d \sim D} \sum_{n,n_1 \sim N_1} \sum_{u \geq U} \left| \sum_{r \in I} \omega(n,n_1;r) e \left( C_6 \left( \frac{urx}{dn_1} \right)^{1/2} \right) \right| \]
\[ + v^2x^{-10\eta}, \]
where \( I \) is some subinterval of \([R,2R]\), which may depend on the variables outside the absolute value symbol. By Lemma 1.6 we get
\[ x^{-4\eta}|S(M,N)|^2 \]
\[ \ll \frac{NM^5/2}{(RDx)^{1/2}} \sum_{d \sim D} \sum_{n,n_1 \sim N_1} \sum_{u \geq U} \left| \sum_{r \geq R} \omega(n,n_1;r) e \left( C_6 \left( \frac{urx}{dn_1} \right)^{1/2} + \theta r \right) \right| \]
\[ + v^2x^{-10\eta}, \]
with some \( \theta \), which is independent of the variables \( r, u, n, n_1 \) and \( d \). Now we have arrived at (6.1) of [3], p. 325. The argument in what follows is exactly the same as in [3], and we get (cf. p. 329 of [3]), by the assumption of Lemma 2,
\[ x^{-10\eta}|S(M,N)|^4 \ll v^2 \left( \left( \frac{Hx}{v} \right)^{1/2} H^2N^4 \right) \]
\[ + \left( \frac{Hx}{v} \right)^{1/2} H^{3/2}N^{7/2} + \left( \frac{Hx}{v} \right)^{1/2} N^3H^2M^{1/2} \]
\[ + \left( \frac{Hx}{v} \right)^{1/2} H^{4/3}M^{1/3}N^{8/3} \]
\[ + (v^2x^{-10\eta})^2 \]
\[ \ll v^4x^{-20\eta}. \]
Lemma 2 is proven.
Lemma 3. For $v \equiv MN$, $H = vx^{-1/2+\eta}$, $x^{0.6} \ll vx^{\varepsilon} \ll x^{3/4}$, $|a(n)|, |b(m)| \leq x^n$, $(k, \lambda)$ an exponent pair, and $vx^{-1/2+10\eta} \leq N \leq \min((v^{1-\lambda+k}x^{-k+\lambda/2-1/4})^{1/(1-k+\lambda)}, x^{1/4})^{x^{-20\eta}}$, we have

$$\sum_{1 \leq h \leq H} \sum_{m \sim M} \sum_{n \sim N} a(n)b(m)e\left(\frac{hx}{mn}\right) \ll vx^{-\eta}.$$

Proof. Note that (2.4) holds provided that (2.5)

$$v \leq x^{3/4-\varepsilon}, \quad N \leq x^{1/4-20\eta}.$$

In view of Lemma 2.1, (2.4) and (2.5), we get

$$x^{-4\eta}|S(M, N)|^2 \ll \frac{N^3 M^{3/2}}{(RD)^{1/2}} \sum_{r \sim R} e(F(r, d, n, n_1, u)) + v^2 x^{-10\eta},$$

where $F(r, d, n, n_1, u) = C_0(wx/(dnn_1))^{1/2} + \theta r$, for certain $d, n, n_1, u$, and $\theta$ with $|\theta| \leq 1$. It is easy to see that $F'(r, d, n, n_1, u) \equiv xD/(vN) \gg 1$, thus

$$\sum_{r \sim R} e(F(r, d, n, n_1, u)) \ll R^\lambda \left(\frac{Dx}{vN}\right)^k.$$

Lemma 3 follows from (2.6), (2.7) and the fact that $R \ll HN$.

The last result of the section is

Lemma 4. For $v \equiv MN$, $H = vx^{-1/2+\eta}$, $x^{0.6} \ll vx^{\varepsilon} \ll x^{3/4}$, $|a(n)| \leq x^\eta$ and $N \leq x^{3/8-\varepsilon}$, we have

$$\sum_{1 \leq h \leq H} \sum_{m \sim M} \sum_{n \sim N} a(n)e\left(\frac{hx}{mn}\right) \ll vx^{-\eta}.$$

To prove Lemma 4 we need again two lemmata.

Lemma 2.3. Let $\alpha, \beta, \gamma$ be real constants such that $(\alpha - 1)\beta \gamma \neq 0$. Let $M, R, S, x \geq 1$ and let $\phi_m$ and $\psi_{rs}$ be complex numbers with modulus not exceeding $1$. Then

$$\sum_{m \sim M} \sum_{r \sim R} \sum_{s \sim S} \phi_m \psi_{rs} e\left(\frac{xm^{\alpha}r^{\beta}S^\gamma}{M^\alpha R^\beta S^\gamma}\right) \ll (x^{1/4}M^{1/2}(RS)^{3/4} + M^{7/10}RS + M(RS)^{3/4}) + x^{-1/4}M^{11/10}(\ln(10MRS))^5.$$

Proof. This is Theorem 3 of [3].

Lemma 2.4. Let $X \geq 100$, $Y \geq 100$, $A > 0$, $f(x, y) = Ax^\alpha y^\beta$, $F = AX^\alpha Y^\beta$, $\alpha$ and $\beta$ being rational numbers (not positive integers). Suppose
\( F^{-2}X^4 \leq Y^3N^{-\eta}, \) and for \( a_x \) and \( b_y \) being complex numbers with modulus not exceeding 1, define

\[
S(X, Y) := \sum_{(x,y) \in D} a_x b_y e(f(x, y)),
\]

where \( D \) is some region contained in the rectangle \( \{(x, y) \mid x \sim X, y \sim Y\} \) such that for a fixed \( \tilde{x} \), \( \tilde{x} \sim X \), the intersection \( D \cap \{(\tilde{x}, y) \mid y \sim Y\} \) has at most \( O(1) \) segments. Then, for \( W = X^5 + Y^5 \),

\[
(2.8) \quad S(X, Y) \ll (\sqrt[5]{F^4X^{24}Y^{27}W} + \sqrt[5]{F^{-4}X^{28}Y^{39}W} + \sqrt[5]{F^{-1}X^{3}Y^{5}} + \sqrt[5]{F^{4}X^{13}Y^{19}} + \sqrt[5]{F X^{2}Y^{5}} + \sqrt[5]{F^{4}X^{13}Y^{7}} + \sqrt[5]{F^{-3}X^{16}Y^{13}} + \sqrt[5]{F^{-3}X^{6}Y^{13}})(FXY)^{\theta} =: E.
\]

**Proof.** Put \( \delta = \eta^2 \). By Lemma 1.6, we have (with \( N = FXY \))

\[
N^{-\delta}|S(X, Y)| \ll \sum_{x \sim X} \sum_{y \sim Y} C(y)e(f(x, y)),
\]

where \( C(y) = b_y e(\theta y) \) with some real number \( \theta \) (which is independent of \( x \) and \( y \)). We choose

\[
Q = (F^{-1}X^2Y)^{2/5} \leq YN^{-\delta} \quad \text{(by assumption)}.
\]

If \( Q \leq N^{\theta} \), then we trivially get

\[
|S(X, Y)| \ll N^{\theta/2}XYQ^{-1/2} \ll N^{\theta/2}(\sqrt[5]{F X^2Y^3}) \ll E.
\]

Assume that \( Q \geq N^{\theta} \). By Cauchy’s inequality and Lemma 1.3, we get

\[
(2.9) \quad N^{-3\theta}|S(X, Y)|^2 \ll \left( \frac{XY}{Q} \right)^2 + \frac{XY}{Q} \sum_{(y, q) \in D_1} C(y) C(y + q) \sum_{x \sim X} e(Ax^{\alpha}t(y, q)),
\]

where \( D_1 = \{(y, q) \mid y, y + q \sim Y, q \sim Q_1\} \) for some \( Q_1 \), \( 1 \leq Q_1 \leq Q \), and \( t(y, q) = (y + q)^\beta - y^\beta \). Applying Lemma 1.4 to the innermost sum, we get

\[
(2.10) \quad \sum_{x \sim X} e(Ax^{\alpha}t) = \sum_{u \in I} C_7[|At|^\gamma u^{-1/2 - \gamma} |e(C_8(At)^{2\gamma} u^{1-2\gamma})
\]

\[
+ O\left( \min \left( \left( \frac{X^2Y}{Q_1F} \right)^{1/2}, \frac{1}{\|g_1(y, q)\| + \|g_2(y, q)\|} \right) + \frac{XY}{Q_1F} + \ln N + R(y, q) \right),
\]

where \( I = (C_8AX^{\alpha-1}|t|, C_10AX^{\alpha-1}|t|) \), \( \gamma = 1/(2(1 - \alpha)) \), \( g_1 = \alpha AX^{\alpha-1}t \), \( g_2 = \alpha A(2X)\alpha-1t \), and \( R(y, q) = 0 \) or \( O((X^2YQ_1^{-1}F^{-1})^{1/2}) \) according as
none of the end points of $I$ is an integer or otherwise. For each fixed $q$, $q \sim Q$, we consider the sum
\[ \Phi = \sum_{y \sim Y} \min((X^2YQ_1^{-1}F^{-1})^{1/2}, 1/\|g(y)\|), \]
where $g(y)$ is either $g_1(y, q)$ or $g_2(y, q)$. As both $g(y)$ and $g'(y)$ are monotonic, and $g'(y) \equiv FQ_1X^{-1}Y^{-2}$, we can classify $y$ according to the integer part of $g(y)$, and it is easy to get
\[ \Phi \ll (1 + FQ_1X^{-1}Y^{-1})((X^2YQ_1^{-1}F^{-1})^{1/2} + F^{-1}Q_1^{-1}XY^2 \ln N), \]
which contributes to the RHS of (2.9) at most
\[
(2.11) \quad \ll \frac{10}{\sqrt{F^{-3}X^{16}Y^{13}}} + \frac{5}{\sqrt{F^{-3}X^{6}Y^{13}}} + \frac{10}{\sqrt{F^{-3}X^{14}Y^{7}}} + \sqrt{XY^2} \ll E^2N^{-2\eta},
\]
and similarly for the contribution of $R(y, q)$. From (2.9)–(2.11), after changing the order of summation, and estimating the sum over $u$ trivially, we get, with some $u$, $|u| \equiv FQ_1X^{-1}Y^{-1}$, the estimate
\[
(2.12) \quad N^{-\delta}|S(X, Y)|^2 \ll \frac{XY}{Q} \left(1 + \frac{FQ}{XY}\right) A^7 \left(\frac{XY}{FQ_1}\right)^{\gamma+1/2} |S_1| + N^{-\eta}E^2,
\]
\[ S_1 = S_1(u) = \sum_{(y, q) \in D_2} C(y) C(y + q) t^\gamma e(C_8 (At)^{2\gamma} u^{1-2\gamma}), \]
\[ D_2 = D_1 \cap \{(y, q) \mid (C_0 \alpha X^{\alpha - 1}) |t| \geq u \geq (C_9 \alpha X^{\alpha - 1}) |t|\}. \]
If $Q_1 \leq N^\eta$, then we trivially have
\[
\frac{XY}{Q} \left(1 + \frac{FQ}{XY}\right) A^7 \left(\frac{XY}{FQ_1}\right)^{\gamma+1/2} |S_1| \ll \frac{XY}{Q} \left(1 + \frac{FQ}{XY}\right) XY^{3/2}F^{-1/2}N^{1/2} \ll \sqrt{F^{-1}X^{12}Y^{21}} + \sqrt{FX^2Y^3} \ll \sqrt{F^{-1}X^{3}Y^{5}} + \frac{10}{\sqrt{FX^{15}Y^{19}}} + \sqrt{FX^2Y^3} \ll N^{-\eta}E^2.
\]
Assume that $Q_1 \geq N^\eta$. By Lemma 1.6 we get
\[
(2.13) \quad N^{-\delta} |S_1| \ll \sum_{y \sim Y} \left| \sum_{q \sim Q_1} C(y + q) e(\varphi q) t^\gamma e(C_8 (At)^{2\gamma} u^{1-2\gamma}) \right|,
\]
with some real number $\varphi$, independent of $y$ and $q$. Assume that $t(y, q) > 0$ (otherwise we consider $-t$). Applying Cauchy’s inequality and Lemma 1.3
to (2.13), we get, with $Q_2 = Q_1^{1/2}N^{-\delta}$, $G = (Q_1Y^{\beta-1})^\gamma$,

$$N^{-3\delta}|S_1|^2 \ll \frac{(YQ_1G)^2}{Q_2} + \frac{YQ_1}{Q_2} \sum_{1 \leq |q_1| \leq Q_2} |S_2(q_1)|,$$

(2.14)

$$S_2(q_1) = \sum_{(y, q) \in D_3} C(y + q)C(y + q + q_1)t_1(y, q, q_1)e(t_2(y, q, q_1)),$$

$$D_3 = \{(y, q) \mid y \sim Y, q + q_1 \sim Q_1\},$$

$$t_1(y, q, q_1) = (t(y, q)I(y, q + q_1))^{\gamma},$$

$$t_2(y, q, q_1) = C_q u \left( \frac{A}{u} \right)^{2\gamma} (t(y, q + q_1)^{2\gamma} - t(y, q)^{2\gamma}).$$

By writing $y + q = z$ we get

$$S_2(q_1) = \sum_{(z, q) \in D_4} C(z)C(z + q_1)T_1(z, q, q_1)e(T_2(z, q, q_1)),$$

$$D_4 = \{(z, q) \mid q + q_1 \sim Q_1, z - q \sim Y\},$$

$$T_1(z, q, q_1) = (t(z - q, q)t(z - q, q + q_1))^{\gamma},$$

$$T_2(z, q, q_1) = C_q u \left( \frac{A}{u} \right)^{2\gamma} (t(z - q, q + q_1)^{2\gamma} - t(z - q, q)^{2\gamma}).$$

Again byLemma 1.6, we get, with $I_1 = [0.5Y, 2.5Y]$, 

$$N^{-\delta}|S_2(q_1)| \ll \sum_{z \in I_1} \left| \sum_{q \sim Q_1} T_1(z, q, q_1)e(\xi q)e(T_2(z, q, q_1)) \right|,$$

with some real number $\xi$, independent of $z$ and $q$. Applying Cauchy’s inequality and Weyl’s lemma, we obtain, with $Q_3 = Q_1N^{-\delta}$,

$$N^{-3\delta}|S_2(q_1)|^2 \ll (YQ_1G^2)^2Q_3^{-1} + YQ_1Q_3^{-1} \sum_{1 \leq |q_2| \leq Q_3} \sum_{q \sim Q_1} |S_3(q, q_1, q_2)|,$$

(2.15)

$$S_3(q, q_1, q_2) = \sum_{z \in I_1} T_3(z, q, q_1, q_2)e(T_4(z, q, q_1, q_2)),$$

$$T_3(z, q, q_1, q_2) = T_1(z, q, q_1)T_1(z, q + q_1, q_2),$$

$$T_4(z, q, q_1, q_2) = T_2(z, q + q_2, q_1) - T_2(z, q, q_1).$$

Let

$$U(z) = T_3(z, q, q_1, q_2), \quad V(z) = T_4(z, q, q_1, q_2).$$

It is an easy exercise to verify that, for $z \in I_1$, 

$$U(z) \equiv G^4, \quad U'(z) \equiv G^4Y^{-1},$$

$$V(z) \equiv \langle Fq_1q_2 \rangle Y^{-1}Q_1^{-1}, \quad V'(z) \equiv \frac{Fq_1q_2}{YQ_1Y}.$$
As $V(z)$ has nice properties with respect to the variable $z$, by partial summation and the exponent pair $(1/2, 1/2)$, we obtain

$$S_3(q, q_1, q_2) \ll G^4((|Fq_1q_2|Y^{-2}Q_1^{-1})^{1/2}Y^{1/2} + Y^2Q_1(|Fq_1q_2|)^{-1}).$$

In view of (2.14) and (2.15), the first term in (2.16) contributes to (2.12) at most

$$\ll N^{\eta/2} \frac{XY}{Q} \left(1 + \frac{FQ}{XY}\right) A^7 \left(\frac{XY}{FQ_1}\right)^{1/2+\gamma} G^{16} \sqrt{Y^{10}Q_1^{17}F^2},$$

$$\ll \left(1 + \frac{FQ}{XY}\right)^{\psi} F^{-6} X^{32} Y^{34} Q^{-7} N^{\eta/2},$$

$$\ll \left(\frac{2}{4} F^8 X^{24} Y^{27} W + \frac{3}{4} F^{-4} X^{28} Y^{39} W\right) N^{\eta/2}$$

$$\ll E^2 N^{-\eta};$$

and the second term in (2.16) contributes to (2.12) at most

$$\ll N^{\eta/2} \frac{XY}{Q} \left(1 + \frac{FQ}{XY}\right) A^7 \left(\frac{XY}{FQ_1}\right)^{1/2+\gamma} G(Q_1^7 Y^{10} F^{-2})^{1/8}$$

$$\ll \frac{\psi}{Q} F^{-5} X^{16} Y^{22} F^{-6} \left(1 + \frac{FQ}{XY}\right) N^{\eta/2}$$

$$\ll \left(\sqrt{F^{-1} X^3 Y^5} + \sqrt{F X^{13} Y^{19}}\right) N^{\eta/2} \ll E^2 N^{-\eta}.$$}

Finally, the term $(Y Q_1 G^2)^2 Q_3^{-1}$ together with the term $(Y Q_1 G)^2 Q_2^{-1}$ contributes to (2.12) at most

$$N^{\eta/2} \frac{XY}{Q} \left(1 + \frac{FQ}{XY}\right) A^7 \left(\frac{XY}{FQ_1}\right)^{1/2+\gamma} Y G Q_1^{3/4}$$

$$\ll X^2 Y^{5/2} F^{-1/2} Q^{-3/4} \left(1 + \frac{FQ}{XY}\right) N^{\eta/2} \ll E^2 N^{-\eta}.$$}

The estimate (2.8) follows from the above observations.

**Proof of Lemma 4.** By Lemma 1.4, we obtain

$$\sum_{m \sim M} e(-\frac{h_x}{mn}) = \sum_{\alpha \in I} C_{11}(m\nu^{-3} h^{-1})^{1/4} e\left(C_{12}\left(\frac{h_x}{n}\right)^{1/2}\right)$$

$$+ O\left(\left(\frac{M^3 N}{h_x}\right)^{1/2}\right) + O\left(\frac{M^2 N}{h_x}\right) + O(ln x),$$

where $I = (\alpha, \beta)$, $\alpha = h_x/(4M^2n)$, $\beta = h_x/(M^2n)$. We divide the sum in question into subsums of the form

$$\sum_{h \sim H_1} \left|\sum_{n \sim N} a(n) \sum_{m \sim M} e\left(-\frac{h_x}{mn}\right)\right| = S(H_1), \quad \text{say},$$

$$\sum_{n \sim N} a(n) \sum_{m \sim M} e\left(-\frac{h_x}{mn}\right) \ll H_1 \frac{1}{\varphi(\rho)} \left(\frac{N}{M}\right)^{1/2} H_1^{1/2} + H_1 \frac{1}{\varphi(\rho)} \left(\frac{N}{M}\right)^{1/2} H_1^{1/2} \ll H_1 \frac{1}{\varphi(\rho)} \left(\frac{N}{M}\right)^{1/2} H_1^{1/2},$$

$$\ll H_1 \sqrt{\frac{N}{M}} \ll H_1 \sqrt{\frac{N}{M}}.$$}

In view of (2.14) and (2.15), the first term in (2.16) contributes to (2.12) at most
where $H_1 \leq H$. By substituting (2.17) in (2.18), and exchanging the order of summation, we get, with an application of Lemma 1.6,

$$S(H_1) \ll \left( \frac{M^3 N}{H_1 x} \right)^{1/2} \sum_{w \sim W} \left| \sum_{n \sim N} b(n)e(\theta n)e \left( C_{12} \left( \frac{w x}{n} \right) \right) \right| x^\eta + v x^{-10\eta},$$

where $W = H_1^2 x v^{-1} M^{-1}$ and $\theta$ is some number (independent of the variables).

If $H_1^8 \leq x^\eta (N x^{-5} v^8)$, then by Lemma 2.3 with $(r, s, m) = (1, w, n)$, we get the estimate

$$x^{-3\eta} |S(H_1)| \ll \left( \frac{M^3 N}{H_1 x} \right)^{1/2} \left( \frac{40}{H_1} H_1^{15} x^{3^5} N^7 v^{-5} + \frac{40}{H_1} H_1^{1} x^{N^3} + \frac{40}{H_1} H_1^{25} x^{N^2} v^{-5} \right)^x$$

$$\ll v \left( \sqrt[30]{N^{15} x^{-35}} + \sqrt[10]{N^7 x^{-3}} + \sqrt[30]{N^{201} x^{-85}} + \sqrt[30]{N^{13} x^{-9}} \right) + v x^{-\varepsilon} \ll v x^{-\varepsilon}.$$

If $H_1^8 > x^\eta (N x^{-5} v^8)$, then we can apply Lemma 2.4, with $(x, y) = (n, w)$, to get

$$x^{-3\eta} |S(H_1)| \ll \left( \frac{M^3 N}{H_1 x} \right)^{1/2} \left( \frac{40}{H_1} H_1^{14} x^{3^5} N^7 v^{-74} N^{2} + \frac{40}{H_1} H_1^{1} x^{N^2} v^{-84} N^{72} + \frac{40}{H_1} H_1^{1} x^{N^2} v^{-3} N^{32} + \frac{40}{H_1} H_1^{1} x^{N^2} v^{-7} N^{5} + \frac{40}{H_1} H_1^{1} x^{N^2} v^{-10} N^{19} \right)$$

$$\ll \left( \frac{40}{H_1} H_1^{14} x^{3^5} N^7 v^{-74} N^{2} + \frac{40}{H_1} H_1^{1} x^{N^2} v^{-84} N^{72} + \frac{40}{H_1} H_1^{1} x^{N^2} v^{-3} N^{32} + \frac{40}{H_1} H_1^{1} x^{N^2} v^{-7} N^{5} + \frac{40}{H_1} H_1^{1} x^{N^2} v^{-10} N^{19} \right) x^{\eta}$$

$$\ll v x^{-10\eta}.$$

The proof of Lemma 4 is finished.

### 3. Sieve methods

#### 3.1. Outline of setting

Let $v$ be a number such that

$$x^{0.6 - \varepsilon} \ll v \ll x^{0.723}.$$
The sequence $\mathcal{A}$ is defined as

$$\mathcal{A} = \{ n \mid v \leq n \leq ev \text{ and there exists an } m \text{ with } x \leq mn \leq x + x^{1/2} \},$$

where $e = 2.71828 \ldots$ is the base of the natural logarithms. As usual, $S(\mathcal{A}, z)$ denotes the number of elements in $\mathcal{A}$ having no prime factors less than $z$. For $r$ a positive integer, let

$$|\mathcal{A}_r| = \sum_{n \in \mathcal{A}, n \equiv 0 \pmod{r}} 1.$$

It is easy to see that

$$|\mathcal{A}_r| = x^{1/2}/r + R(\mathcal{A}, r),$$

$$R(\mathcal{A}, r) = \sum_{\nu \leq r \leq ev} \left( \psi \left( \frac{x + x^{1/2}}{rs} \right) - \psi \left( \frac{x}{rs} \right) \right) + O \left( \frac{x^{1/2}}{v} \right).$$

Let $V(z) = \prod_{p<z} (1 - 1/p)$. With the above property of $\mathcal{A}$, we have

**Lemma 3.1.1.** We have

$$(3.1.1) \quad S(\mathcal{A}, z) \leq x^{1/2}V(z) \left( F \left( \frac{\ln D}{\ln z} \right) + O(\varepsilon) \right) + R^+ \quad \text{if } 2 \leq z \leq D,$$

where

$$R^+ = \sum_{(D)} \sum_{r \leq D^n \atop r \mid P(D^n)} C(D)(r, \eta) \sum_{D_i \leq p_i \leq \min(z, D_i^{1+\eta}) \atop 1 \leq i \leq t} R(\mathcal{A}, rp_1 \ldots p_t),$$

$t \gg 1$, $\sum_{(D)}$ is summation over all sequences $\{D_i\}_{i=1}^t$ with each $D_i$ of the form

$$D_i^{n^2(1+\eta^2)^n}, \quad n = 0, 1, 2, \ldots,$$

such that

$$D_1 \geq \ldots \geq D_t \geq D^n, \quad D_1 \ldots D_{2s}D_{2s+1}^3 \leq D \quad \text{for all } 0 \leq s \leq (t-1)/2,$$

and

$$P(D^n) = \prod_{p<D^n} p, \quad |C(D)(r, \eta)| \leq 1.$$

Also, for $p \sim \frac{1}{2}M$, $2 \leq M \leq D^{1/2}$, we have

$$(3.1.2) \quad S(\mathcal{A}_p, M) \geq \frac{x^{1/2}}{p} V(M) \left( f \left( \frac{\ln D}{\ln M} \right) + O(\varepsilon) \right) - \sum_{d<D \atop d \mid P(M)} \lambda_d R(\mathcal{A}, pd),$$

$\lambda_d$ being some numbers with $|\lambda_d| \leq D^n$. The functions $F$ and $f$ are the well-known functions in the linear sieve.

**Proof.** Both (3.1.1) and (3.1.2) come from [6].
We choose \( P = vx^{-1/2} + 50\eta \), and
\[
x^{10r} D = \begin{cases} (x^{10}v^{-15})^{1/2} & \text{for } x^{0.6} \ll vx^\epsilon \leq x^{11/18}, \\ x^{3/8} & \text{for } x^{11/18} \leq vx^\epsilon \leq x^{0.723+.}, \\
\end{cases}
\]
\[
x^{50n} Q = \begin{cases} v^{-3}x^2 & \text{for } x^{0.6} \ll vx^\epsilon \leq x^{27/44}, \\ (vx^{1/2})^{1/7} & \text{for } x^{27/44} \leq vx^\epsilon \leq x^{67/104}, \\ (v^{20}x^{-7})^{1/36} & \text{for } x^{67/104} \leq vx^\epsilon \leq x^{0.665}, \\ (v^{50}x^{-21})^{1/70} & \text{for } x^{0.665} \leq vx^\epsilon \leq x^{0.7}. \\
\end{cases}
\]

**Lemma 5.** We have
\[
S(A, P) \leq x^{1/2}V(P)\left( F\left( \frac{\ln D}{\ln P} \right) + O(\epsilon) \right).
\]

**Proof.** By (3.1.1) it suffices to show that
\[
\sum_{p_1 \sim A_1} \ldots \sum_{p_r \sim A_r} \sum_{v' p_1 \ldots p_r n \leq v'} \left( \psi\left( \frac{x + x^{1/2}}{p_1 \ldots p_r n} \right) - \psi\left( \frac{x}{p_1 \ldots p_r n} \right) \right) \ll x^{1/2 - 0.75\eta},
\]
where

\[
A_1 \gg A_2 \gg \ldots \gg A_t,
\]
\[
A_1 \ldots A_{2s} A_{2s+1}^{3} \leq D^{1+\eta} \quad \text{for } 0 \leq s \leq (t - 1)/2,
\]
\[
v' = v^{-1}, \quad r < D^\eta.
\]

Then, by a standard reduction, it is enough to establish
\[
(3.1.3) \quad \sum_{1 \leq s \leq H} \sum_{p_1 \sim A_1} \ldots \sum_{p_r \sim A_r} \sum_{v' p_1 \ldots p_r n \leq v'} e\left( \frac{hx'}{p_1 \ldots p_r n} \right) \ll vx^{-\eta},
\]
where \( H = vx^{-1/2+\eta}, \quad x \equiv x' \). Our aim is to arrange \( \{r, p_1, \ldots, p_t, n\} \) into two subsets, so that we can produce from (3.1.3) an exponential sum of the type of Section 2, and then (3.1.3) will follow from the estimate given there.

We claim that either
\[
(3.1.4) \quad \text{there exists a subset } S \text{ of } \{1, 2, \ldots, t\} \text{ with}
\]
\[
P \leq \prod_{i \in S} A_i \leq Qx^{-2\eta} := Q_1,
\]
or
\[
(3.1.5) \quad A_1 \ldots A_t \leq x^{3/8 - 22\eta} := A_0.
\]
We assume the contrary, that is, neither (3.1.4) nor (3.1.5) is true, and deduce a contradiction. For \( v \geq x^{11/18 - \epsilon} \), (3.1.5) is obvious. For \( v < x^{11/18 - \epsilon} \) we reason as follows. From \( A_1^3 \leq D^{1+\eta} < Q_1^3 \), we have \( A_1 < P \). As \( A_0 < A_1 \ldots A_t \ll P^2 A_3 \ldots A_t \), we have \( A_3 \ldots A_t > Q_1 \). If \( A_3 A_4 < P \), there
must be a least \( j \) such that \( A_3A_4 \ldots A_j > P \), hence \( A_3A_4 \ldots A_j > Q_1 \); but then

\[
Q_1 < A_3A_4 \ldots A_j \leq (A_3 \ldots A_{j-1})(A_3A_4)^{1/2} < P^{3/2},
\]
a contradiction. If \( A_3A_4 \geq P \), then \( A_3A_4 > Q_1 \), thus \( A_3 > Q_1^{1/2} \). But now

\[
Q_1^{3/2} \leq A_1A_2A_3^2 \leq D^{1+\eta},
\]
also a contradiction. The proof of Lemma 5 is finished.

**Lemma 6.** We have

\[
\sum_{P \leq p < Q'} S(A_p, p) \geq \sum_{1 \leq j \leq J} \sum_{P \sim 2^{j-1}P} S(A_p, 2^j P).
\]

By (3.1.2), we have, with \( D_j = (v/(2^j P))x^{-\epsilon} \), \( P_j = 2^j P \),

\[
S(A_p, 2^j P) \geq \frac{x^{1/2}}{P} V(2^j P) \left( f \left( \frac{\ln D_j}{\ln P_j} \right) + O(\epsilon) \right) - \sum_{d \leq D_j} \lambda_d R(A, pd).
\]

We now show that for each fixed \( j \), \( 1 \leq j \leq J \), we have

\[
\sum_{P \sim P_j/2, d \leq D_j} \lambda_d R(A, pd) \ll x^{1/2-\eta/4}.
\]
Again, by the standard argument using the Fourier expansion of \( \psi(x) \), to prove (3.1.8) it suffices to prove

\[
\sum_{1 \leq h \leq H} \sum_{P \sim P_j/2} \sum_{d \leq D_j} \sum_{v \leq pdfs \leq \varepsilon} e \left( \frac{hx'}{pdfs} \right) \ll vx^{-\eta},
\]
where \( H = vx^{-1/2+\eta}, x \equiv x' \). At this stage the condition \( v \leq pdfs \leq \varepsilon \) can be removed by a familiar lemma (cf. [1]). Thus it is enough to get

\[
\sum_{1 \leq h \leq H} \sum_{n \sim N} \sum_{m \sim M} a(n)b(m)e \left( \frac{hx}{mn} \right) \ll vx^{-\eta},
\]
where \( N = P_j/2, MN \equiv v, \xi \equiv x, |a(n)| \leq x^{\eta}, |b(m)| \leq x^{\eta} \). The above exponential sum is just of the form we considered in Section 2. As \( P \ll
N = P_j/2 \ll Q for x^{0.6} \ll vx^e \leq x^{0.7}, we find that

\begin{align}
&\text{(3.1.9) follows from } \\
&\begin{cases}
\text{Lemma 1.2 when } x^{0.6} \ll vx^e \leq x^{27/44}; \\
\text{Lemma 2 when } x^{27/44} < vx^e \leq x^{67/104}; \\
\text{Lemma 3 with } (k, \lambda) = (2/7, 4/7) \text{ when } x^{67/104} < vx^e \leq x^{0.665}; \\
\text{Lemma 3 with } (k, \lambda) = (11/30, 16/30) \text{ when } x^{0.665} < vx^e \leq x^{0.7}.
\end{cases}
\end{align}

Hence (3.1.8) is true. Using the asymptotic formula for V(\cdot) and the well-known property of f(\cdot), we have, for p \sim P_j/2 = 2^{j-1} P,

\begin{align}
V(2^j P) &= V(p) \left(1 + O\left(\frac{1}{\ln x}\right)\right), \\
f\left(\frac{\ln D_j}{\ln P_j}\right) &= f\left(\frac{\ln(v/p)}{\ln p}\right) + O(\varepsilon).
\end{align}

Lemma 6 follows from (3.1.6)–(3.1.8) and (3.1.10).

3.2. The contribution of the range (0.6 - \varepsilon, 11/18 - \varepsilon). We prove

**Proposition 2.** We have

\begin{align}
\sum_{Q' \leq p_1 \leq \min(p_2, Q)} 21 := \sum_{x^{0.6 - \varepsilon} \leq p \leq x^{11/18 - \varepsilon}} N(p) \ln p < 0.02278 x^{1/2} \ln x.
\end{align}

Unless otherwise specified, all symbols have the same meaning as in Subsection 3.1. Now, v satisfies

\begin{align}
e^{-1} x^{0.6 - \varepsilon} \leq v \leq x^{11/18 - \varepsilon}.
\end{align}

**Lemma 3.2.1.**

\begin{align}
\sum_{p_1 p_2 \in A \atop Q' \leq p_1 \leq \min(p_2, Q)} 1 &= x^{1/2} \ln v \left(\ln \left(\frac{5.4 - 8.1 s}{4s - 2}\right) + O(\varepsilon)\right),
\end{align}

where s = (\ln v)/(\ln x).

**Proof.** It is clear that

\begin{align}
\sum_{p_1 p_2 \in A \atop Q' \leq p_1 \leq \min(p_2, Q)} 1 &= \sum_{Q' \leq p_1 \leq Q} \sum_{v \leq p_1 p_2 \leq ev} \sum_{x \leq p_1 p_2 m \leq x + x^{1/2}} 1 = U + V,
\end{align}

where

\begin{align}
U &= \sum_{Q' \leq p_1 \leq Q} \sum_{v \leq p_1 p_2 \leq ev} \frac{x^{1/2}}{p_1 p_2}, \\
V &= \sum_{Q' \leq p_1 \leq Q} \sum_{v \leq p_1 p_2 \leq ev} \left(\frac{x + x^{1/2}}{p_1 p_2} - \frac{x}{p_1 p_2}\right).
\end{align}
By the Prime Number Theorem, we easily deduce that
\[
U = x^{1/2} \left( \sum_{Q' \leq p_1 \leq Q} \frac{1}{p_1 \ln(v/p_1)} + O((\ln x)^{-2}) \right)
\]
\[
= \frac{x^{1/2}}{\ln v} \left( \ln \left( \frac{5.4 - 8.1s}{4s - 2} \right) + O(\varepsilon) \right);
\]
and the argument in the proof of Lemma 6 gives
\[
V = O(x^{1/2}(\ln x)^{-2}).
\]
This proves Lemma 3.2.1.

Proof of Proposition 2. Let \( A = 0.6 - \varepsilon, B = 11/18 - \varepsilon, \) and \( L = \ln x. \) Then
\[
\sum_{21} \leq \sum_{A L - 1 \leq k \leq B L} (k + 1) \sum_{e^k \leq p \leq e^{k+1}} N(p).
\]
For \( v = e^k, \) by Buchstab’s identity it is easy to verify that
\[
\sum_{v \leq p \leq ev} N(p) \leq S(A, P) - \sum_{P \leq p < Q'} S(A_p, p) - \sum_{p_1, p_2 \in A} S(p_1 p_2, p_1 p_2),
\]
which, in conjunction with Lemmata 5, 6, and 3.2.1, leads to the estimate
\[
\sum_{21} \leq x^{1/2} L(I_1 - I_2 - I_3 + O(\varepsilon)),
\]
where (\( \gamma \) being the Euler constant)
\[
e^\gamma I_1 = \int_A^B \frac{s}{s - 1/2} F \left( \frac{10 - 15s}{2s - 1} \right) ds,
\]
\[
e^\gamma I_2 = \int_A^B g(s) f(t) dt ds, \quad g(s) = 1/(2s - 1),
\]
\[
I_3 = \int_A^B \ln \left( \frac{5.4 - 8.1s}{4s - 2} \right) ds.
\]
The following formulae are well known:

\[(3.2.1) \quad F(u) = \frac{2e^\gamma}{u} (1 + B(u)), \]
\[
B(u) = \begin{cases} 
0 & \text{for } 0 < u \leq 3, \\
\int_2^{u-1} \frac{\ln(t-1)}{t} dt & \text{for } 3 \leq u \leq 5;
\end{cases}
\]
\[(3.2.2) \quad f(u) = \frac{2e^\gamma \ln(u - 1)}{u} \quad \text{for } 2 \leq u \leq 4;\]
(3.2.3) \[ f(u) = \frac{2e^\gamma}{u} \left( \ln(u) + \int_3^u \frac{dt}{t} \right) + \int_2^{u-1} \frac{\ln(s-1)}{s} ds \]

for \( 4 \leq u \leq 6 \).

et \( C = 11/4 \). By (3.2.1), after changing the order of integration, we find that

\[ I_1 = I_{11} + I_{12} + I_{13} + I_{14} + O(\varepsilon), \]

where

\[ I_{11} = \int_A^B \frac{4u}{10 - 15u} \, du < 0.02945, \]

\[ I_{12} = \left( \int_2^C \frac{\ln(t-1)}{t} \, dt \right) \left( \int_A^B \frac{4u}{10 - 15u} \, du \right) < 0.00273, \]

\[ I_{13} = \frac{4}{75} \left( \int_C^4 \frac{(t-4) \ln(t-1)}{t(17+2t)} \, dt \right) < -0.00042, \]

\[ I_{14} = \frac{40}{225} \left( \int_C^4 \frac{\ln(t-1)}{t} \ln\left( \frac{17 + 2t}{5 + 5t} \right) \, dt \right) < 0.00452; \]

thus

(3.2.4) \[ I_1 < 0.03628. \]

By (3.2.2) and (3.2.3), after changing the order of integration, we deduce that

\[ I_2 > I_{21} + I_{22}, \]

where

\[ I_{21} = \frac{1}{45} \left( \int_{2.7}^{10} \frac{\ln(t)}{t} \, dt \right) > 0.01026, \]

\[ I_{22} = e^{-\gamma}f(4.5) \int_{4.5}^5 \left( \frac{1}{2t} - \frac{1}{10} \right) \, dt > 0.00268e^{-\gamma}f(4.5). \]

Since

\[ f(4.5) = \frac{2e^\gamma}{4.5} \left( \ln3.5 + \int_2^{2.5} \frac{\ln(s-1)}{s} \ln\left( \frac{3.5}{s+1} \right) ds \right) > 0.55788e^\gamma, \]

we have

\[ I_{22} > 0.00268 \cdot 0.55788 > 0.00149, \]

and thus

(3.2.5) \[ I_2 > 0.01175. \]
Finally,

\[
(3.2.6) \quad I_3 = \int_A^B \ln \left( \frac{5.4 - 8.1s}{4s - 2} \right) ds > 0.00175.
\]

Combining (3.2.4), (3.2.5) and (3.2.6) proves Proposition 2.

**3.3.** The range \((11/18, 27/44)\). For \(11/18 \leq \alpha < \beta \leq 0.723\), let

\[
e^\gamma I(\alpha, \beta) = \int_\alpha^\beta \frac{t}{t - 1/2} F\left(\frac{3}{8t - 4}\right) dt.
\]

**Proposition 3.** We have

\[
\sum_{\gamma} : = \sum_{x^{11/18 - \varepsilon} \leq p \leq x^{27/44 - \varepsilon}} N(p) \ln p
\]

\[
\leq \left( I\left(\frac{11}{18}, \frac{27}{44}\right) - 0.00218 \right) x^{1/2} \ln x.
\]

**Proof.** Let \(A = 11/18 - \varepsilon, B = 27/44 - \varepsilon, L = \ln x\). Then

\[
\sum_{\gamma} \leq \sum_{AL-1 \leq k \leq BL} (k + 1) \sum_{e^k \leq p \leq e^{k+1}} N(p).
\]

For \(v = e^k\), Buchstab’s identity gives

\[
\sum_{v \leq p \leq v^v} N(p) \leq S(A, P) - \sum_{P \leq P' \leq Q} S(A_{p'}, p),
\]

which, in conjunction with Lemmata 5, 6 and the Prime Number Theorem, leads to the estimate

\[
\sum_{\gamma} \leq x^{1/2} L(I_1 - I_2 + O(\varepsilon)),
\]

where

\[
e^\gamma I_1 = \int_A^B \frac{t}{t - 1/2} F\left(\frac{3}{8t - 4}\right) dt = \left( I\left(\frac{11}{18}, \frac{27}{44}\right) + O(\varepsilon) \right) e^\gamma,
\]

\[
e^\gamma I_2 = \int_A^B \int_{G(s)} f(t) dt ds, \quad g(s) = \frac{1}{2s - 1}, \quad G(s) = \frac{4s - 2}{2 - 3s}.
\]

By (3.2.2) and (3.2.3), after changing the order of integration, we find that

\[
I_2 > I_{21} + I_{22} + I_{23},
\]
where, with $C = \frac{8}{3}$, $D = \frac{20}{7}$, $E = \frac{22}{5}$, $F = 4.5$,

\[
I_{21} = 2 \int_{C}^{D} \frac{\ln(t-1)}{t} \left( \frac{2(1+t)}{4+3t} - \frac{11}{18} \right) dt > 0.0001,
\]
\[
I_{22} = 2 \int_{D}^{E} \frac{\ln(t-1)}{t} \left( \frac{27}{44} - \frac{11}{18} \right) dt > 0.00202,
\]
\[
I_{23} = 2 \int_{E}^{F} \frac{\ln(t-1)}{t} \left( \frac{t+1}{2t} - \frac{11}{18} \right) dt > 0.00006.
\]

Thus

\[
(3.3.2) \quad I_2 > 0.00218.
\]

Proposition 3 is proved in view of (3.3.1) and (3.3.2).

**3.4. The range (27/44, 67/104).** We prove

**PROPOSITION 4.** We have

\[
\sum_{23} := \sum_{x^{27/44} - \varepsilon \leq p \leq x^{67/104} - \varepsilon} N(p) \ln p
\]
\[
\leq \left( I \left( \frac{27}{44}, \frac{67}{104} \right) - 0.01552 \right) x^{1/2} \ln x.
\]

**Proof.** Let $A = \frac{27}{44} - \varepsilon$, $B = \frac{67}{104} - \varepsilon$, $L = \ln x$. Then

\[
\sum_{23} \leq \sum_{AL - 1 \leq k \leq BL} (k + 1) \sum_{\varepsilon \leq p \leq \varepsilon+1} N(p).
\]

For $v = e^k$, Buchstab’s identity gives

\[
\sum_{v \leq p \leq e^v} N(p) = \sum_{p \in A} 1 \leq S(A, P) - \sum_{P \leq p < Q'} S(A_p, p),
\]

which, in conjunction with Lemmata 5, 6 and the Prime Number Theorem, leads to the estimate

\[
\sum_{23} \leq x^{1/2} L(I_1 - I_2 + O(\varepsilon)),
\]

where

\[
e^\gamma I_1 = \int_{\frac{B}{A}}^{\frac{t}{t-1/2}} F \left( \frac{3}{8t-4} \right) dt = e^\gamma \left( I \left( \frac{27}{44}, \frac{67}{104} \right) + O(\varepsilon) \right),
\]

(3.4.1)

\[
e^\gamma I_2 = \int_{A}^{B} g(s) f(t) dt \, ds, \quad g(s) = \frac{1}{2s-1}, \quad G(s) = \frac{12s-1}{2s+1}.
\]
By (3.2.2) and (3.2.3), after changing the order of integration, we find that

\[ I_2 > I_{21} + I_{22} + I_{23}, \]

where, with \( C = 20/7, D = 50/17, E = 52/15, F = 22/5, \)

\[ I_{21} = \int_C^D \frac{2 \ln(t-1)}{t} \left( \frac{t+1}{12-2t} - \frac{27}{44} \right) dt > 0.00056, \]

\[ I_{22} = \int_D^E \frac{2 \ln(t-1)}{t} \left( \frac{67}{104} - \frac{27}{44} \right) dt > 0.00789, \]

\[ I_{23} = \int_E^F \frac{2 \ln(t-1)}{t} \left( \frac{t+1}{2t} - \frac{27}{44} \right) dt > 0.00707. \]

Hence

\[(3.4.2) \quad I_2 > 0.01552.\]

Proposition 4 follows from (3.4.1) and (3.4.2).

### 3.5. The range \((67/104, 0.665)\). We prove

**Proposition 5.** We have

\[
\sum_{24} \sum_{x^{67/104-\varepsilon} \leq p \leq x^{0.665-\varepsilon}} N(p) \ln p < \left( I \left( \frac{67}{104}, 0.665 \right) - 0.00359 \right) x^{1/2} \ln x.
\]

**Proof.** Let \( A = 67/104 - \varepsilon, B = 0.665 - \varepsilon, L = \ln x. \) Then

\[
\sum_{24} \sum_{A \leq p \leq B} (k+1) \sum_{\varepsilon^k \leq p \leq \varepsilon^{k+1}} N(p).
\]

For \( v = \varepsilon^k, \) by Buchstab’s identity, we easily get

\[
\sum_{v \leq p \leq v^v} N(p) \leq S(A, P) - \sum_{P \leq p < Q} S(A_p, p),
\]

which, in conjunction with Lemmata 5, 6 and the Prime Number Theorem, gives the estimate

\[
\sum_{24} \leq x^{1/2} L(I_1 - I_2 + O(\varepsilon)),
\]
where
\[ e^\gamma I_1 = \int_A^B \frac{t}{t-1/2} F \left( \frac{3}{8t-4} \right) dt = \left( I \left( \frac{67}{104}, 0.665 \right) + O(\varepsilon) \right) e^\gamma, \]
(3.5.1)
\[ e^\gamma I_2 = \int_A^B \int_A^B f(t) \frac{g(s)}{G(s)} ds dt, \quad g(s) = \frac{1}{2s-1}, \quad G(s) = \frac{16s+7}{20s-7}. \]

By (3.2.2) and (3.2.3), after changing the order of integration, we find that
\[ I_2 = I_{21} + I_{22} + I_{23} + O(\varepsilon), \]
where, with \( C = \frac{14}{5}, D = \frac{50}{17}, E = \frac{100}{33}, F = \frac{52}{15}, \)
\[ I_{21} = 2 \int_C^D \frac{\ln(t-1)}{t} \left( 0.665 - \frac{7 + 7t}{20t-16} \right) dt > 0.00065, \]
\[ I_{22} = 2 \int_D^E \frac{\ln(t-1)}{t} \left( 0.665 - \frac{67}{104} \right) dt > 0.00084, \]
\[ I_{23} = 2 \int_E^F \frac{\ln(t-1)}{t} \left( \frac{1}{2} \left( 1 + \frac{1}{t} \right) - \frac{67}{104} \right) dt > 0.0021. \]

Hence
(3.5.2) \[ I_2 > 0.00359. \]

Proposition 5 is proved by virtue of (3.5.1) and (3.5.2).

3.6. The range \((0.665, 0.7)\). We show

**PROPOSITION 6.** We have
\[ \sum_{25} := \sum_{x^{0.665-\varepsilon} \leq p \leq x^{0.7-\varepsilon}} N(p) \ln p \leq (I(0.665, 0.7) - 0.00149)x^{1/2}\ln x. \]

**Proof.** Let \( A = 0.665 - \varepsilon, B = 0.7 - \varepsilon, L = \ln x. \) Then
\[ \sum_{25} \leq \sum_{AL-1 \leq k \leq BL} (k+1) \sum_{e^k \leq p \leq e^{k+1}} N(p). \]

For \( v = e^k, \) by Buchstab’s identity, we deduce that
\[ \sum_{v \leq p \leq ev} N(p) \leq S(A, P) - \sum_{P \leq p < Q'} S(A_p, p), \]
which, in conjunction with Lemmata 5, 6 and the Prime Number Theorem, gives
\[ \sum_{25} \leq x^{1/2}L(I_1 - I_2 + O(\varepsilon)), \]
where
\[ e^\gamma I_1 = \int_A^B \frac{t}{t - 1/2} F\left(\frac{3}{8t - 4}\right) dt, \]
and
\[ e^\gamma I_2 = \int_A^B \int_A^G f(t) dt ds, \quad g(s) = \frac{1}{2s - 1}, \quad G(s) = \frac{20s + 21}{50s - 21}. \]

By (3.2.2) and (3.2.3), after changing the order of integration, we get, with
\[ C = 2.5, \quad D = 2.8, \quad E = 100/33, \]

\[ I_2 = I_{21} + I_{22} + O(\varepsilon), \]

\[ I_{21} = 2 \int_C^D \frac{\ln(t - 1)}{t} \left(\frac{1}{2} \left(1 + \frac{1}{t}\right) - \frac{21(1 + t)}{50t - 20}\right) dt > 0.00083, \]

\[ I_{22} = 2 \int_D^E \frac{\ln(t - 1)}{t} \left(\frac{1}{2} \left(1 + \frac{1}{t}\right) - 0.665\right) dt > 0.00066. \]

Thus
\[ I_2 > 0.00149. \]

By (3.6.1) and (3.6.2) we get the proposition.

**3.7. The range (0.7, 0.723).** We prove

**Proposition 7.** We have
\[ \sum_{26} : = \sum_{x^{0.7 - \varepsilon} \leq p \leq x^{0.723}} N(p) \ln p < (I(0.7, 0.723) + O(\varepsilon)) x^{1/2} \ln x. \]

**Proof.** Let \( A = 0.7 - \varepsilon, \quad B = 0.723, \quad L = \ln x. \) Then
\[ \sum_{26} \leq \sum_{AL^{-1} \leq k \leq BL} (k + 1) \sum_{e^k \leq p \leq e^{k+1}} N(p). \]

For \( v = e^k, \) it is obvious that \( \sum_{v \leq p \leq e^v} N(p) \leq S(A, P), \) which, in conjunction with Lemma 5, gives
\[ \sum_{26} \leq x^{1/2} L(I_1 + O(\varepsilon)), \]

where
\[ e^\gamma I_1 = \int_A^B \frac{t}{t - 1/2} F\left(\frac{3}{8t - 4}\right) dt = (I(0.7, 0.723) + O(\varepsilon)) e^\gamma, \]

which gives the proposition.
Proof of Theorem. From Propositions 1 to 7, we get
\[ \sum_{p \leq x^{0.723}} N(p) \ln p \leq x^{1/2} \ln x(I(11/18, 0.723) + 0.6). \]

Let \( A = 11/18 \), \( B = 0.723 \) and \( C = 19/8 \). By (3.2.1) we obtain
\[
I(11/18, 0.723) = \frac{16}{3} \int_A^B t \ln(t) \, dt + \frac{8}{3} \int_A^C \frac{\ln(t-1)}{t} \left( \frac{(4t + 7)^2}{64(t+1)^2} - A^2 \right) \, dt
< 0.39807 + 0.00043 = 0.3985,
\]
thus \( I(11/18, 0.723) + 0.6 < 0.9985 < 1 \), and our Theorem is proved in view of (1).

References


Received on 4.5.1992
and in revised form on 15.6.1993