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Squares in products from a block of consecutive integers

by

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1. Let $k \ge t \ge 2$, $m \ge 0$, $y \ge 1$ be integers and write $d_1, \ldots, d_t$ for distinct positive integers not exceeding k. The letter b denotes a positive integer such that the greatest prime factor of b is less than or equal to k. We put

$$F(k) = k(\log k)/(\log\log k) \quad \text{for } k \ge 3\,.$$

For a real number θ and $k \ge 27$, we define

$$\mu_k(\theta) = k\left(1 - \frac{\log\log k}{\log k} + \frac{\log\log\log k}{\log k} + \frac{\theta}{\log k}\right).$$

Finally, we recall that γ is Euler's constant.

We consider the equation

$$(m + d_1)\ldots(m + d_t) = by^2\,. \tag{1}$$

It follows from a theorem of Baker [1] that equation (1) with $t \ge 3$ implies that $\max(b, m, y)$ is bounded by an effectively computable number depending only on k. Erdős [2] and Rigge [8], independently, proved in 1939 that equation (1) with $t = k$ and $b = 1$ is not possible. Thus, the product of two or more consecutive positive integers is never a square. In fact, Erdős [3, p. 88] observed in 1955 that his method allows to show that there exists an absolute constant $C_1 > 0$ such that equation (1) with $b = 1$ and

$$m > k^2, \quad t \ge k - C_1 k/(\log k)$$

implies that k is bounded by an effectively computable absolute constant. Further, Erdős [3, p. 88] stated in 1955 that he had no proof of the following sharpening of the preceding result:

Let $\varepsilon > 0$. The equation (1) *with $b = 1$ and*

$$m > k^2, \quad t \ge k - (1-\varepsilon)k\frac{\log\log k}{\log k} \tag{2}$$

implies that k is bounded by an effectively computable number depending only on ε.

Shorey [9] applied in 1986 Brun's sieve and an estimate of Sprindžuk [12] on the magnitude of integral solutions of a hyperelliptic equation to prove that equation (1) with

$$m > k^2, \quad t \geq k - (1-\varepsilon)k\frac{\log\log\log k}{\log k} \tag{3}$$

implies that k is bounded by an effectively computable number depending only on ε. Further, Shorey [10] relaxed in 1987 the assumption (3) to (2). In this paper, we combine the arguments for the proofs of the preceding results of Shorey to obtain a further relaxation of the assumption (2).

Theorem 1. *Let $k \geq 27$. There exist effectively computable absolute constants θ_0 and C_2 such that equation* (1) *with*

$$m > k^2 \tag{4}$$

and

$$t \geq \mu_k(\theta_0) \tag{5}$$

implies that

$$k \leq C_2\,.$$

Since $\mu_k(\theta)$ is an increasing function of θ, we observe that the assumption (5) can be replaced by $t \geq \mu_k(\theta)$ for any $\theta > \theta_0$. For an integer $\nu > 1$, we define $P(\nu)$ to be the greatest prime factor of ν and we write $P(1) = 1$. If equation (1) with $P(y) > k$ is valid, we can find an integer i with $1 \leq i \leq k$ such that $m + d_i \geq (k+1)^2$, which implies that $m > k^2$. Consequently, we observe that the assumptions (4) and (5) in Theorem 1 can be replaced by

$$P(y) > k, \quad t \geq \mu_k(\theta_0)\,.$$

If $P(y) \leq k$, we observe from (1) that

$$P(m + d_i) \leq k \quad \text{for } 1 \leq i \leq t\,,$$

which implies that

$$t \leq \frac{k\log k}{\log m} + \pi(k)\,, \tag{6}$$

by a well-known argument of Erdős [3, Lemma 3]; see also [4, Lemma 2.1]. In (6), we write $\pi(k)$ for the number of distinct primes not exceeding k. From now onward, we shall always understand that θ_0 is an effectively computable absolute constant given by Theorem 1. Now, we combine Theorem 1 and (6) to derive the following result.

Corollary 1. *Let $\varepsilon > 0$ and $k \geq 27$. The equation* (1) *with*

$$m \geq e^{1-\theta_0+\varepsilon}F(k) \tag{7}$$

and (5) *implies that* k *is bounded by an effectively computable number depending only on* ε.

On the other hand, we show that Corollary 1 with $b = 1$ is close to best possible in each of the assumptions (7) and (5). For this, we prove the following more general result.

Theorem 2. *Let* $\varepsilon > 0$. *There exist effectively computable numbers* C_3, C_4 *and* C_5 *depending only on* ε *such that for every pair* k, m *with* $k \geq C_3$ *and*

$$m \leq k^{17/12-\varepsilon}, \tag{8}$$

we can find distinct integers $d_1, \ldots, d_t$ *in* $[1, k]$ *with*

$$t \geq \min\left(k - C_4\frac{k}{\log k}, k - \frac{k}{\log k}\left(1 + \frac{C_5}{\log k}\right)\left(\log\left(\frac{m+k}{k}\right) + 1 + \gamma + \varepsilon\right)\right) \tag{9}$$

and

$$(m + d_1)\ldots(m + d_t) \tag{10}$$

is a square.

If $m \leq k$, Erdős and Turk [4, p. 167] proved the assertion of Theorem 2 with (9) replaced by $t \geq k - 4k/(\log k)$. As an immediate consequence of Theorem 2, we obtain the following result.

Corollary 2. (a) *Let* $\varepsilon > 0$, $k \geq 3$ *and*

$$m < e^{-1-\gamma-\theta_0-\varepsilon}F(k).$$

There exists an effectively computable number C_6 *depending only on* ε *such that for* $k \geq C_6$ *there are distinct positive integers* $d_1, \ldots, d_t$ *not exceeding* k *with* t *satisfying* (5) *and the product is a square.*

(b) *Let* $\varepsilon > 0$, $k \geq 3$ *and*

$$m < e^{1-\theta_0-\varepsilon}F(k).$$

The assertion of Corollary 2(a) *is valid with* t *satisfying*

$$t \geq \mu_k(\theta_0) - (2 + \gamma)k/(\log k) \tag{11}$$

in place of (5).

By Corollary 2(a), we observe that the assumption (7) in Corollary 1 with $b = 1$ cannot be replaced by

$$m \geq e^{-1-\gamma-\theta_0-\varepsilon}F(k).$$

Further, we see from Corollary 2(b) that we cannot relax the assumption (5) to (11) in Corollary 1 with $b = 1$.

2. Proof of Theorem 1. We shall choose later θ_0, a suitable absolute positive constant. We may suppose that $k \geq c_1$ where c_1 is a sufficiently large effectively computable number depending only on θ_0. Thus

$$\varepsilon =: \frac{\theta_0}{\log\log k} \tag{12}$$

satisfies $0 < \varepsilon \leq 1/2$. By (1), we have

$$m + d_i = A_i X_i^2 \quad \text{for } 1 \leq i \leq t\,, \tag{13}$$

where A_i and X_i are positive integers such that $P(A_i) \leq k$ and A_i is square free. Further, by (4), we observe that the elements of $S_1 =: \{A_1, \ldots, A_t\}$ are pairwise distinct. By a well-known argument of Erdős [3, Lemma 3], we find a subset S_2 of S_1 with $|S_2| \geq t - \pi(k)$ such that

$$\prod_{A_i \in S_2} A_i \leq k^k\,.$$

Then, we apply [10, Lemma 6] with $\eta = \varepsilon$ and

$$g = \log\log k - \log\log\log k - (\theta_0 - 2)$$

to conclude that there exists a subset S_3 of S_2 with

$$|S_3| \geq \varepsilon k/2 \tag{14}$$

and

$$A_i \leq 4e^2 F(k) \quad \text{if } A_i \in S_3\,. \tag{15}$$

By (13), (4) and (15), we derive that

$$X_i > k^{1/4} \quad \text{if } A_i \in S_3\,. \tag{16}$$

We write S_4 for the set of all $A_i \in S_3$ with $A_i \leq 3k$ and let S_5 be the complement of S_4 in S_3. Now, we follow the proof of [10, Theorem 2] to derive from Erdős [3, Lemma 4] and (15) that

$$|S_5| \leq \frac{12e^2 k}{\log\log k}\,. \tag{17}$$

By taking $\theta_0 > 48e^2$, we observe from (14), (17) and (12) that

$$|S_4| > \varepsilon k/4\,. \tag{18}$$

Let C be as in the proof of [9, Theorem 2] to which we refer in this paragraph without explicit mention. We write $b_1, \ldots, b_s$ for all the integers between $k/(\log k)^{2C}$ and $3k$ such that every proper divisor of b_i is less than or equal to $k/(\log k)^{2C}$. By Brun's sieve, we derive that

$$s \leq \frac{c_2 k}{\log\log k} \tag{19}$$

where c_2 is an effectively computable absolute constant. By taking θ_0 sufficiently large, we derive from (18), (19) and (12) that

$$B_2B_3(X_2X_3)^2 = (B_1X_1^2 + R)(B_1X_1^2 + R') \tag{20}$$

where B_1, B_2, B_3 and R, R' are integers of absolute values not exceeding $(\log k)^{3C}$. For this assertion, we may permute the subscripts of $d_1, \ldots, d_t$ and this involves no loss of generality. Finally, we apply a theorem of Sprindžuk [12] (see also [9, Lemma 4]) to equation (20) to conclude from (16) that k is bounded by an effectively computable absolute constant. Finally, we fix θ_0 sufficiently large so that the arguments of the proof of Theorem 1 are valid. ∎

3. In this section, we shall prove Theorem 2. For this, we require the following lemmas.

LEMMA 1. *For $x \geq 2$, we have*

$$\sum_{n \leq x} n^{-1} = \log x + \gamma + O(x^{-1}).$$

Proof. See Nagell [6, p. 276]. In particular, there is an effectively computable absolute constant $c_3 > 0$ satisfying

$$\sum_{n \leq x} n^{-1} \leq \log x + \gamma + c_3 x^{-1}. \tag{21}$$

Let $\mathcal{G}$ be a set of positive integers and denote by $\omega(\mathcal{G})$ the number of prime divisors of all the elements of $\mathcal{G}$. Then, we have

LEMMA 2. *There is a subset $\mathcal{G}'$ of $\mathcal{G}$ with*

$$|\mathcal{G}'| \geq |\mathcal{G}| - \omega(\mathcal{G})$$

such that the product of all elements of $\mathcal{G}'$ is a square.

Proof. See Erdős and Turk [4, Lemma 6.2].

Finally, we state the following well-known result on the number of prime factors in short intervals.

LEMMA 3. *Let $\varepsilon > 0$. There exists $x_0 \geq 2$ depending only on ε such that for every $x \geq x_0$ and $h \geq x^{7/12+\varepsilon}$, we have*

$$\pi(x+h) - \pi(x) = \frac{h}{\log x} + O\left(\frac{h}{(\log x)^2}\right). \tag{22}$$

Proof. This is due to Huxley [5]; an upper bound given by (22) is enough for our purpose. For the error term in (22), see Ramachandra [7].

Proof of Theorem 2. We put $S_6 = \{m+1, \ldots, m+k\}$. Let

$$\varepsilon_1 = (2c_3)^{-1}\varepsilon \tag{23}$$

where c_3 is the absolute positive constant appearing in (21). We write c_4, c_5 and c_6 for effectively computable positive numbers depending only on ε. We may assume that $k \geq c_4$ with c_4 sufficiently large. If $m \leq k/\varepsilon_1$, we observe that

$$\omega(S_6) \leq c_5 k/(\log k)$$

and we apply Lemma 2 with $\mathcal{G} = S_6$ to obtain the assertion of Theorem 2. Thus, we may suppose that

$$m > k/\varepsilon_1 \,. \tag{24}$$

We write S_7 for the set of all $\nu \in S_6$ such that $P(\nu) \leq k$. Further, we denote by S_8 the complement of S_7 in S_6. An element of S_8 is of the form λp where $p > k$ is a prime number and λ is an integer satisfying $1 \leq \lambda \leq (m+k)/k$. For an integer λ with $1 \leq \lambda \leq (m+k)/k$, we write T_λ for all the elements of S_8 of the form λp where $p > k$ is a prime number. Further, we write

$$t_\lambda = |T_\lambda| \quad \text{for } 1 \leq \lambda \leq (m+k)/k \,.$$

Thus

$$|S_8| \leq \sum_{1 \leq \lambda \leq (m+k)/k} t_\lambda \,. \tag{25}$$

It is clear from the definition of T_λ that

$$t_\lambda = \pi\left(\frac{m+k}{\lambda}\right) - \pi\left(\frac{m}{\lambda}\right) \quad \text{for } 1 \leq \lambda \leq (m+k)/k \,. \tag{26}$$

For $1 \leq \lambda \leq (m+k)/k$, we derive from (8) that

$$\frac{k}{\lambda} > \left(\frac{m}{\lambda}\right)^{7/12+\varepsilon/2} \tag{27}$$

and, by (24),

$$\frac{m}{\lambda} \geq \frac{mk}{m+k} > \frac{k}{1+\varepsilon_1}. \tag{28}$$

Now, we apply Lemma 3 with $x = m/\lambda, h = k/\lambda$ to derive from (26)–(28) that

$$t_\lambda \leq \left(1 + \frac{c_6}{\log k}\right) \frac{k}{\lambda \, (\log k)} \,. \tag{29}$$

Next, we combine (25), (29), (21), (24) and (23) to conclude that

$$|S_8| \leq G(k)$$

where

$$G(k) = \left(1 + \frac{c_6}{\log k}\right) \frac{k}{\log k} \left(\log\left(\frac{m+k}{k}\right) + \gamma + \frac{\varepsilon}{2}\right).$$

Therefore, we obtain

$$|S_7| \ge k - G(k)\,.$$

Consequently, we apply Lemma 2 with $\mathcal{G} = S_7$ to conclude that there exists a subset S_9 of S_7 with

$$|S_9| \ge k - G(k) - \pi(k) \tag{30}$$

such that the product of all the elements of S_9 is a square. Finally, we observe that the right hand side of (30) is at least the right hand side of (9) with $C_5 = c_6$ to complete the proof of Theorem 2. ■

Remarks. (i) Without applying Lemma 3, it is possible to obtain a slightly weaker estimate for $|S_8|$. By definition of S_8, we have

$$k^{|S_8|} \le \frac{(m+1)\ldots(m+k)}{k!} \le \left(\frac{e(m+k)}{k}\right)^k,$$

which implies that

$$|S_8| \le \frac{k}{\log k}\left(\log\left(\frac{m+k}{k}\right) + 1\right).$$

(ii) Slight improvements of the exponent $7/12$ in Lemma 3 are known. Consequently, the assumption (8) in Theorem 2 can be relaxed slightly.

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