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The Pólya–Vinogradov inequality for totally real algebraic number fields

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The Pólya–Vinogradov inequality states that for any primitive character $\chi \bmod q$,

$$\sum_{n\le x}\chi(n)\ll q^{1/2}\log q\,. \tag{1}$$

Conversely, there is a $1\le x\le q$ satisfying

$$\Big|\sum_{n\le x}\chi(n)\Big|\gg q^{1/2} \tag{2}$$

(see Montgomery and Vaughan [6]).

Here a generalization of these inequalities to totally real algebraic number fields is given. So let K be a totally real field of degree n over $\mathbb{Q}$ with ramification ideal $\mathfrak{d}$, absolute value of discriminant $d=N\mathfrak{d}$ and ring of integers $\mathbb{Z}_K$. All constants implied by the $\ll$-notation depend only on n, if no other dependence is explicitly noted. The nature of the difficulties in making the dependence of the constants on n explicit seems to be purely technical. One has to substitute formula (6) below by a result similar to Lemma 2 of [1].

Let $\mathfrak{f}\subset\mathbb{Z}_K$ be an ideal, χ a primitive character of the multiplicative group $(\mathbb{Z}_K/\mathfrak{f})^*$ extended to $\mathbb{Z}_K$ in the usual manner.

Finally, let $\mathbf{x}\in\mathbb{R}_+^n$ satisfy $X:=\prod_{q=1}^n x_q\ge 2$ and let $\mathbf{y}\in\mathbb{R}^n$.

By means of Siegel's summation formula and an additional argument Hinz [3] succeeded in showing

$$\sum_{\substack{\nu\in\mathbb{Z}_K\\ 0<\nu^{(q)}\le x_q}}\chi(\nu)=E(\chi)X+O_\varepsilon(N\mathfrak{f}^{1-1/(2(n+1))}X^\varepsilon) \tag{3}$$

where ε is an arbitrary positive number and $E(\chi)$ equals $1/\sqrt{d}$ if $\mathfrak{f}=\mathbb{Z}_K$, and 0 otherwise.

A similar estimate was given by Lee [4] who had the exponent 1 on $N\mathfrak{f}$. Our result is

THEOREM 1.

$$\sum_{\substack{\nu\in\mathbb{Z}_K\\ y_q<\nu^{(q)}\le y_q+x_q}} \chi(\nu) = E(\chi)X + O(d^{n/2}N\mathfrak{f}^{1/2}\log^n(dX))\,.$$

This sharpens (3) for any value of X and $N\mathfrak{f}$ and is up to logarithms the same as (1). Moreover, arbitrary values of $\mathbf{y}$ may be chosen, while (3) needs $\mathbf{y}=0$.

Recently Rausch ([8], (1.9)) proved this result (with constants depending on d) using a different method.

In the opposite direction we have

THEOREM 2. *For any* $\mathbf{y}\in\mathbb{R}^n$ *there exists* $\mathbf{x}\in\mathbb{R}^n_+$, $\max_{1\le q\le n} x_q \ll_K N\mathfrak{f}^{1/n}$, *subject to*

$$\Big|\sum_{\substack{\nu\in\mathbb{Z}_K\\ y_q<\nu^{(q)}\le y_q+x_q}} \chi(\nu) - E(\chi)X\Big| \gg_K N\mathfrak{f}^{1/2}\left(\frac{1}{\omega(2\mathfrak{f})\log\omega(6\mathfrak{f})}\right).$$

Here $\omega(\mathfrak{a})$ *denotes the number of prime divisors of* $\mathfrak{a}$. *In particular, the right-hand side is* $\gg_{K,\varepsilon} N\mathfrak{f}^{1/2}(\log 2N\mathfrak{f})^{-1-\varepsilon}$.

In the case of the ideal $\mathfrak{d}\mathfrak{f}$ *being principal one has for some* $\mathbf{x}\in\mathbb{R}^n_+$,

$$\Big|\sum_{\substack{\nu\in\mathbb{Z}_K\\ y_q<\nu^{(q)}\le y_q+x_q}} \chi(\nu) - E(\chi)X\Big| \ge \frac{(dN\mathfrak{f})^{1/2}}{(2\pi)^n}\,.$$

Only minor additional work has to be done to extend Theorems 1 and 2 to non-primitive characters χ.

An easy corollary of the proof of Theorem 1 is given by

PROPOSITION 1. *Let* $\nu_0\in\mathbb{Z}_K$. *Then*

$$|\{\nu\in\mathbb{Z}_K \mid \nu\equiv\nu_0 \bmod \mathfrak{f},\ y_q<\nu^{(q)}\le y_q+x_q,\ 1\le q\le n\}| = \frac{X}{d^{1/2}N\mathfrak{f}} + O(d^{n/2}\log^n(XdN\mathfrak{f}))\,.$$

The right-hand side coincides with the number of lattice points in a parallelotope (see (7) below). The problem of counting these is similar to that of counting the lattice points of a polyhedron of volume $\sim X$. For the polyhedron $\{\mathbf{w}\in\mathbb{R}^n \mid w_j\ge 1,\ \sum w_j\omega_j \le X^{1/n}\}$ it was shown by Spencer [11] that for almost all (in the sense of Lebesgue measure) coefficients $\omega_1,\dots,\omega_n$ the remainder does not exceed $O_\varepsilon(\log^{n+\varepsilon} X)$.

In the case of $n = 2$ and ω_1/ω_2 being a quadratic irrationality, Hardy and Littlewood proved that the remainder is $O(\log X)$ which is best possible ([2], Theorems A3 and A4). Thus the remainder in Proposition 1 is $O_{d,\mathfrak{f}}(\log X)$ for real-quadratic K. Skriganov [10] gives a proof of Proposition 1 with remainders $O_{\mathfrak{f},d}(\log^n X)$, $n \geq 3$, and $O_{\mathfrak{f},d}(\log X)$, $n = 2$. Nevertheless, it seems impossible to use his approach based on the inequality (3.18) of [10] to estimate character sums.

Our method of proof goes back to Pólya's original proof ([7]; see also [6]). The most important tool in it is

$$\sum_{\substack{0<k\leq x\\ k\equiv l \bmod q}} 1 = \left[\frac{x-l}{q}\right] - \left[\frac{-l}{q}\right] \tag{4}$$

$$= \frac{x}{q} + \sum_{0<|m|\leq H} \frac{1}{2\pi i m}\Big(e\Big(\frac{mx}{q}\Big) - 1\Big)e\Big(-\frac{ml}{q}\Big) + O\bigg(\min\bigg(1, \frac{1}{H\|\frac{x-l}{q}\|} + \frac{1}{H\|\frac{l}{q}\|}\bigg)\bigg),$$

where $\|x\| := \min(|x-k| \mid k \in \mathbb{Z})$ and $e(x) := e^{2\pi i x}$.

Theorem 3 below gives an adequate generalization of (4).

Minkowski's convex body theorem shows that there is a $\beta' \in \mathbb{Z}_K - \{0\}$ subject to

$$|\beta'^{(q)}| \leq c_1 d^{1/(2n)} X^{1/(2n)} x_q^{-1/2}, \quad 1 \leq q \leq n.$$

$\beta := \beta'^2$ satisfies

$$0 < \beta^{(q)} \leq c_1^2 d^{1/n} X^{1/n} x_q^{-1}, \quad 1 \leq q \leq n. \tag{5}$$

By Theorem 1 of Mahler [5] there is a $\mathbb{Z}$-basis $\{\alpha_1, \ldots, \alpha_n\}$ of $\beta\mathfrak{f}$ subject to

$$|\alpha_q^{(p)}| \leq c_2 d^{1/2} N(\beta\mathfrak{f})^{1/n} \leq c_3 d N\mathfrak{f}^{1/n}, \quad 1 \leq p, q \leq n. \tag{6}$$

We use it to define the functions

$$\alpha : \mathbb{R}^n \to \mathbb{R}^n, \quad \alpha(\mathbf{t}) = \Big(\sum_{q=1}^n t_q \alpha_q^{(p)}\Big)_{p=1}^n \quad (\text{thus } \alpha(\mathbb{Z}^n) = \beta\mathfrak{f})$$

and

$$\eta := \alpha^{-1\top} : \mathbb{R}^n \to \mathbb{R}^n \quad (\text{thus } \eta(\mathbb{Z}^n) = 1/(\mathfrak{d}\beta\mathfrak{f})).$$

Moreover, for $\mathbf{u}, \mathbf{v}, \mathbf{z} \in \mathbb{R}^n$ with

$$0 < v_p \leq 2c_1^2 d^{1/n} X^{1/n}, \quad 1 \leq p \leq n, \tag{7}$$

we define

$$F(\mathbf{u}, \mathbf{z}) := F(\mathbf{u}, \mathbf{z}; \mathbf{v}, \alpha) := |\{\mathbf{m} \in \mathbb{Z}^n \mid z_p < \alpha^{(p)}(\mathbf{m} + \mathbf{u}) \leq z_p + v_p\}|.$$

In Sections 1–3 we fix $\mathbf{z}$ and work with the Fourier series of F with respect to $\mathbf{u}$. This will prove Theorem 1.

In Section 4, $\mathbf{u}$ is fixed and the Fourier expansion of F with respect to $\mathbf{z}$ is used to derive lower bounds. Here only L^2-convergence of the series is needed, so that the proof is easily compared to that of the upper bounds requiring a result similar to (4).

We make use of the notations

$$|\mathbf{t}|_\infty := \max(|t_j| \mid 1 \le j \le k) \quad \text{and} \quad \langle \mathbf{s}, \mathbf{t}\rangle := \sum_{j=1}^{k} s_j t_j\,, \quad \mathbf{s}, \mathbf{t} \in \mathbb{R}^k\,;$$

in particular,

$$|\nu|_\infty = \max(|\nu^{(q)}| \mid 1 \le q \le n) \quad \text{and} \quad \langle \nu, \mu\rangle = S(\nu\mu) \quad \text{for } \nu, \mu \in K\,.$$

1. Preliminary lemmas. First we need

LEMMA 1. *For a natural number N and reals $v < w$ one has*

$$\int_v^w \sum_{N<|k|\le 2N} e(kt)\,dt \ll \frac{1}{N}\min\left(\frac{1}{\|v\|} + \frac{1}{\|w\|}, N\right).$$

Proof. Obviously, it suffices to prove the lemma assuming $v, w \notin \mathbb{Z}$. The integral equals

$$\sum_{N<|k|\le 2N} \frac{1}{2\pi i k}(e(kw) - e(kv))$$

and is, therefore, by trivial estimation, $\ll 1$, and is

$$\ll \frac{1}{N}\min\left(\frac{1}{\|v\|} + \frac{1}{\|w\|}\right)$$

by use of partial summation and of $\sum_{a<k<b} e(kt) \ll 1/\|t\|$. ∎

LEMMA 2. *Let $M, T \ge 2$, $C \ge 1$ and $\beta \in \mathbb{R}$. Then*

$$\int_{-C}^{C} \min\left(\frac{1}{\|t\|}, M\right)\min\left(\frac{1}{|t+\beta|}, T\right)dt \ll \log(MT)\sum_{|m|\le 2C}\min\left(\frac{1}{|m+\beta|}, MT\right).$$

Proof. The left-hand side is less than

$$\sum_{|m|\le 2C}\int_{m-1/2}^{m+1/2}\min\left(\frac{1}{|t-m|}, M\right)\min\left(\frac{1}{|t+\beta|}, T\right)dt\,.$$

For fixed m, the integral can be estimated in a trivial way by MT.

For $|m+\beta| \ge 1$ one has

$$\min\left(\frac{1}{|t+\beta|}, T\right) \ll \min\left(\frac{1}{|m+\beta|}, T\right) \qquad (m-1/2 \le t \le m+1/2)$$

and

$$\int\limits_{m-1/2}^{m+1/2} \min\left(\frac{1}{|t-m|}, M\right) dt = 2\log\left(\frac{eM}{2}\right) \ll \log(MT)\,.$$

Otherwise, let

$$I_1 := \left]m - \frac{1}{MT}, m + \frac{1}{MT}\right[\cup \left] -\beta - \frac{1}{MT}, -\beta + \frac{1}{MT}\right[$$

and

$$I_2 := [m-1/2, m+1/2] - I_1\,.$$

The integral taken over I_1 does not exceed $4 \ll \min(1/|m+\beta|, T)$. I_2 is the union of at most 3 subintervals. Let $[v_1, v_2]$ be one of them. Then

$$\int\limits_{v_1}^{v_2} \frac{dt}{|t-m|\,|t+\beta|} = \left| \int\limits_{v_1}^{v_2} \frac{dt}{(t-m)(t+\beta)} \right|$$

(note that the integrand does not change its sign on $[v_1, v_2]$)

$$= \left| \frac{1}{m+\beta} \int\limits_{v_1}^{v_2} \left(\frac{1}{t-m} - \frac{1}{t+\beta}\right) dt \right|$$

$$\ll \frac{1}{|m+\beta|} \log(MT)\,. \ \blacksquare$$

LEMMA 3. *Let $k \in \mathbb{N}$, $\mathbf{a} \in (\mathbb{R} - \{0\})^k$, $M \ge 2$, $\beta \in \mathbb{R}$ and $C \ge 1$. Assume*

$$T \ge M + 2 \max_{1 \le j \le k} (|a_j| + |a_j^{-1}|)\,.$$

Then

$$\int\limits_{[-C,C]^k} \prod_{q=1}^{k} \min\left(\frac{1}{\|t_q\|}, M\right) \min\left(\frac{1}{|\langle \mathbf{a}, \mathbf{t}\rangle + \beta|}, T\right) dt$$

$$\ll_k (\log T)^k \sum_{\substack{\mathbf{m} \in \mathbb{Z}^k \\ |\mathbf{m}|_\infty \le 2^k C}} \min\left(\frac{1}{|\langle \mathbf{a}, \mathbf{m}\rangle + \beta|}, TM^k\right).$$

P r o o f. We use induction on k and the formula

$$\int_{-C}^{C} \min\left(\frac{1}{\|t_k\|}, M\right) \min\left(\frac{1}{|\langle \mathbf{a}, \mathbf{t}\rangle + \beta|}, T\right) dt_k$$
$$= \frac{1}{|a_k|} \int_{-C}^{C} \min\left(\frac{1}{\|t_k\|}, M\right)$$
$$\times \min\left(\frac{1}{|t_k + (\sum_{j\le k-1} t_j a_j + \beta)a_k^{-1}|}, T|a_k|\right) dt_k$$
$$\ll \log T \sum_{|m_k|\le 2C} \min\left(\frac{1}{|\sum_{j\le k-1} t_j a_j + (a_k m_k + \beta)|}, TM\right)$$
$$\forall (t_1, \ldots, t_{k-1}) \in [-C, C]^{k-1}$$

by Lemma 2. ∎

2. Fourier expansion of $F(\cdot, \mathbf{z})$. Obviously, one has

$$F(\cdot, \mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^n} a_{\mathbf{n}} e(\langle \mathbf{n}, \cdot\rangle) \quad \text{in } L^2([0,1]^n)\,, \tag{8}$$

where

$$\begin{aligned} a_{\mathbf{n}} = a_{\mathbf{n}}(\mathbf{z}; \mathbf{v}, \alpha) &= \int_{[0,1]^n} F(\mathbf{u}, \mathbf{z}) e(-\langle \mathbf{n}, \mathbf{u}\rangle)\, d\mathbf{u} \\ &= \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n \\ \mathbf{m}+\mathbf{u}\in V}} \int_{[0,1]^n} e(-\langle \mathbf{n}, \mathbf{m}+\mathbf{u}\rangle)\, d\mathbf{u} = \int_V e(-\langle \mathbf{n}, \mathbf{u}\rangle)\, d\mathbf{u}\,, \\ V &:= \alpha^{-1}\Big(\bigtimes_{p=1}^{n}]z_p, z_p + v_p]\Big)\,. \end{aligned} \tag{9}$$

For brevity, let

$$\begin{cases} \tau := c_4 d^{n/2}\left(\left(\dfrac{X}{N\mathfrak{f}}\right)^{1/n} + 1\right), \\ \tau' := c_5 d^n (X^{1/n} + N\mathfrak{f}^{1/n})\,, \\ \mathbf{k} \in \mathbb{Z}^n \text{ given by } k_q := [(\alpha^{-1}(\mathbf{z}))^{(q)}]\,, \text{ thus } |\mathbf{k} - \alpha^{-1}(\mathbf{z})|_\infty \le 1\,. \end{cases} \tag{10}$$

LEMMA 4. $\mathbf{t} \in V$, $\mathbf{u} \in [0,1]^n \Rightarrow |\mathbf{t} - \mathbf{u} - \mathbf{k}|_\infty \le \tau$.

P r o o f. Cramer's rule and (6) imply

$$\max_{|\mathbf{w}|_\infty\le 1}|\alpha^{-1}(\mathbf{w})|_\infty \ll |\det(\alpha_p^{(q)})|^{-1}(\max_{p,q}|\alpha_p^{(q)}|)^{n-1} \ll d^{n/2-1}N(\beta\mathfrak{f})^{-1/n} \ll d^{n-1}N\mathfrak{f}^{-1/n}\,. \tag{11}$$

This proves the assertion since (7) and (6) yield

$$\begin{aligned}|\alpha(\mathbf{t}-\mathbf{u}-\mathbf{k})|_\infty &= |(\alpha(\mathbf{t})-\mathbf{z})+\alpha(\alpha^{-1}(\mathbf{z})-\mathbf{u}-\mathbf{k})|_\infty\\ &\le |\mathbf{v}|_\infty + \max_{|\mathbf{w}|_\infty\le 2}|\alpha(\mathbf{w})|_\infty\\ &\ll (dX)^{1/n}+dN\mathfrak{f}^{1/n}\,. \ \blacksquare\end{aligned}$$

PROPOSITION 2. *Let* $N\ge 2\tau'$ *and* $\mathbf{u}\in[0,1]^n$, $\varrho:=\alpha(\mathbf{u})$. *Then*

$$\Big|\sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ N<|\mathbf{n}|_\infty\le 2N}} a_{\mathbf{n}}e(\langle\mathbf{n},\mathbf{u}\rangle)\Big| \ll \frac{\log^n N}{N}dN\mathfrak{f}^{1/n}\sum_{\substack{\nu\in\beta\mathfrak{f}\\ |\nu-\mathbf{z}|_\infty\le\tau'}}\sum_{c=0}^{1}\sum_{p=1}^{n}\min\left(\frac{1}{|z_p+cv_p-\varrho^{(p)}-\nu^{(p)}|},N^n\tau\right).$$

P r o o f. Our approach should be compared to the proof of Theorem 1 of Tatuzawa [12]. We divide the left-hand side into 2^n-1 subsums taken over the sets

$$W_I=\{\mathbf{n}\in\mathbb{Z}^n \mid N<|n_q|\le 2N\ \forall q\in I, |n_q|\le N\ \forall q\notin I\}$$

corresponding to the nonempty sets $I\subset\{1,\dots,n\}$. Let I be one of these sets; to simplify the notation we assume $n\in I$.

(9) leads to

$$\begin{aligned}\Big|\sum_{\mathbf{n}\in W_I}a_{\mathbf{n}}e(\langle\mathbf{n},\mathbf{u}\rangle)\Big| &= \Big|\int_V\sum_{\mathbf{n}\in W_I}e(\langle\mathbf{n},\mathbf{u}-\mathbf{t}\rangle)\,d\mathbf{t}\Big|\\ &= \Big|\int_V\prod_{p\in I}\sum_{N<|n_p|\le 2N}e(n_p(u_p-t_p))\prod_{p\notin I}\sum_{|n_p|\le 2N}e(n_p(u_p-t_p))\,d\mathbf{t}\Big|\\ &\ll \int_{[-\tau,\tau]^{n-1}}\prod_{p=1}^{n-1}\min\left(\frac{1}{\|s_p\|},N\right)\\ &\qquad\times\Big|\int_{\{s_n\mid(s_1,\dots,s_n)^\top+\mathbf{u}+\mathbf{k}\in V\}}\sum_{N<|n_n|\le 2N}e(n_ns_n)\,ds_n\Big|d^{n-1}\mathbf{s}\end{aligned}$$

by means of the substitution $\mathbf{s}=\mathbf{t}-\mathbf{u}-\mathbf{k}$ and of Lemma 4.

Now $\overline{\{s_n \mid (s_1,\dots,s_n)+\mathbf{u}+\mathbf{k}\in V\}} =: [\xi_1,\xi_2]$ is an interval, and

$$\alpha^{(p)}((s_1,\dots,s_{n-1},\xi_j)+\mathbf{u}+\mathbf{k}) = z_p + cv_p$$
$$\Rightarrow \xi_j = \Big(z_p + cv_p - \varrho^{(p)} - \alpha^{(p)}(k) - \sum_{j=1}^{n-1} s_j\alpha_j^{(p)}\Big)\alpha_n^{(p)^{-1}} =: \xi_{cp}(\mathbf{s})$$

for some $c\in\{0,1\}$, $p\in\{1,\dots,n\}$.

Since

$$\begin{aligned}\xi_{cp}(\mathbf{s}) &= (cv_p + \alpha^{(p)}(\alpha^{-1}(\mathbf{z}) - \mathbf{u} - \mathbf{k} + O(\tau)))\alpha_n^{(p)^{-1}}\\ &\ll ((dX)^{1/n} + dN\mathfrak{f}^{1/n}\tau)\Big|\prod_{q\neq p}\alpha_n^{(q)}N\alpha_n^{-1}\Big|\\ &\ll (X^{1/n} + N\mathfrak{f}^{1/n}\tau)d^{n/2}N\mathfrak{f}^{-1/n}\\ &\ll d^{n/2}\tau \quad \text{by (6)\&(7)},\end{aligned}$$

the inner integral is, by Lemma 1,

$$\begin{aligned}&\ll \frac{1}{N}\sum_{c,p}\min\Big(\frac{1}{\|\xi_{cp}(\mathbf{s})\|}, N\Big)\\ &\ll \frac{1}{N}\sum_{c,p}\sum_{|m_n|\le c_6 d^{n/2}\tau}\min\Big(\frac{1}{|\xi_{cp}(\mathbf{s}) - m_n|}, N\Big).\end{aligned}$$

This gives

$$\begin{aligned}&\Big|\sum_{\mathbf{n}\in W_I} a_{\mathbf{n}} e(\langle\mathbf{n},\mathbf{u}\rangle)\Big|\\ &\ll \frac{1}{N}\sum_{c,p}\sum_{|m_n|\le c_6 d^{n/2}\tau}\int_{[-\tau,\tau]^{n-1}}\prod_{j=1}^{n-1}\min\Big(\frac{1}{\|s_j\|}, N\Big)\\ &\quad\times\min\Big(\frac{1}{|\xi_{cp}(\mathbf{s}) - m_n|}, N\Big)d^{n-1}\mathbf{s}\\ &\ll \frac{\log^n N}{N}\sum_{c,p}\sum_{|\mathbf{m}|_\infty\le c_7 d^{n/2}\tau}\min\Big(\frac{1}{|\xi_{cp}((m_1,\dots,m_{n-1})^\top) - m_n|}, \tau N^n\Big)\end{aligned}$$

by Lemma 3, which is applicable because (6) imply

$$\begin{aligned}\max_{h,i,j}|(\alpha_j/\alpha_h)^{(i)}| &= \max_{h,i,j}\Big|\alpha_j^{(i)}\prod_{k\neq i}\alpha_h^{(k)}/N\alpha_h\Big|\\ &\ll (d^{1/2}N\mathfrak{f}^{1/n})^n N\beta\mathfrak{f}^{-1} \ll d^{n/2} \ll \tau.\end{aligned}$$

Now

$$(\xi_{cp}((m_1,\ldots,m_{n-1})^\top)-m_n)\alpha_n^{(p)} = z_p+cv_p-\varrho^{(p)}-\alpha^{(p)}(\mathbf{k})-\alpha^{(p)}(\mathbf{m}) = z_p+cv_p-\varrho^{(p)}-\nu^{(p)}\,,$$

where $\nu := \alpha(\mathbf{k}+\mathbf{m}) \in \beta\mathfrak{f}$ satisfies

$$|\nu-\mathbf{z}|_\infty = |\alpha(\mathbf{k}-\alpha^{-1}(\mathbf{z})+\mathbf{m})|_\infty \le \max(|\alpha(\mathbf{t})|_\infty \mid |\mathbf{t}|_\infty \le c_7\tau d^{n/2}) \le \tau'$$

for sufficiently large c_5.

Moreover, $\mathbf{m} \to \nu$ is injective, and the assertion follows since $|\alpha_n^{(p)}| \ll dN\mathfrak{f}^{1/n}$. ■

Proposition 3. *Let $N \ge 2\tau$. Then*

$$F(\mathbf{u},\mathbf{z}) = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ |\mathbf{n}|_\infty\le N}} a_{\mathbf{n}} e(\langle\mathbf{n},\mathbf{u}\rangle) + O\bigg(\frac{\log^n N}{N} dN\mathfrak{f}^{1/n} \sum_{\substack{\nu\in\beta\mathfrak{f}\\ |\nu-\mathbf{z}|_\infty\le\tau'}} \sum_{c=0}^{1}\sum_{p=1}^{n} \frac{1}{|z_p+cv_p+\alpha^{(p)}(\mathbf{u})-\nu^{(p)}|}\bigg)$$

for any $\mathbf{u} \in [0,1]^n$.

Remark. For certain values of $\mathbf{u}$ the expression inside $O(\cdot)$ is not finite. It is easy to show (but not needed in this paper) that the remainder does not exceed

$$O\bigg(\bigg(\frac{X}{N\mathfrak{f}}\bigg)^{1-1/n} + d^{n/2}\log^n NX\bigg).$$

One has to combine Lemma 5 below and a result similar to Hilfssatz 10 of [9].

Proof of Proposition 3. Define

$$K := \{\mathbf{u}\in[0,1]^n \mid \exists\nu\in\beta\mathfrak{f},\ |\nu-\mathbf{z}|_\infty\le\tau',\ \exists c,p : z_p+cv_p=\nu^{(p)}+\alpha^{(p)}(\mathbf{u})\}\,,$$

$$G := [0,1]^n - K \quad \text{and} \quad F_m(\mathbf{u}) := \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ 2^{m-1}N<|\mathbf{n}|_\infty\le 2^m N}} a_{\mathbf{n}} e(\langle\mathbf{n},\mathbf{u}\rangle)\,.$$

Proposition 1 yields, for $\mathbf{u} \in G$,

$$(12)\quad \sum_{m=1}^{\infty}|F_m(\mathbf{u})| \ll \sum_{m=1}^{\infty}\frac{\log^n(2^mN)}{2^mN} dN\mathfrak{f}^{1/n}\sum_{c,p}\sum_{|\nu-\mathbf{z}|_\infty\le\tau'}\frac{1}{|z_p+cv_p-\alpha^{(p)}(\mathbf{u})-\nu^{(p)}|}$$

$$\ll \frac{\log^n N}{N} dN\mathfrak{f}^{1/n} \sum_{\substack{\nu\in\beta\mathfrak{f}\\ |\nu-\mathbf{z}|_\infty\le\tau'}} \sum_{c=0}^{1}\sum_{p=1}^{n} \frac{1}{|z_p+cv_p+\alpha^{(p)}(\mathbf{u})-\nu^{(p)}|}\,.$$

Therefore, $\sum_{m=1}^{\infty} F_m$ converges uniformly on any compact set $\widetilde{G}\subset G$. It coincides by (8) with

$$F(\cdot,\mathbf{z}) - \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ |\mathbf{n}|_\infty\le N}} a_{\mathbf{n}} e(\langle \mathbf{n},\cdot\rangle) \quad \text{in } L^2(\widetilde{G})\,.$$

Since K is closed and both the functions are continuous, equality holds at every point of G, and the assertion follows (of course it is trivial for $\mathbf{u}\in K$). ■

3. Upper bounds. From Proposition 3 we derive our generalization of (4):

THEOREM 3. *There are complex numbers $b_{\mathbf{n}} = b_{\mathbf{n}}(\mathbf{x},\mathbf{y},\alpha)$ satisfying*

$$|b_{\mathbf{n}}| \ll \frac{1}{\sqrt{d}N\beta\mathfrak{f}} \prod_{p=1}^{n} \min\left(\frac{1}{|\eta^{(p)}(\mathbf{n})|}, X^{1/n}\right)$$

and

$$|\{\nu\in\mathbb{Z}_K \mid \nu\equiv\nu_0 \bmod \mathfrak{f},\ y_q<\nu^{(q)}\le y_q+x_q\,,\ 1\le q\le n\}| = \frac{X}{\sqrt{d}N\mathfrak{f}} + \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ 0<|\mathbf{n}|_\infty\le N}} b_{\mathbf{n}} e(S(\eta(\mathbf{n})\beta\nu_0)) + O(N^{-1/3})$$

for any $\nu_0\in\mathbb{Z}_K$ and any $N\ge c_8(dN\mathfrak{f}X)^{c_9}$.

Proof. Let $\widetilde{z}_q := \beta^{(q)}y_q$, $\widetilde{v}_q := \beta^{(q)}x_q$, $1\le q\le n$. For different integers ν_1, ν_2 of K satisfying $|\nu_j-\widetilde{\mathbf{z}}|_\infty\le 2\tau'$,

$$\min_{1\le p\le n} |\nu_1^{(p)}-\nu_2^{(p)}| \ge |N(\nu_1-\nu_2)|/|\nu_1-\nu_2|_\infty^{n-1} \ge (4\tau')^{1-n}\,.$$

Thus at least one of the intervals

$$]\widetilde{z}_p+c\widetilde{v}_p-(8\tau')^{1-n}, \widetilde{z}_p+c\widetilde{v}_p] \quad \text{and} \quad]\widetilde{z}_p+c\widetilde{v}_p, \widetilde{z}_p+c\widetilde{v}_p+(8\tau')^{1-n}]$$

does not contain the pth conjugate of any $\nu\in\mathbb{Z}_K$, $|\nu-\mathbf{z}|_\infty\le\tau'$.

This allows us to choose $a_{cp}, b_{cp}\in\{0,1\}$ so that

$$z_p := \widetilde{z}_p + (-1)^{a_{cp}}(8\tau')^{1-n}N^{-1/3}$$

and

$$v_p := \widetilde{v}_p + (\widetilde{z}_p - z_p) + (-1)^{b_{cp}}(8\tau')^{1-n}N^{-1/3}$$

satisfy

$$|z_p+cv_p-\nu^{(p)}| \ge (8\tau')^{1-n}N^{-1/3} \quad \forall\nu\in\mathbb{Z}_K : |\nu-\mathbf{z}|_\infty\le\tau'\ \forall c\in\{0,1\}\ \forall p$$

and (since all elements of the counted sets are integers ν subject to $|\nu-\mathbf{z}|_\infty \le 2\tau'$)

$$\begin{aligned}F(\alpha^{-1}(\beta\nu_0),\mathbf{z};\mathbf{v},\alpha) &= F(\alpha^{-1}(\beta\nu_0),\widetilde{\mathbf{z}};\widetilde{\mathbf{v}},\alpha)\\ &= |\{\mathbf{m}\in\mathbb{Z}^n \mid \widetilde{z}_p < \alpha^{(p)}(\mathbf{m}+\alpha^{-1}(\beta\nu_0)) \le \widetilde{z}_p+\widetilde{v}_p\}|\\ &= |\{\mu\in\beta\mathfrak{f} \mid \beta^{(p)}y_p < (\mu+\beta\nu_0)^{(p)} \le \beta^{(p)}(y_p+x_p)\}|\\ &= |\{\nu\in\mathbb{Z}_K \mid \nu\equiv\nu_0 \bmod \mathfrak{f},\ y_p<\nu^{(p)}\le y_p+x_p\}|\,.\end{aligned}$$

(7) holds because of (5) and of $v_p=\widetilde{v}_p+O(N^{-1/3})=\beta^{(p)}x_p+O(N^{-1/3})$. Thus Proposition 3 can be used to obtain

$$\begin{aligned}&|\{\nu\in\mathbb{Z}_K \mid \nu\equiv\nu_0 \bmod \mathfrak{f},\ y_p<\nu^{(p)}\le y_p+x_p\}|\\ &\quad= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ |\mathbf{n}|_\infty\le N}} a_{\mathbf{n}}(\mathbf{z},\mathbf{v},\alpha)e(\langle\mathbf{n},\alpha^{-1}(\beta\nu_0)\rangle)\\ &\qquad+O\Big(\frac{\log^n N}{N}dN\mathfrak{f}^{1/n}\sum_{\substack{\nu\in\beta\mathfrak{f}\\ |\nu-\mathbf{z}|_\infty\le\tau'}}(\tau'^{1-n}N^{-1/3})^{-1}\Big)\\ &\quad= \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ |\mathbf{n}|_\infty\le N}} a_{\mathbf{n}}(\mathbf{z},\mathbf{v},\alpha)e(\langle\eta(\mathbf{n}),\beta\nu_0\rangle)+O\Big(\frac{\log^n N}{N}d^{1/2}\tau'^{2n-1}N^{1/3}\Big)\end{aligned}$$

by use of

$$\begin{aligned}|\{\nu\in\beta\mathfrak{f} \mid |\nu-\mathbf{z}|_\infty\le\tau'\}| &= |\{\mathbf{m}\in\mathbb{Z}^n \mid |\alpha(\mathbf{m})-\alpha(\mathbf{z})|_\infty\le\tau'\}|\\ &\le \operatorname{Vol}(\mathbf{t}\in\mathbb{R}^n \mid |\alpha(\mathbf{t})|_\infty\le 2\tau') \ll \frac{\tau'^n}{\sqrt{d}N\beta\mathfrak{f}}\,.\end{aligned}$$

For sufficiently large c_9 the remainder is $\ll N^{-1/3}$ (see (10)).

Moreover, by means of the substitution $\mathbf{t}=\alpha(\mathbf{v})$, (9) gives

$$b_{\mathbf{n}} := a_{\mathbf{n}}(\mathbf{z},\mathbf{v},\alpha) = \frac{1}{\sqrt{d}N\beta\mathfrak{f}}\prod_{p=1}^{n}\int_{z_p}^{z_p+v_p} e(-\eta^{(p)}(\mathbf{n})t_p)\,dt_p\,,$$

which shows the estimate for the $b_{\mathbf{n}}$, $\mathbf{n}\ne\mathbf{0}$, and

$$\begin{aligned}b_{\mathbf{0}} :&= \frac{1}{\sqrt{d}N\beta\mathfrak{f}}\prod_{p=1}^{n}v_p = \frac{1}{\sqrt{d}N\beta\mathfrak{f}}\prod_{p=1}^{n}(\widetilde{v}_p+O(\tau'^{1-n}N^{-1/3}))\\ &= \frac{1}{\sqrt{d}N\beta\mathfrak{f}}\Big(\prod_{p=1}^{n}\beta^{(p)}x_p+O\Big(\Big(\frac{(dX)^{1/n}}{\tau'}\Big)^{n-1}N^{-1/3}\Big)\Big) \quad \text{by (5)}\\ &= \frac{N\beta X}{\sqrt{d}N\beta\mathfrak{f}}+O(N^{-1/3}) \quad \text{by (10)}\,. \ \blacksquare\end{aligned}$$

Lemma 5. *Let $\mathfrak{c}$ denote a* (*not necessarily integral*) *ideal of K and let $M \ge 2 + N\mathfrak{c}$. Then*

$$\sum_{\substack{\gamma\in\mathfrak{c}\\ 0<|\gamma|_\infty\le M}} \frac{1}{|N\gamma|} \ll d^{(n-1)/2} N\mathfrak{c}^{-1} (\log M)^n .$$

Proof. Given $\mathbf{z} \in \mathbb{R}_+^n$, $Z = \prod_{q=1}^n z_q$, we obtain from Theorem 1 of [5] the existence of a linear mapping $\gamma = \gamma_{\mathbf{z}} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\gamma(\mathbb{Z}^n) = \mathfrak{c} \quad \text{ and } \quad \sup_{|\mathbf{t}|_\infty\le 1} |\gamma^{(q)}(\mathbf{t})| \le c_{10} d^{1/2} N\mathfrak{c}^{1/n} z_q Z^{-1/n} .$$

This implies

$$\begin{aligned}\sum_{\substack{\gamma\in\mathfrak{c}\\ z_q<|\gamma^{(q)}|\le 2z_q}} 1 &= \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n\\ z_q<|\gamma^{(q)}(\mathbf{m})|\le 2z_q}} \int\limits_{\mathbf{m}+[0,1]^n} d\mathbf{t} \\ &\le \operatorname{Vol}(\mathbf{t}\in\mathbb{R}^n \mid |\gamma^{(q)}(\mathbf{t})| \le 2z_q + O(d^{1/2}N\mathfrak{c}^{1/n} z_q Z^{-1/n})) \\ &\ll \frac{1}{d^{1/2}N\mathfrak{c}} \prod_{q=1}^n (2z_q + O(d^{1/2}N\mathfrak{c}^{1/n} z_q Z^{-1/n})) \\ &\ll \frac{Z}{d^{1/2}N\mathfrak{c}} + d^{(n-1)/2} .\end{aligned}$$

Since $\gamma \in \mathfrak{c}$ and $0 < |\gamma|_\infty \le M$ imply

$$|N\gamma| \ge N\mathfrak{c} \quad \text{ and } \quad |\gamma^{(q)}| = |N\gamma| \prod_{p\ne q} |\gamma^{(p)}|^{-1} \ge N\mathfrak{c} M^{1-n}$$

we conclude that

$$\begin{aligned}\sum_{\substack{\gamma\in\mathfrak{c}\\ 0<|\gamma|_\infty\le M}} \frac{1}{|N\gamma|} &\le \sum_{0\le k_1,\dots,k_n\le \frac{\log(M^{n-1}/N\mathfrak{c})}{\log 2}} \sum_{\substack{\gamma\in\mathfrak{c}\\ M2^{-k_q-1}<|\gamma^{(q)}|\le M2^{-k_q}}} \frac{1}{|N\gamma|} \\ &\le \sum_{0\le k_1,\dots,k_n\le \frac{\log(M^{n-1}/N\mathfrak{c})}{\log 2}} \min\Big(\frac{1}{N\mathfrak{c}}, M^{-n}2^{\Sigma k_q+n}\Big) \sum_{\substack{\gamma\in\mathfrak{c}\\ M2^{-k_q-1}<|\gamma^{(q)}|\le M2^{-k_q}}} 1 \\ &\ll \sum_{0\le k_1,\dots,k_n\le \frac{\log(M^{n-1}/N\mathfrak{c})}{\log 2}} \min\Big(\frac{1}{N\mathfrak{c}}, M^{-n}2^{\Sigma k_q}\Big)\Big(\frac{M^n 2^{-\Sigma k_q}}{d^{1/2}N\mathfrak{c}} + d^{(n-1)/2}\Big) \\ &\ll \log^n(M^n/N\mathfrak{c})(d^{-1/2}N\mathfrak{c}^{-1} + d^{(n-1)/2}N\mathfrak{c}^{-1}) . \ \blacksquare\end{aligned}$$

Let $G(\gamma)$ denote the Gaussian sum $\sum_{\varrho \bmod \mathfrak{f}} \chi(\varrho) e(S(\gamma\varrho))$, $\gamma \in 1/(\mathfrak{d}\mathfrak{f})$. Since χ is primitive one has the well-known

LEMMA 6.
$$|G(\gamma)| = \begin{cases} 0, & (\gamma\mathfrak{d}\mathfrak{f},\mathfrak{f}) \neq 1, \\ N\mathfrak{f}^{1/2}, & (\gamma\mathfrak{d}\mathfrak{f},\mathfrak{f}) = 1. \end{cases}$$

Proof of Theorem 1. One has

$$\sum_{y_q<\nu^{(q)}\le y_q+x_q} \chi(\nu) = \sum_{\nu_0 \bmod \mathfrak{f}} \chi(\nu_0)|\{\nu\in\mathbb{Z}_K \mid \nu\equiv\nu_0 \bmod \mathfrak{f},\ y_q<\nu^{(q)}\le y_q+x_q\}|$$
$$= \Big(\sum_{\nu_0 \bmod \mathfrak{f}} \chi(\nu_0)\Big)\frac{X}{d^{1/2}N\mathfrak{f}} + \sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ 0<|\mathbf{n}|_\infty\le N}} b_{\mathbf{n}} \sum_{\nu_0 \bmod \mathfrak{f}} \chi(\nu_0)e(S(\eta(\mathbf{n})\beta\nu_0)) + O(1)$$

by Theorem 3, with $N := c_8(dN\mathfrak{f}X)^{c_9} \ge N\mathfrak{f}^3$. Analogously to (11),
$$\max_{|\mathbf{t}|_\infty\le 1} |\eta(\mathbf{t})| \le c_{11}d^{n-1}N\mathfrak{f}^{-1/n}$$
follows. This yields

$$(13)\qquad \{\eta(\mathbf{n}) \mid \mathbf{n}\in\mathbb{Z}^n,\ 0<|\mathbf{n}|_\infty\le N\} \subset \{\eta\in 1/(\mathfrak{d}\beta\mathfrak{f}) \mid 0<|\eta|_\infty\le N^2\}$$

since $N \ge c_{12}d^{n-1}$.

From Lemma 6 one infers

$$\sum_{\substack{\mathbf{n}\in\mathbb{Z}^n\\ 0<|\mathbf{n}|_\infty\le N}} |b_{\mathbf{n}}|\Big|\sum_{\nu_0 \bmod \mathfrak{f}} \chi(\nu_0)e(S(\eta(\mathbf{n})\beta\nu_0))\Big|$$
$$\ll \frac{N\mathfrak{f}^{1/2}}{d^{1/2}N\beta\mathfrak{f}} \sum_{\substack{\eta\in 1/(\mathfrak{d}\beta\mathfrak{f})\\ 0<|\eta|_\infty\le N^2\\ (\eta\mathfrak{d}\beta\mathfrak{f},\mathfrak{f})=1}} \prod_{q=1}^{n} \min\Big(\frac{1}{|\eta^{(q)}|}, X^{1/n}\Big)$$
$$\ll \frac{1}{d^{1/2}N\beta N\mathfrak{f}^{1/2}} \sum_{\substack{\eta\in 1/(\mathfrak{d}\beta\mathfrak{f})\\ 0<|\eta|_\infty\le N^2}} \frac{1}{|N\eta|} \ll d^{n/2}N\mathfrak{f}^{1/2}(\log N)^n$$

by Lemma 5.

So Theorem 1 follows directly for $N\mathfrak{f}^{1/2} \le X$ (implying $\log(dN\mathfrak{f}X) \ll \log(dX)$); otherwise it is trivial (use Theorem 1 with $\mathfrak{f} = \mathbb{Z}_K$). ∎

The proof of Proposition 1 follows in the same way.

4. Lower bounds. To derive lower bounds we fix $\nu_0 \in \mathbb{Z}_K$, replace β by 1 and work with the Fourier series of $F(\alpha^{-1}(\nu_0), \mathbf{z}; \mathbf{v}, \alpha)$ with respect to $\mathbf{z}$.

From (6) follows the existence of $\mathbf{w} \in \mathbb{R}^n$, $|\mathbf{w}|_\infty \ll d^{1/2} N\mathfrak{f}^{1/n}$, satisfying

$$\varDelta := \mathbf{w} + \alpha([0,1]^n) \subset \mathbb{R}^n_+ .$$

In $L^2(\varDelta)$,

$$F(\alpha^{-1}(\nu_0), \cdot; \mathbf{v}, \alpha) = \sum_{\gamma \in 1/(\mathfrak{d}\mathfrak{f})} c_\gamma e(-\langle \gamma, \cdot \rangle) \tag{14}$$

holds where the coefficients are given by

$$\begin{aligned} c_\gamma = c_\gamma(\nu_0, \mathbf{v}, \alpha) &= \frac{1}{\operatorname{Vol} \varDelta} \int_{\varDelta} F(\alpha^{-1}(\nu_0), \mathbf{z}) e(-\langle \gamma, \mathbf{z} \rangle)\, d\mathbf{z} \\ &= \frac{e(-S(\gamma\nu_0))}{d^{1/2} N\mathfrak{f}} \int_{\varDelta} \sum_{\substack{\nu \in \mathfrak{f} \\ z_p < \nu^{(p)} + \nu_0^{(p)} \le z_p + v_p}} e(\langle \gamma, \nu + \nu_0 - \mathbf{z} \rangle)\, d\mathbf{z} \\ &= \frac{e(-S(\gamma\nu_0))}{d^{1/2} N\mathfrak{f}} \sum_{\substack{\nu \in \mathbb{Z}_K \\ \nu \equiv \nu_0 \bmod \mathfrak{f}}} \int_{\varDelta \cap \{\mathbf{z} \mid 0 < \nu^{(p)} - z_p \le v_p\}} e(\langle \gamma, \nu - \mathbf{z} \rangle)\, d\mathbf{z} \\ &= \frac{e(-S(\gamma\nu_0))}{d^{1/2} N\mathfrak{f}} \int_{\{\mathbf{z} \mid 0 < z_p \le v_p\}} e(\langle \gamma, \mathbf{z} \rangle)\, d\mathbf{z} \\ &= \frac{e(-S(\gamma\nu_0))}{d^{1/2} N\mathfrak{f}} \prod_{p=1}^{n} \int_0^{v_p} e(\gamma^{(p)} t_p)\, dt_p \\ &= \begin{cases} \dfrac{1}{(2\pi i)^n} \dfrac{e(-S(\gamma\nu_0))}{d^{1/2} N\mathfrak{f}} \dfrac{1}{N\gamma} \displaystyle\prod_{p=1}^{n} (e(\gamma^{(p)} v_p) - 1), & \gamma \ne 0, \\ \dfrac{X}{d^{1/2} N\mathfrak{f}}, & \gamma = 0. \end{cases} \end{aligned} \tag{15}$$

R e m a r k. $c_\gamma(\nu_0, \mathbf{v}, \alpha) e(S(\eta\nu_0)) = \overline{a_{\eta^{-1}(\gamma)}(\mathbf{z}, \mathbf{v}, \alpha)} e(-\langle \gamma, \mathbf{z} \rangle)$.

PROPOSITION 4. *Let* $\gamma_0 \in 1/(\mathfrak{d}\mathfrak{f}) - \{0\}$ *satisfy* $(\gamma_0 \mathfrak{d}\mathfrak{f}, \mathfrak{f}) = 1$. *For any* $\mathbf{y} \in \mathbb{R}^n$ *there is an* $\mathbf{x} \in \mathbb{R}^n_+$, $|\mathbf{x}|_\infty \ll |1/\gamma_0|_\infty + d^{1/2} N\mathfrak{f}^{1/n}$, *satisfying*

$$\Big| \sum_{y_q < \nu^{(q)} \le y_q + x_q} \chi(\nu) - E(\chi) X \Big| \ge \frac{1}{(2\pi)^n d^{1/2}} \frac{1}{N\mathfrak{f}^{1/2} |N\gamma_0|} .$$

P r o o f. One has

$$\begin{aligned} h(\mathbf{z}) :&= \sum_{z_q + y_q < \nu^{(q)} \le y_q + z_q + v_q} \chi(\nu) - E(\chi) \prod_{q=1}^{n} v_q \\ &= \sum_{\nu \bmod \mathfrak{f}} \chi(\nu) \sum_{\gamma \in 1/(\mathfrak{d}\mathfrak{f}) - \{0\}} c_\gamma(\nu, \mathbf{v}, \alpha) e(-\langle \gamma, \mathbf{z} + \mathbf{v} \rangle) \end{aligned}$$

in $L^2(\varDelta)$ by (14).

Parseval's equation and (15) lead to

$$\max_{\mathbf{z}\in\Delta}|h(\mathbf{z})|^2 \ge \frac{1}{\operatorname{Vol}\Delta}\int\limits_{\Delta}|h(\mathbf{z})|^2\,d\mathbf{z}$$

$$= \sum_{\gamma\in 1/(\mathfrak{d}\mathfrak{f})-\{0\}}\Big|\sum_{\nu \bmod \mathfrak{f}} c_\gamma(\nu,\mathbf{v},\alpha)\chi(\nu)\Big|^2$$

$$\ge \frac{1}{(4\pi^2)^n}\frac{1}{dN\mathfrak{f}^2}\frac{1}{|N\gamma_0|^2}|G(-\gamma_0)|^2\prod_{p=1}^{n}|e(\gamma_0^{(p)}v_p)-1|^2\,.$$

The product is 4^n if we choose v_p to be $(2|\gamma_0^{(p)}|)^{-1}$.

By use of Lemma 6 we obtain the existence of $\mathbf{z}\in\Delta$ (thus $0<z_q\ll d^{1/2}N\mathfrak{f}^{1/n}$) satisfying

$$\frac{1}{\pi^n d^{1/2}N\mathfrak{f}}\frac{1}{|N\gamma_0|}N\mathfrak{f}^{1/2} \le |h(\mathbf{z})|$$

$$= \Big|\sum_{z_q+y_q<\nu^{(q)}\le y_q+z_q+v_q}\chi(\nu) - E(\chi)\operatorname{Vol}(\mathbf{v}\in\mathbb{R}^n \mid y_q+z_q<v_q\le y_q+z_q+v_q)\Big|$$

$$= \Big|\sum_{c_1,\ldots,c_n=0}^{1}(-1)^{n-\Sigma c_q}\Big(\sum_{y_q<\nu^{(q)}\le y_q+z_qc_q+v_q}\chi(\nu) - E(\chi)\operatorname{Vol}(\mathbf{v}\in\mathbb{R}^n \mid y_q<v_q\le y_q+z_qc_q+v_q)\Big)\Big|\,.$$

So at least one of the 2^n values of $(z_qc_q+v_q)_{q=1}^n$ can be chosen to be $\mathbf{x}$. ∎

Proof of Theorem 2. The ideal class generated by $\mathfrak{d}\mathfrak{f}$ contains at least $2\omega(\mathfrak{f})$ prime ideals of norm less than $c_{13}(K)\omega(2\mathfrak{f})\log(\omega(6\mathfrak{f}))$. Thus one of these ideals, say $\mathfrak{p}$, does not divide $\mathfrak{f}$. Any generator γ_0 of the principal ideal $\mathfrak{p}/(\mathfrak{d}\mathfrak{f})$ satisfying

$$|\gamma_0^{(q)}| \ll_K |N\gamma_0|^{1/n} \ll_K \frac{\omega(2\mathfrak{f})\log(\omega(6\mathfrak{f}))}{N\mathfrak{f}} \qquad \text{(see e.g. (81) of [9])}$$

is admissible in Proposition 3 since

$$(\gamma_0\mathfrak{d}\mathfrak{f},\mathfrak{f}) = (\mathfrak{p},\mathfrak{f}) = 1\,.$$

This proves Theorem 2. If $\mathfrak{d}\mathfrak{f}=(\varrho)$ is principal one applies Proposition 4 with $\gamma_0 = 1/\varrho$. ∎

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