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Proof of a conjecture of Selmer

by

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1. Introduction. The following definitions, results, and conjecture are taken from [7], to which the reader is referred for background.

Let $k \ge 1$ be an integer. Consider a *basis* $A_k = \{a_1, \ldots, a_k\}$, where $1 = a_1 < \ldots < a_k$ are integers ("stamp denominations"). Let $h \ge 1$ be an integer ("size of the envelope"). Consider all combinations

$$x_1 a_1 + \ldots + x_k a_k, \qquad \text{where } x_i \ge 0 \text{ are integers and } \sum_{i=1}^{k} x_i \le h. \tag{1.1}$$

Let $N_h(A_k)$ = smallest positive integer not represented by (1.1) and let $n_h(A_k) = N_h(A_k) - 1$ = the *h-range of* A_k. To emphasize the role of h in these considerations, we will call A_k an *h-basis* in this paper. Let $h_0 = h_0(A_k) = \min\{h \mid n_h(A_k) \ge a_k\}$.

Remark 1.1. The importance of h_0 stems from the fact that we are (basically) interested only in the case $h \ge h_0$. For if $h < h_0$, then in the representations (1.1) of the integers n, $1 \le n \le n_h(A_k)$, we cannot use a_k at all.

The "local" problem: given h and A_k, determine $n_h(A_k)$. The "global" problem: given h and k, find the *extremal h-range* $n_h(k) = \max\{n_h(A_k)\}$ and *extremal h-bases* A_k^* such that $n_h(A_k^*) = n_h(k)$.

If, in addition to (1.1), we have

$$x_1 a_1 + \ldots + x_j a_j < a_{j+1} \qquad \text{for } j = 1, \ldots, k-1, \tag{1.2}$$

then we have a *regular* representation, introduced by Hofmeister [1]. Thinking of stamps, (1.2) means that we use first the largest stamp a_k as often as possible, then the next stamp a_{k-1} as often as possible, etc.

We can then define analogously the *regular h-range* $g_h(A_k)$, $\widetilde{h}_0 = \widetilde{h}_0(A_k) = \min\{h \mid g_h(A_k) \ge a_k\}$, *extremal regular h-range* $g_h(k)$ and *extremal regular h-bases* $\widetilde{A}_k^*$: $g_h(\widetilde{A}_k^*) = g_h(k)$. In this paper, only regular representations will be considered.

Since the case $k = 1$ is trivial, and also the case $k = 2$ is under control (see [7, p. 9.1]), we suppose, in this section, that $k \geq 3$. We denote by $\lfloor x \rfloor$ the greatest integer $\leq x$ and by $\lceil x \rceil$ the least integer $\geq x$ (so that $\lceil x \rceil = -\lfloor -x \rfloor$ for all $x \in \mathbb{R}$).

The following result of Hofmeister [1] (see [7, p. 9.1]) solves the local problem in the regular case.

THEOREM 1.2 (Hofmeister). *If* $h \geq \widetilde{h}_0$, *then*

$$g_h(A_k) = \mu_1 a_1 + \ldots + \mu_k a_k, \tag{1.3}$$

where the coefficients μ_j *are given recursively by*

$$\mu_j = \left\lfloor \frac{a_{j+1} - 2 - \sum_{i=1}^{j-1} \mu_i a_i}{a_j} \right\rfloor \quad \textit{for } j = 1, \ldots, k-1, \tag{1.4}$$

$$\mu_k = h - \sum_{i=1}^{k-1} \mu_i. \tag{1.5}$$

The representation (1.3) *is regular. We also have*

$$\widetilde{h}_0 = \sum_{i=1}^{k-1} \mu_i + 1. \tag{1.6}$$

It follows immediately from (1.5) and (1.6) that

$$h = \widetilde{h}_0 \quad \text{if and only if} \quad \mu_k = 1. \tag{1.7}$$

Suppose now that we are given a sequence $(\nu_1, \ldots, \nu_k)$ of nonnegative integers with $\nu_k \geq 1$ and

$$\nu_1 + \ldots + \nu_k = h. \tag{1.8}$$

We can define a basis $A_k = \{a_1, \ldots, a_k\}$ by

$$a_{i+1} = (\nu_i + 2) a_i - a_{i-1} \quad \text{for } i = 1, \ldots, k-1 \quad (a_0 = 0). \tag{1.9}$$

It is then easy to prove by induction that

$$\mu_i = \nu_i \quad \text{for } i = 1, \ldots, k-1.$$

In fact, if we suppose that $\mu_i = \nu_i$ for $i = 1, \ldots, j-1$, then we have, by (1.9) and (1.4),

$$\mu_j = \left\lfloor \frac{(\nu_j + 2) a_j - a_{j-1} - 2 - \sum_{i=1}^{j-1} \nu_i a_i}{a_j} \right\rfloor = \left\lfloor \frac{(\nu_j + 1) a_j - a_1}{a_j} \right\rfloor$$
$$= (\nu_j + 1) - \left\lceil \frac{a_1}{a_j} \right\rceil = \nu_j.$$

By (1.6) and (1.8) with $\nu_k \geq 1$, we have $h \geq \widetilde{h}_0$, and using Theorem 1.2 we get $\mu_k = \nu_k$ and (1.3).

The following result (see [7, p. 9.5]), conjectured by Hofmeister [1] and proved by Mrose [4, p. 68], solves the global problem completely in the regular case.

THEOREM 1.3 (Mrose). *The two sets of coefficients*

$$\mu_i^{(1)} = \begin{cases} \left\lfloor \frac{ih}{k} \right\rfloor - \left\lfloor \frac{(i-1)h}{k} \right\rfloor & \text{for } i = 1, \ldots, k-1, \\ \left\lceil \frac{h}{k} \right\rceil & \text{for } i = k, \end{cases} \tag{1.10}$$

$$\mu_i^{(2)} = \begin{cases} \left\lceil \frac{h}{k} \right\rceil - 1 & \text{for } i = 1, \\ \left\lceil \frac{ih}{k} \right\rceil - \left\lceil \frac{(i-1)h}{k} \right\rceil & \text{for } i = 2, \ldots, k-1, \\ \left\lfloor \frac{h}{k} \right\rfloor + 1 & \text{for } i = k, \end{cases} \tag{1.11}$$

give the extremal regular h-bases $\widetilde{A}_k^*$ through

$$a_{i+1} = (\mu_i + 2)a_i - a_{i-1} \quad \text{for } i = 1, \ldots, k-1 \quad (a_0 = 0), \tag{1.12}$$

and the extremal regular h-range through (1.3). *The two bases are equal if and only if* $\gcd(h,k) = 1$.

For ordinary h-ranges $n_h(A_k)$, the ("classical") extremal problem is much more difficult. It is simple only for $k = 2$, but very complicated already for $k = 3$, a case which was settled by Hofmeister [3]. He also solved the analogous problem, already suggested by Salié [6], of determining the bases A_3 for which $n_{h_0}(A_3)$ is extremal for given h_0.

Inspired by this, Selmer [7, p. 9.7] raised the analogous problem for regular h-ranges. That is, given h and k, we define

$$\mathcal{A} = \{A_k \mid \widetilde{h}_0(A_k) = h\} \quad \text{and} \quad g_{\tilde{h}_0=h}(k) = \max\{g_h(A_k) \mid A_k \in \mathcal{A}\}.$$

Supported by extensive computer calculations by Svein Mossige, Selmer formulated the following

CONJECTURE 1.4 (Selmer). *If $h \geq k \geq 3$, then the problem of finding $g_{\tilde{h}_0=h}(k)$ is uniquely solved by*

$$\mu_i = \begin{cases} \left\lfloor \frac{i(h-1)}{k-1} \right\rfloor - \left\lfloor \frac{(i-1)(h-1)}{k-1} \right\rfloor & \text{for } i = 1, \ldots, k-2, \\ \left\lceil \frac{h-1}{k-1} \right\rceil & \text{for } i = k-1, \\ 1 & \text{for } i = k, \end{cases} \tag{1.13}$$

in connection with (1.12) *and* (1.3).

Remark 1.5.

• If $h < k$, then the problem is solved by Theorem 1.3. This follows from (1.7) because then $\mu_k^{(1)} = \mu_k^{(2)} = 1$.

• If $h = k$, then (1.13) gives $\mu_i = 1$ for $i = 1, \ldots, k$. Then also $\mu_i^{(1)} = 1$ for $i = 1, \ldots, k$ while $\mu_k^{(2)} = 2$. It follows that Conjecture 1.4 is true in the case $h = k$. We note that in this case $a_i = F_{2i}$ for $i = 1, \ldots, k$, where the F_{2i} are *Fibonacci numbers*, defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n = 1, 2, \ldots$

• If $h > k$, then $\mu_k^{(1)} > 1$ and $\mu_k^{(2)} > 1$, so that, by (1.7), we have $h > \widetilde{h}_0$ for both h-bases coming from Theorem 1.3. These bases, therefore, do not solve Selmer's problem.

In this paper we prove Selmer's Conjecture 1.4. The proof is based on Mrose's thesis [4]. In Section 2 we introduce certain determinants and indicate how the problem can be reformulated. The proof is then carried out in two steps, in Sections 3 and 4. Finally, some asymptotic estimates are given in Section 5.

2. Continuants and reformulation of the conjecture. The following definitions and results are taken from [4].

DEFINITION 2.1. If $(x_1, \ldots, x_k)$ is a sequence of real numbers, then by a *continuant* $C(x_1, \ldots, x_k)$ we mean the determinant

$$C(x_1, \ldots, x_k) = \begin{vmatrix} x_1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & x_2 & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & x_3 & -1 & \ldots & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & 0 & 0 & \ldots & -1 & x_k \end{vmatrix}.$$

We also define

$$C(x_r, \ldots, x_s) = \begin{cases} 1 & \text{when } s = r - 1, \\ 0 & \text{when } s \le r - 2. \end{cases}$$

The following result [4, p. 2] follows immediately.

LEMMA 2.2. *For $k \ge 1$ and real numbers $x_1, \ldots, x_k$ we have*

$$\begin{aligned} C(x_1, \ldots, x_k) &= x_1 C(x_2, \ldots, x_k) - C(x_3, \ldots, x_k) \\ &= x_k C(x_1, \ldots, x_{k-1}) - C(x_1, \ldots, x_{k-2}), \end{aligned} \tag{2.1}$$

$$C(x_1, \ldots, x_k) = C(x_k, \ldots, x_1). \tag{2.2}$$

Remark 2.3. In what follows, the *elements* $x_1, \ldots, x_k$ of a continuant $C(x_1, \ldots, x_k)$ will always be *integers* ≥ 2.

Remark 2.4. The continuants are connected with continued fractions. If we denote by

$$\langle a_0, a_1, a_2, \ldots\rangle = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \ddots}}$$

a *reduced-regular* ("reduziert-regelmässig" [5, p. 163]) continued fraction, then (see [4, p. 1] and also [5, p. 8])

$$\langle a_0, a_1, \ldots, a_n\rangle = \frac{C(a_0, a_1, \ldots, a_n)}{C(a_1, \ldots, a_n)}.$$

Definition 2.5. A continuant $C(x_1, \ldots, x_k)$ with x_i integers ≥ 2 is called *extremal* if for any continuant $C(y_1, \ldots, y_k)$ with y_i integers ≥ 2 and $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i$, we have $C(y_1, \ldots, y_k) \leq C(x_1, \ldots, x_k)$.

Definition 2.6. A sequence $(x_1, \ldots, x_k)$ of positive integers is called *homogeneous* if the following conditions are satisfied.

- (a) There exists a positive integer x such that $x_i \in \{x, x+1\}$ for $i = 1, \ldots, k$.
- (b_1) If $x_i = x$ and $x_{i+1} = x+1$ and if there exists a positive integer t such that $x_{i-t} \neq x_{i+1+t}$, then for the smallest such t we have $x_{i-t} = x+1$ and $x_{i+1+t} = x$.
- (b_2) If there is no such t, then $i \leq k/2$.
- (c_1) If $x_i = x+1$ and $x_{i+1} = x$ and if there exists a positive integer t such that $x_{i-t} \neq x_{i+1+t}$, then for the smallest such t we have $x_{i-t} = x$ and $x_{i+1+t} = x+1$.
- (c_2) If there is no such t, then $i \geq k/2$.

Remark 2.7. The condition (c) above is equivalent with the requirement that the sequence $(x_k, \ldots, x_1)$ satisfies (b). A sequence $(x_1, \ldots, x_k)$ is therefore homogeneous if and only if $(x_k, \ldots, x_1)$ is homogeneous.

The following remarkable result [4, pp. 33 and 44] plays a key role in the proof of Theorem 1.3 and occupies a central position also in this paper.

Theorem 2.8 (Mrose). *Let $k \geq 1$ and $s \geq 2k$ be given positive integers. In case $\gcd(k, s+1) = 1$ there is exactly one homogeneous sequence $(x_1, \ldots, x_k)$ and extremal continuant $C(x_1, \ldots, x_k)$ with $\sum_{i=1}^k x_i = s$. In case $\gcd(k, s+1) > 1$ there are exactly two homogeneous sequences $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ and extremal continuants $C(x_1, \ldots, x_k)$ and $C(y_1, \ldots, y_k)$ with $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i = s$. Let $x = \lfloor s/k \rfloor$, $m = s - \lfloor s/k \rfloor k$, and $n = k - m = k\lceil (s+1)/k \rceil - s$. The sequences $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$*

are given by

$$(2.3)\quad x_i = \begin{cases} \left\lfloor \frac{i(s+1)}{k} \right\rfloor - \left\lfloor \frac{(i-1)(s+1)}{k} \right\rfloor & \textit{for } i = 1, \ldots, k-1, \\ s - \left\lfloor \frac{(k-1)(s+1)}{k} \right\rfloor & \textit{for } i = k, \end{cases}$$

$$= \begin{cases} x & \textit{for } i = \left\lceil \frac{(r-1)k+1}{n-1} \right\rceil \ (r = 1, \ldots, n-1) \textit{ and } i = k, \\ x+1 & \textit{for } i = \left\lceil \frac{kr}{m+1} \right\rceil \ (r = 1, \ldots, m), \end{cases}$$

$$(2.4)\quad y_i = \begin{cases} \left\lceil \frac{s+1}{k} \right\rceil - 1 & \textit{for } i = 1, \\ \left\lceil \frac{i(s+1)}{k} \right\rceil - \left\lceil \frac{(i-1)(s+1)}{k} \right\rceil & \textit{for } i = 2, \ldots, k, \end{cases}$$

$$= \begin{cases} x & \textit{for } i = 1 \textit{ and } i = \left\lceil \frac{k(r-1)}{n-1} \right\rceil \ (r = 2, \ldots, n), \\ x+1 & \textit{for } i = \left\lceil \frac{kr+1}{m+1} \right\rceil \ (r = 1, \ldots, m). \end{cases}$$

The sequences $(x_1, \ldots, x_k)$ *and* $(y_1, \ldots, y_k)$ *satisfy the conditions*

$$(2.5)\qquad x_{k+1-i} = y_i \quad \textit{for } i = 1, \ldots, k,$$

$$(2.6)\qquad (x_1, \ldots, x_k) = (y_1, \ldots, y_k) \quad \textit{if and only if} \quad \gcd(k, s+1) = 1.$$

Using Theorem 1.2 and other results of Hofmeister [1], Mrose [4, p. 68] proved

THEOREM 2.9 (Mrose). *If* $h \ge 1$ *and* $k \ge 1$ *are given positive integers, then* $A_k = \{a_1, \ldots, a_k\}$ *is an extremal regular* h*-basis* $\widetilde{A}_k^*$ *if and only if there exists a sequence* $(x_1, \ldots, x_k)$ *of integers* ≥ 2 *having the following properties.*

(a)

$$(2.7)\qquad a_i = C(x_1, \ldots, x_{i-1}) \quad \textit{for } i = 1, \ldots, k,$$

(b)

$$(2.8)\qquad \sum_{i=1}^{k} x_i = h + 2k - 1,$$

(c) *the continuant* $C(x_1, \ldots, x_k)$ *is extremal.*

If these conditions are satisfied, then

$$(2.9)\qquad g_h(k) = g_h(\widetilde{A}_k^*) = C(x_1, \ldots, x_k) - 1.$$

Remark 2.10. Given a basis $A_k = \{a_1, \ldots, a_k\}$, Mrose [4, p. 49] defined recursively

$$m_j = \left\lfloor \frac{a_{j+1} - \sum_{i=2}^{j-1} m_i a_i}{a_j} \right\rfloor \quad \text{for } j = 1, \ldots, k-1. \tag{2.10}$$

Comparing (2.10) with (1.4), we easily deduce that $\mu_1 = m_1 - 2$, $\mu_2 = m_2 - 1$, and $\mu_i = m_i$ for $i = 3, \ldots, k-1$. From this it follows [4, p. 59] that for the sequence $(x_1, \ldots, x_k)$ of Theorem 2.9 we have

$$x_i = \mu_i + 2 \quad \text{for } i = 1, \ldots, k-1, \tag{2.11}$$

$$x_k = \mu_k + 1. \tag{2.12}$$

In preparation of our reformulation of Conjecture 1.4, we now briefly sketch the proof of Theorem 1.3. If we use Theorem 2.8 with $s = h + 2k - 1$ and associate, by (2.11) and (2.12), the $\mu_i^{(1)}$ with the x_i and the $\mu_i^{(2)}$ with the y_i, we easily get (1.10) and (1.11). Using (2.11) and (2.1), (1.12) follows from (2.7). Finally, (1.3) follows easily by induction from (2.9), using (2.11) and (2.12). We have thus seen how Theorem 1.3 follows from Theorems 2.8 and 2.9.

We now turn our attention to Conjecture 1.4. We find it convenient to introduce the following definitions (not to be found in [4]).

Definition 2.11. We call the sequences $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ of Theorem 2.8 *Mrose's first and second sequence* (*for* s), respectively.

Definition 2.12. We call a continuant $C(x_1, \ldots, x_{k-1}, 2)$ with x_i integers ≥ 2 a 2-*extremal continuant* if for every continuant $C(y_1, \ldots, y_{k-1}, 2)$ with y_i integers ≥ 2 and $\sum_{i=1}^{k-1} x_i = \sum_{i=1}^{k-1} y_i$ we have

$$C(y_1, \ldots, y_{k-1}, 2) \leq C(x_1, \ldots, x_{k-1}, 2).$$

It follows from Theorem 2.9, (1.7), and (2.12) that the problem behind Conjecture 1.4 can be reformulated in the following way.

Problem 2.13. If $h \geq k \geq 3$, find 2-extremal continuants $C(x_1, \ldots, x_{k-1}, 2)$ such that $\sum_{i=1}^{k-1} x_i = h + 2k - 3$ (and then use (2.7) and (2.9)).

Comparing (1.13) with (1.10), and taking the above sketch of the proof of Theorem 1.3 into consideration, we see that Conjecture 1.4 is equivalent with the following result.

Theorem 2.14. *Let* $h \geq k \geq 3$. *There is exactly one* 2-*extremal continuant* $C(x_1, \ldots, x_{k-1}, 2)$ *with* $\sum_{i=1}^{k-1} x_i = h + 2k - 3$, *and it satisfies the condition that the sequence* $(x_1, \ldots, x_{k-2}, x_{k-1} - 1)$ *is Mrose's first sequence* (2.3) (*for* $s = h + 2k - 4$).

Theorem 2.14, that is, Selmer's Conjecture 1.4, will be proved in two steps, as Theorems 2.15 and 2.16 below.

THEOREM 2.15. *Let $h \geq k \geq 3$. If $C(x_1, \ldots, x_{k-1}, 2)$ is a 2-extremal continuant with $\sum_{i=1}^{k-1} x_i = h + 2k - 3$, then the sequence $(x_1, \ldots, x_{k-2}, x_{k-1} - 1)$ is homogeneous.*

This first step leaves us, according to Theorem 2.8, at most two possibilities for a 2-extremal continuant of the desired kind. The second step is then

THEOREM 2.16. *Let $k \geq 3$. If $(x_1, \ldots, x_{k-1})$ and $(y_1, \ldots, y_{k-1})$ are Mrose's first and second sequence, respectively (for $s \geq 2(k-1)$), then, if $(x_1, \ldots, x_{k-1}) \neq (y_1, \ldots, y_{k-1})$, we have*

$$C(x_1, \ldots, x_{k-2}, x_{k-1} + 1, 2) > C(y_1, \ldots, y_{k-2}, y_{k-1} + 1, 2).$$

EXAMPLE 2.17. Let $h = 15$, $k = 9$. Calculating the coefficients μ_i of Conjecture 1.4 we get

$$\mu_1 = 1,\ \mu_2 = 2,\ \mu_3 = 2,\ \mu_4 = 2,\ \mu_5 = 1,\ \mu_6 = 2,\ \mu_7 = 2,\ \mu_8 = 2,\ \mu_9 = 1.$$

Using (1.10) and (1.3) we get the regular h-basis

$$A_k = \{1, 3, 11, 41, 153, 418, 1519, 5658, 21113\} \tag{2.13}$$

and the regular h-range

$$g_h(A_k) = 36567. \tag{2.14}$$

Now $s = h + 2k - 4 = 29$ and denoting by $(x_1, \ldots, x_8)$ and $(y_1, \ldots, y_8)$ Mrose's first and second sequence, respectively (for $s = 29$), we find

$$(x_1, \ldots, x_8) = (3, 4, 4, 4, 3, 4, 4, 3),$$
$$(y_1, \ldots, y_8) = (3, 4, 4, 3, 4, 4, 4, 3).$$

Using (2.7) with the sequence $(x_1, \ldots, x_7, x_8 + 1, 2) = (3, 4, 4, 4, 3, 4, 4, 4, 2)$ we get (2.13) and using (2.9) we get (2.14), since $C(3, 4, 4, 4, 3, 4, 4, 4, 2) = 36568$. Note that

$$C(y_1, \ldots, y_7, y_8 + 1, 2) = C(3, 4, 4, 3, 4, 4, 4, 4, 2) = 36567,$$

illustrating Theorem 2.16.

3. First step: proof of Theorem 2.15. The first step is taken on a well-trodden path, since we can take as a model Mrose's Satz 3.1 [4, p. 19] (which says that an extremal continuant is homogeneous).

Remark 3.1. Leaving now [7] and staying with [4] for the rest of this paper, we find it convenient to make a slight *change in the notation*: the letter k, which from now on will denote an integer ≥ 2, will lose its former meaning.

Theorem 2.15 will be proved in the following form. Let

$$h \geq k + 1 \tag{3.1}$$

and let $C(x_1, \dots, x_k, 2)$ be a 2-extremal continuant with

$$\sum_{i=1}^{k} x_i = h + 2k - 1. \tag{3.2}$$

Then

$$(x_1, \dots, x_{k-1}, x_k - 1) \text{ is homogeneous.} \tag{3.3}$$

The proof of (3.3) will be preceded by a series of lemmas. Before starting with the proofs, we introduce the following

DEFINITION 3.2. If a sequence $(x_1, \dots, x_k)$ satisfies the conditions (a), (b_1), and (c_1) of Definition 2.6, we say that it is *weakly homogeneous*.

LEMMA 3.3. *Suppose that* $C(x_1, \dots, x_k, 2)$ *is a* 2*-extremal continuant. Then the sequence* $(x_1, \dots, x_k)$ *is weakly homogeneous.*

Proof. The proof is similar to the proof of Satz 3.1 in [4, p. 19] (see Remark 3.4 below). ∎

Remark 3.4. Mrose was able to make certain short cuts, using the symmetry of the situation (our (2.2) and Remark 2.7). Note that while the analogue of Remark 2.7 clearly holds for weakly homogeneous sequences, we do not have the full symmetry of the situation here, caused by the last element 2 of the continuant. We therefore define, for $1 \le p < q \le k$,

$$y_i = \begin{cases} x_i & \text{if } i \ne p, q, \\ x_p - 1 & \text{if } i = p, \\ x_q + 1 & \text{if } i = q. \end{cases} \tag{3.4}$$

Then we have, analogously to (9) in [4, p. 20],

$$\begin{aligned} &C(x_1, \dots, x_k, 2) - C(y_1, \dots, y_k, 2) \\ &\quad = (1 - x_p + x_q) C(x_1, \dots, x_{p-1}) C(x_{p+1}, \dots, x_{q-1}) C(x_{q+1}, \dots, x_k, 2) \\ &\qquad - C(x_1, \dots, x_{p-1}) C(x_{p+1}, \dots, x_{q-2}) C(x_{q+1}, \dots, x_k, 2) \\ &\qquad - C(x_1, \dots, x_{p-1}) C(x_{p+1}, \dots, x_{q-1}) C(x_{q+2}, \dots, x_k, 2) \\ &\qquad + C(x_1, \dots, x_{p-2}) C(x_{p+1}, \dots, x_{q-1}) C(x_{q+1}, \dots, x_k, 2) \\ &\qquad + C(x_1, \dots, x_{p-1}) C(x_{p+2}, \dots, x_{q-1}) C(x_{q+1}, \dots, x_k, 2). \end{aligned} \tag{3.5}$$

LEMMA 3.5. *If* $C(x_1, \dots, x_k, 2)$ *is a* 2*-extremal continuant satisfying* (3.1) *and* (3.2), *then* $x_i \ge 3$ *for* $i = 1, \dots, k$.

Proof. According to Lemma 3.3, we have, for some integer x, $x_i \in \{x, x+1\}$ for $i = 1, \dots, k$. If, for some $j \in \{1, \dots, k\}$, we have $x_j = 2$, then $x = 2$ and so $h + 2k - 1 = \sum_{i=1}^{k} x_i < 3k$, contradicting (3.1). ∎

LEMMA 3.6. *If $x_1, \ldots, x_j$ are any integers ≥ 2 and $x_j \geq 3$, then*

$$C(x_1, \ldots, x_j) > 2C(x_1, \ldots, x_{j-1}). \tag{3.6}$$

Proof. Let $x_j = 2 + t_j$, where $t_j \geq 1$. From (2.1) and Satz 1.7 [4, p. 7] we get

$$\begin{aligned} C(x_1, \ldots, x_j) &- 2C(x_1, \ldots, x_{j-1}) \\ &= t_j C(x_1, \ldots, x_{j-1}) - C(x_1, \ldots, x_{j-2}) \\ &\geq C(x_1, \ldots, x_{j-1}) - C(x_1, \ldots, x_{j-2}) > 0. \ \blacksquare \end{aligned}$$

LEMMA 3.7. *Let $C(x_1, \ldots, x_k, 2)$ be a 2-extremal continuant satisfying, for some integer $x \geq 3$, the condition $x_i \in \{x, x+1\}$ for $i = 1, \ldots, k$. If, for some $j \in \{1, \ldots, k\}$, we have $x_j = x+1$, then $x_k = x+1$.*

Proof. Suppose that $x_k = x$. Let p be the greatest number such that $x_p = x + 1$. We let $q = k$ and define a sequence $(y_1, \ldots, y_k)$ by (3.4). Now $1 - x_p + x_q = 0$ and so (3.5) gives

$$\begin{aligned} C(x_1, \ldots, x_k, 2) &- C(y_1, \ldots, y_k, 2) \\ &= - C(x_1, \ldots, x_{p-1})C(x_{p+1}, \ldots, x_{q-2})C(2) \\ &\quad - C(x_1, \ldots, x_{p-1})C(x_{p+1}, \ldots, x_{q-1}) \cdot 1 \\ &\quad + C(x_1, \ldots, x_{p-2})C(x_{p+1}, \ldots, x_{q-1})C(2) \\ &\quad + C(x_1, \ldots, x_{p-1})C(x_{p+2}, \ldots, x_{q-1})C(2). \end{aligned} \tag{3.7}$$

By the definition of p, if $p < i < q$, then $x_i = x$. It follows that $C(x_{p+1}, \ldots, x_{q-2}) = C(x_{p+2}, \ldots, x_{q-1})$, so that (3.7) and (3.6) give

$$\begin{aligned} C(x_1, \ldots, x_k, 2) &- C(y_1, \ldots, y_k, 2) \\ &= C(x_{p+1}, \ldots, x_{q-1})(2C(x_1, \ldots, x_{p-2}) - C(x_1, \ldots, x_{p-1})) < 0, \end{aligned}$$

which contradicts the fact that $C(x_1, \ldots, x_k, 2)$ is a 2-extremal continuant. ■

Proof of (3.3). We suppose that $C(x_1, \ldots, x_k, 2)$ is a 2-extremal continuant satisfying (3.1) and (3.2). We have to prove that the sequence $(x_1, \ldots, x_{k-1}, x_k - 1)$ satisfies the conditions (a), (b_1), (b_2), (c_1), and (c_2) of Definition 2.6.

(a) It follows immediately from Lemmas 3.3, 3.5, and 3.7 that $(x_1, \ldots, x_{k-1}, x_k - 1)$ satisfies (a).

If $x_i = x$ for $i = 1, \ldots, k$, then the sequence $(x_1, \ldots, x_{k-1}, x_k - 1)$ is clearly homogeneous. For the rest of the proof, we may therefore suppose that for an integer $x \geq 3$, we have $x_i \in \{x, x+1\}$ for $i = 1, \ldots, k$ and that

$$x_k - 1 = x. \tag{3.8}$$

Moreover, since the sequence $(x, x, \ldots, x)$ is clearly homogeneous, we may suppose that for some $j \in \{1, \ldots, k-1\}$, $x_j = x + 1$.

(b_1) We know from Lemma 3.3 that $(x_1,\ldots,x_k)$ satisfies (b_1). It follows immediately from (3.8) that also $(x_1,\ldots,x_{k-1},x_k-1)$ satisfies (b_1).

(b_2) Suppose that (b_2) fails. Then there exists an integer r, $k/2<r<k-1$, such that $x_r=x$, $x_{r+1}=x+1$, $x_{r-i}=x_{r+1+i}$ for $i=1,\ldots,k-r-2$, and

$$x_{2r-k+1}=x_k-1. \tag{3.9}$$

We define a sequence $(y_1,\ldots,y_k)$ by (8) in [4, p. 19] with $p=r$, $q=r+1$. Using (10) in [4, p. 23], (3.9), and (3.8), we obtain

$$\begin{aligned}
&C(x_1,\ldots,x_k,2)-C(y_1,\ldots,y_k,2)\\
&\qquad=2C(x_1,\ldots,x_{2r-k+1})-C(x_1,\ldots,x_{2r-k})C(x_k,2)\\
&\qquad=-2C(x_1,\ldots,x_{2r-k-1})-C(x_1,\ldots,x_{2r-k})<0,
\end{aligned}$$

a contradiction. This proves (b_2).

(c_1) and (c_2) can be proved in a similar manner, using (3.4). ∎

We have proved Theorem 2.15.

4. Second step: proof of Theorem 2.16. In this section we take a closer look at Mrose's sequences (2.3) and (2.4). First of all, we have to consider the following question: *If Mrose's two sequences are different, how do they differ from each other?*

Let $(x_1,\ldots,x_k)$ and $(y_1,\ldots,y_k)$ be Mrose's first and second sequence, respectively (for $s\ge 2k$), and let $x=\lfloor s/k\rfloor$. Theorem 2.8 implies

LEMMA 4.1 (Mrose). (a) *If* $s\equiv -1 \pmod k$, *we have*

$$\begin{aligned}(x_1,\ldots,x_k)&=(x+1,x+1,\ldots,x+1,x),\\ (y_1,\ldots,y_k)&=(x,x+1,x+1,\ldots,x+1).\end{aligned} \tag{4.1}$$

(b) *If* $s\not\equiv -1 \pmod k$, *we have*

$$x_1=y_1=x_k=y_k=x. \tag{4.2}$$

LEMMA 4.2. *We have*

$$\begin{aligned}
&C(x_1,\ldots,x_{k-1},x_k+1,2)-C(y_1,\ldots,y_{k-1},y_k+1,2)\\
&\qquad=C(x_1,\ldots,x_{k-1},x_k+1)-C(y_1,\ldots,y_{k-1},y_k+1)\\
&\qquad=C(x_1,\ldots,x_{k-1})-C(y_1,\ldots,y_{k-1}).
\end{aligned}$$

Proof. This can easily be proved using (2.1), (2.2), (2.5), and Lemma 4.1. ∎

Using Lemma 4.2 (and keeping Remark 3.1 in mind), we prove Theorem 2.16 in the following form. If $(x_1,\ldots,x_k)$ and $(y_1,\ldots,y_k)$ are Mrose's first and second sequence, respectively (for $s \geq 2k$), then

$$C(x_1, \ldots, x_{k-1}) > C(y_1, \ldots, y_{k-1}). \tag{4.3}$$

If $s \equiv -1 \pmod{k}$, then we see immediately from Lemma 4.1 that (4.3) holds. (Alternatively, we could use Lemma 3.3 and note that as the sequence $(y_1, \ldots, y_{k-1}, y_k+1)$ does not satisfy the condition (a) of Definition 2.6, then $C(y_1, \ldots, y_{k-1}, y_k + 1, 2)$ is not a 2-extremal continuant.) From now on, we will therefore suppose that

$$s \not\equiv -1 \pmod{k}. \tag{4.4}$$

The following result provides a fundamental tool for our investigations.

Lemma 4.3. *Let $s \geq 2k$, assume* (4.4), *and let $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ be Mrose's first and second sequence*, *respectively. Let* $\gcd(k, s+1) = d$ *and* $x = \lfloor s/k \rfloor$. *Consider the box*

$$\boxed{\begin{array}{l} x_1, \ldots, x_k \\ y_1, \ldots, y_k \end{array}} \tag{4.5}$$

This box has the following form.

(a) *At both ends of the box* (4.5) *there are subboxes of the form*

$$\boxed{\begin{array}{c} x \\ x \end{array}}$$

(b) *There are $d - 1$ subboxes*

$$\boxed{\begin{array}{cc} x+1 & x \\ x & x+1 \end{array}} \tag{4.6}$$

where the corresponding elements of $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ differ from each other.

(c) *The boxes in* (a) *and* (b) *are separated by* (*possibly empty*) *identical*, *symmetrical subboxes with the same corresponding elements.*

Example 4.4. We take $k = 21$, $s = 74$ (see [4, p. 71]) to illustrate the situation. In this case $d = \gcd(21, 75) = 3$ and (4.5) is

3	4 3 4 3 4	4 3	4 3 4 3 4	4 3	4 3 4 3 4	3
3	4 3 4 3 4	3 4	4 3 4 3 4	3 4	4 3 4 3 4	3

The following proof was suggested by Veikko Ennola. It has the merit of being simpler and shorter than the author's original proof, which proceeded by induction on d.

Proof of Lemma 4.3. We note that (a) follows immediately from (4.2). To prove (b) and (c), we use (2.3) and (2.4) in connection with the obvious formula

$$\lceil \xi \rceil - \lfloor \xi \rfloor = \begin{cases} 1 & \text{if } \xi \notin \mathbb{Z}, \\ 0 & \text{if } \xi \in \mathbb{Z}. \end{cases} \tag{4.7}$$

We write $s+1 = da$ and $k = db$, so that $\gcd(a,b) = 1$ and, by (4.4), $b > 1$. If $1 < i < k$, then we have

$$x_i - y_i = \left\lfloor \frac{ia}{b} \right\rfloor - \left\lceil \frac{ia}{b} \right\rceil - \left\lfloor \frac{(i-1)a}{b} \right\rfloor + \left\lceil \frac{(i-1)a}{b} \right\rceil,$$

so that, by (4.7),

$$x_i - y_i = \begin{cases} 0 & \text{if } i \not\equiv 0, 1 \pmod{b}, \\ 1 & \text{if } i \equiv 0 \pmod{b}, \\ -1 & \text{if } i \equiv 1 \pmod{b}. \end{cases} \tag{4.8}$$

In a similar fashion,

$$x_i - x_{i+b} = \left\lfloor \frac{ia}{b} \right\rfloor - \left\lfloor \frac{(i-1)a}{b} \right\rfloor - \left\lfloor \frac{(i+b)a}{b} \right\rfloor + \left\lfloor \frac{(i+b-1)a}{b} \right\rfloor \tag{4.9}$$
$$= 0 \quad \text{if } i \not\equiv 0, 1 \pmod{b},$$

$$x_i - x_{b+1-i} = \left\lfloor \frac{ia}{b} \right\rfloor - \left\lceil \frac{ia}{b} \right\rceil + \left\lceil \frac{(i-1)a}{b} \right\rceil - \left\lfloor \frac{(i-1)a}{b} \right\rfloor \tag{4.10}$$
$$= 0 \quad \text{if } i \not\equiv 0, 1 \pmod{b}.$$

(b) and (c) now follow immediately from (4.8)–(4.10). ■

We introduce the following notation.

DEFINITION 4.5. If $(x_1, \ldots, x_j)$ and $(y_1, \ldots, y_j)$ are any sequences with x_i, y_i integers ≥ 2 for $i = 1, \ldots, j$, we write

$$C(x_1, \ldots, x_j) = u_j, \qquad C(y_1, \ldots, y_j) = v_j, \qquad d_j = u_j - v_j. \tag{4.11}$$

According to this notation, if $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are Mrose's first and second sequence, respectively, then (4.3) can be stated as

$$d_{k-1} > 0. \tag{4.12}$$

We find it convenient to prove a little more than (4.12). In fact, if $t \geq 0$ is the length of the symmetrical subbox in Lemma 4.3, and if $(x_1, \ldots, x_k) \neq (y_1, \ldots, y_k)$, we show that

$$d_1 = \ldots = d_{t+1} = 0, \tag{4.13}$$

$$d_{t+2} > 0, \ \ldots, \ d_{k-1} > 0, \qquad \text{and} \tag{4.14}$$

$$d_k = 0. \tag{4.15}$$

Here (4.13) is clear from Lemma 4.3, and (4.15) follows from (2.5) and (2.2).

R e m a r k 4.6. If $s \equiv -1 \pmod{k}$, then it follows immediately from (4.1) that

$$d_1 > 0, \ \ldots, \ d_{k-1} > 0, \qquad d_k = 0. \tag{4.16}$$

Our goal now is to prove (4.14). The proof is based on Lemma 4.3.

R e m a r k 4.7. Actually, we get the result for a little wider class of pairs of sequences than just for Mrose's sequences, since from the elements of the symmetrical subboxes we need, indeed, only the fact that they form a symmetrical sequence. Their actual values do not concern us. In particular, they do not have to be in the set $\{x, x+1\}$.

LEMMA 4.8. *Let*

$$(x_1, \ldots, x_k) = (z^*, z_1, \ldots, z_t, z+1, z, z_t, \ldots, z_1),$$
$$(y_1, \ldots, y_k) = (z^*, z_1, \ldots, z_t, z, z+1, z_t, \ldots, z_1),$$

with $t \geq 0$, z^*, z, z_i *integers* ≥ 2 *for* $i = 1, \ldots, t$. *Then*

$$d_i = 0 \qquad \textit{for } i = 1, \ldots, t+1, \qquad \textit{and} \tag{4.17}$$

$$d_{k-i} = u_i \qquad \textit{for } i = 0, \ldots, t+1. \tag{4.18}$$

P r o o f. (4.17) is trivial. Note that, by Definitions 2.1 and 4.5, $u_0 = 1$. It follows immediately from (2.1) that

$$d_i = x_i u_{i-1} - y_i v_{i-1} - d_{i-2} \qquad \text{for } i = 1, \ldots, k \tag{4.19}$$

(with $d_0 = d_{-1} = 0$). Using (4.19) and (4.17) we easily obtain

$$d_{t+2} = u_{t+1} \qquad \text{and} \qquad d_{t+3} = u_t. \tag{4.20}$$

We now prove (4.18) in the form

$$d_{t+2+i} = u_{t+1-i} \qquad \text{for } i = 0, \ldots, t+1, \tag{4.21}$$

using induction on i.

1. If $i = 0$ or $i = 1$, (4.21) follows from (4.20).
2. The general induction step (for $t > 0$) follows easily from the obvious formulas

$$u_i = z_{i-1} u_{i-1} - u_{i-2} \qquad \text{for } i = 2, \ldots, t+1,$$
$$d_{t+2+i} = z_{t-i+2} d_{t+i+1} - d_{t+i} \qquad \text{for } i = 2, \ldots, t+1. \ \blacksquare$$

LEMMA 4.9. *Let* x, x_i *be integers* ≥ 2 *for* $i = 1, \ldots, n$. *Let*

$$b_r = C(x_1, \ldots, x_n, x, x_1, \ldots, x_n, x, \ldots, x, x_1, \ldots, x_n),$$

where r *is the number of occurrences of the symbol* x *in the continuant. Let* $a = C(x_1, \ldots, x_n)$. *Then* $a \mid b_r$ *for* $r = 0, 1, \ldots$

P r o o f. Easy by induction on r using Satz 1.2 (2) in [4, p. 3]. $\blacksquare$

LEMMA 4.10. *Let*

$$(x_1, \ldots, x_k) = (x_1, \ldots, x_n, x+1, x),$$
$$(y_1, \ldots, y_k) = (y_1, \ldots, y_n, x, x+1),$$

where x, x_i, y_i *are integers* ≥ 2 *for* $i = 1, \ldots, n$. *If* $d_{n-1} = x d_n$, *then we have*

$$d_{k-1} = u_n, \tag{4.22}$$
$$d_k = u_{n-1} - d_n. \tag{4.23}$$

Proof. Easy calculation. ∎

Our goal, (4.14), will finally be reached by the next result (see Remark 4.7).

LEMMA 4.11. *Let* $t \geq 0$, *let* x, z_i *be integers* ≥ 2 *for* $i = 1, \ldots, t$, *and let* $(z_1, \ldots, z_t)$ *be a symmetrical sequence. Let*

$$(x_1, \ldots, x_k) = (x, z_1, \ldots, z_t, x+1, x, z_1, \ldots, z_t, x+1, \\ x, \ldots, x+1, x, z_1, \ldots, z_t, x),$$
$$(y_1, \ldots, y_k) = (x, z_1, \ldots, z_t, x, x+1, z_1, \ldots, z_t, x, \\ x+1, \ldots, x, x+1, z_1, \ldots, z_t, x)$$

be two sequences such that their box (4.5) *contains* g *subboxes* (4.6). (*If* $(x_1, \ldots, x_k)$ *and* $(y_1, \ldots, y_k)$ *are Mrose's first and second sequence, respectively* (*for* $s \geq 2k$, $s \not\equiv -1 \pmod{k}$), *then, by Lemma* 4.3, $g = d - 1$, *where* $d = \gcd(k, s+1)$.) *Then* $d_1 = \ldots = d_{t+1} = 0$, $d_k = d_{(g+1)(t+2)} = 0$, *and*

$$d_{j(t+2)+i} = r_j u_{t+1-i} \quad \textit{for } j = 1, \ldots, g, \ i = 0, \ldots, t+1, \tag{4.24}$$

where

$$r_j = \frac{u_{j(t+2)-1}}{u_{t+1}} \ \textit{are integers, and} \tag{4.25}$$

$$1 = r_1 < \ldots < r_g. \tag{4.26}$$

Proof. Again, the claim about the zero values of the d_i's is trivial. We note first that (4.25) follows from Lemma 4.9 and (4.26) follows from Satz 1.7 [4, p. 7]. We prove (4.24) by induction on g.

1. If $g = 1$, then (4.24) follows from Lemma 4.8.

2. Suppose now that $g > 1$. We may assume that (4.24) holds for $j = 1, \ldots, g-1$, $i = 0, \ldots, t+1$. Since $u_0 = 1$ and $u_1 = x$, it follows from our induction hypothesis, and formulas (4.22) and (4.23), that

$$d_{g(t+2)} = u_{g(t+2)-1}, \tag{4.27}$$
$$d_{g(t+2)+1} = u_{g(t+2)-2} - d_{g(t+2)-1}. \tag{4.28}$$

We have to prove that

(4.29) $$d_{g(t+2)+i} = r_g u_{t+1-i} \quad \text{for } i = 0, \dots, t+1.$$

We prove (4.29) by induction on i.

(A) If $i = 0$, then (4.29) follows immediately from (4.27) and the definition of r_g.

If $i = 1$, we use (4.28) and the induction hypothesis for g, and get

$$d_{g(t+2)+1} = u_{(g-1)(t+2)+t} - \frac{u_{(g-1)(t+2)-1}}{u_{t+1}}.$$

It follows that (4.29) holds for $i = 1$ if we have

(4.30) $$u_{t+1}u_{(g-1)(t+2)+t} - u_{(g-1)(t+2)-1} = u_{g(t+2)-1}u_t.$$

But if we apply Satz 1.3 [4, p. 3] to the sequence $(x_1, \dots, x_k)$, and use the symmetry of the subsequence $(z_1, \dots, z_t)$, we get (4.30) without difficulty (taking $m = n - 1 = k - t - 2$, $j = t + 1$ in Satz 1.3). So (4.29) holds for $i = 1$.

(B) If $t > 0$ we continue easily by induction on i in a similar manner to the proof of Lemma 4.8.

This completes the proof of (4.24) by induction on g. ∎

Our goal, (4.14), now follows immediately from (4.24) and (4.26) since, by Satz 1.7 [4, p. 7], the numbers u_j are positive. (4.14) implies (4.12), which, in turn, implies (4.3). Theorem 2.16 is now proved. As Theorem 2.15 was proved in the previous section, we have proved Theorem 2.14. This means that we have proved Selmer's Conjecture.

Remark 4.12. (4.13), (4.14), and (4.16) give also information, using (2.7), about the size relations of the corresponding basis elements of the two extremal regular h-bases in Theorem 1.3 (in case the bases are different).

Remark 4.13. Keeping in mind our question from the beginning of this section, we note that if $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ are Mrose's first and second sequence, respectively (for $s \geq 2k$), and if $(x_1, \dots, x_k) \neq (y_1, \dots, y_k)$, then Remark 2.4, (2.5), (2.2), and (4.3) imply that

$$\langle x_1, \dots, x_k \rangle > \langle y_1, \dots, y_k \rangle.$$

We conclude this section by giving an example of Lemma 4.11 (and Remark 4.12).

Example 4.14. We use the sequences of our earlier Example 4.4 (see [4, p. 71]). In this case we have $g = 2$, $t = 5$, $r_1 = 1$, $r_2 = \frac{u_{13}}{u_6} = \frac{3909217}{1079} = 3623$. We give below a table of values of u_i, v_i, and d_i for $i = 1, \dots, 21$.

Table

i	u_i	v_i	d_i
1	3	3	0
2	11	11	0
3	30	30	0
4	109	109	0
5	297	297	0
6	1079	1079	0
7	4019	2940	1079
8	10978	10681	297
9	39893	39784	109
10	108701	108671	30
11	394911	394900	11
12	1076032	1076029	3
13	3909217	3909216	1
14	14560836	10651619	3909217
15	39773291	38697260	1076031
16	144532328	144137421	394907
17	393823693	393715003	108690
18	1430762444	1430722591	39853
19	3898463639	3898452770	10869
20	14163092112	14163088489	3623
21	38590812697	38590812697	0

5. Asymptotic estimates. As suggested by Gerd Hofmeister, we round off the discussion by giving some asymptotic estimates of the number $g_{\tilde{h}_0=h}(k)$, defined in Section 1 (just before Conjecture 1.4). We start by presenting some results of Hofmeister [1], [2] on the extremal regular h-range $g_h(k)$, from which the results on $g_{\tilde{h}_0=h}(k)$ then easily follow.

THEOREM 5.1 (Hofmeister). *We have*

(a) $$g_h(k) \sim \left(\frac{k}{h}\right)^h \quad \textit{for fixed } h \textit{ and } k \to \infty,$$

(b) $$g_h(k) \sim \left(\frac{h}{k}\right)^k \quad \textit{for fixed } k \textit{ and } h \to \infty,$$

(c) $$g_k(k) \sim \frac{\tau}{\sqrt{5}}(\tau^2)^k \quad \textit{for } k \to \infty, \textit{ where } \tau = \frac{1+\sqrt{5}}{2}.$$

P r o o f. (a) and (b) follow from formulas (37b) and (37a) in [1, p. 56], respectively.

(c) Using, in the case $h = k$, the h-basis coming from (1.10), together with results in [2, p. 65], we get

$$g_k(k) = \left\lfloor \frac{\tau}{\sqrt{5}} (\tau^2)^k \right\rfloor . \quad \blacksquare$$

COROLLARY 5.2. *We have*

(a) $$g_{\tilde{h}_0=h}(k) \sim \left(\frac{k}{h}\right)^h \quad \textit{for fixed } h \textit{ and } k \to \infty,$$

(b) $$g_{\tilde{h}_0=h}(k) \sim 2\left(\frac{h}{k-1}\right)^{k-1} \quad \textit{for fixed } k \textit{ and } h \to \infty,$$

(c) $$g_{\tilde{h}_0=k}(k) \sim \frac{\tau}{\sqrt{5}} (\tau^2)^k \quad \textit{for } k \to \infty.$$

Proof. (a) If $h < k$, then, by Remark 1.5, $g_{\tilde{h}_0=h}(k) = g_h(k)$.

(b) Suppose that $h > k$. Let $A_k = \{a_1, \ldots, a_k\}$ be the h-basis associated with the coefficients $\mu_i^{(1)}$ of (1.10). We show that

$$g_{\tilde{h}_0=h+1}(k+1) = 2g_h(k) + a_k + 1. \tag{5.1}$$

Let $B_{k+1} = \{b_1, \ldots, b_{k+1}\}$ be the basis for which $g_{h+1}(B_{k+1}) = g_{\tilde{h}_0=h+1}(k+1)$. This means that we replace h by $h+1$ and k by $k+1$ in (1.13). Comparing (1.10) and (1.13) we then immediately obtain $\mu_i^{(1)} = \mu_i$ for $i = 1, \ldots, k$ and $\mu_{k+1} = 1$, from which it follows that $a_i = b_i$ for $i = 1, \ldots, k$ and

$$g_{\tilde{h}_0=h+1}(k+1) = g_h(k) + b_{k+1}. \tag{5.2}$$

We have

$$b_{k+1} = (\mu_k + 2)b_k - b_{k-1} = (\mu_k^{(1)} + 2)a_k - a_{k-1}$$

and therefore, using (2.9), (2.7), (2.11), and (2.12),

$$g_h(k) = (\mu_k^{(1)} + 1)a_k - a_{k-1} - 1 = b_{k+1} - a_k - 1,$$

which, together with (5.2), implies (5.1).

From (1.12) it follows that $a_{i+1} \le (\mu_i^{(1)} + 2)a_i$ for $i = 1, \ldots, k-1$, and therefore $a_k \le \prod_{i=1}^{k-1} (\mu_i^{(1)} + 2)$. It follows from this (see (2.10), (2.3), and (2.8)) that

$$a_k \le \left(\frac{h + 3k - 1}{k}\right)^{k-1}. \tag{5.3}$$

Using (5.1), (5.3), and Theorem 5.1(b), we complete the proof.

(c) We suppose, finally, that $h = k$. Using Remark 1.5 we see that there is nothing to prove, since $g_{\tilde{h}_0=k}(k) = g_k(k)$. However, we would like to offer a simple alternative proof. Using Remark 1.5 and (1.3) we immediately obtain (see formula (35) in [8, p. 178])

$$g_{\tilde{h}_0=k}(k) = F_{2k+1} - 1.$$

The desired asymptotic estimate then follows from (see formula (62) in [8, p. 180])

$$F_n = \left\lfloor \frac{\tau^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor \quad \text{for } n = 1, 2, \ldots \ \blacksquare$$

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