An application of the Hooley–Huxley contour

by

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To the memory of Professor Helmut Hasse (1898–1979)

1. Introduction and statement of results. This paper is a continuation of our paper [1]. We begin by stating a special case of what we prove in the present paper.

**Theorem 1.** Let \( k \) be any complex constant and \( (\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n)n^{-s} \) in \( \sigma \geq 2 \). Then

\[
\begin{align*}
&\int_1^T |(\zeta(1+it))^k|^2 \, dt = T \sum_{n=1}^{\infty} |d_k(n)|^2 n^{-2} + O((\log T)^{|k|^2}), \\
&\int_1^T \left| \frac{\zeta'(1+it)}{\zeta(1+it)} \right|^2 \, dt = T \sum_{m \geq 1} \sum_{p} (\log p)^2 p^{-2m} + O((\log T)^2),
\end{align*}
\]

and

\[
\int_1^T |\log \zeta(1+it)|^2 \, dt = T \sum_{m \geq 1} \sum_{p} (mp^m)^{-2} + O(\log \log T).
\]

**Remark 1.** In [1] we proved (1) with \( k = 1 \) and studied the error term in great detail.

**Remark 2.** The proof of this theorem and Theorem 3 to follow require the use of the Hooley–Huxley contour as modified by K. Ramachandra in [2] (for some explanations see [3]). We write \( m(HH) \) for this contour.

**Remark 3.** We have an analogue of these results for \( \zeta \) and \( L \)-functions of algebraic number fields. In fact, under somewhat general conditions on
$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ (or even $\sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ and so on) we can show that

$$\int_{1}^{T} |F(1 + it)|^2 \, dt = T \sum_{n=1}^{\infty} |a_n|^2 n^{-2} + O\left( \log \log T + \sum_{n \leq T} |a_n|^2 n^{-1} \right)$$

where $C (> 0)$ is a large constant.

The following theorem is fairly simple to prove.

**Theorem 2.** Let $1 = \lambda_1 < \lambda_2 < \ldots$ be a sequence of real numbers with $C_0^{-1} \leq \lambda_{n+1} - \lambda_n \leq C_0$ where $C_0 (\geq 1)$ is a constant and let $a_1, a_2, \ldots$ be any sequence of complex numbers satisfying the following conditions:

(i) $\sum_{n \leq x} |a_n| n^{-1} = O(x^\varepsilon)$ for all $\varepsilon > 0$ and $x \geq 1$.

(ii) $\sum_{n=1}^{\infty} |a_n|^2 n^{\lambda-2}$ converges for some constant $\lambda$ with $0 < \lambda < 1$.

(iii) $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ (which converges in $\sigma > 1$) is continuable analytically in $\sigma \geq 1 - \delta$, $t \geq t_0$ and there $|F(s)| < t^A$, where $\delta (0 < \delta < 1/10), t_0 (\geq 100)$ and $A (\geq 2)$ are any constants.

Then

$$\int_{t_0 + C_1 \log \log T}^{T} |F(1 + it)|^2 \, dt = T \sum_{n=1}^{\infty} |a_n|^2 \lambda_n^{-2} + O\left( \log \log T + \sum_{n \leq T} |a_n|^2 n^{-1} \right)$$

where $C_1$ and $C_2$ are certain positive constants depending on other constants which occur in the definition of $F(s)$.

We sketch a proof of this theorem. We put $s = 1 + it, \, t \geq t_0$,

$$R(w) = \exp \left( \sin \left( \frac{w}{100} \right) \right),$$

$$\Delta(u) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} u^w R(w) \frac{dw}{w} \quad (u > 0),$$

and

$$\sum_{n=1}^{\infty} a_n \lambda_n^{-s} \Delta \left( \frac{X}{\lambda_n} \right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s + w) X^w R(w) \frac{dw}{w} \quad (X = T^{C_3}),$$

$C_3 (> 0)$ being a large constant. In the integral just mentioned we cut off the portion $|\text{Im } w| \geq C_4 \log \log T$ where $C_4 (> 0)$ is a large constant and in the remaining part we move the line of integration to $\text{Re } w = -\delta$. Observe
that in $|\text{Re } w| \leq 3$ we have
\[
R(w) = O\left(\left(\exp\exp\left(\left|\text{Im } \frac{w}{100}\right|\right)\right)^{-1}\right).
\]
Without much difficulty we obtain
\[
F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \Delta\left(\frac{X}{\lambda_n}\right) + O(T^{-2}) = A(s) + E(s) \quad \text{say}.
\]
Using a well-known theorem of H. L. Montgomery and R. C. Vaughan we have
\[
\int_{t_0 + C_1 \log \log T}^{T} |A(1 + it)|^2 \, dt = \sum_{n=1}^{\infty} |a_n|^2 \lambda_n^{-2} \left|\Delta\left(\frac{X}{\lambda_n}\right)\right|^2 (T - C_1 \log \log T + O(n)).
\]
Now $\Delta(u) = O(u^2)$ always but it is also $1 + O(u^{-2})$ and using these we are led to the theorem.

However, the proof of Theorem 1 (and also that of Theorem 3) is not simple. It has to use the density results $N(\sigma, T) = O(T^{B(1-\sigma)(\log T)^B})$ and $N(\sigma, T) = O(T^{B'(1-\sigma)^{3/2}(\log T)^{B'}})$ (the former is a consequence of the latter if we are not particular to have a small value of $B$) where $B (> 0)$ and $B' (> 0)$ are constants and $1 - \delta \leq \sigma \leq 1$. Also it has to use the zero free region $\sigma \geq 1 - C_3(\log t)^{-2/3}(\log \log t)^{-1/3}$ for the Riemann zeta function (and more general functions). Since the constant $B$ is unimportant in our proof, Remark 3 below Theorem 1 holds. (In fact, as will be clear from our proof, only the portion $\sigma \geq 1 - \delta$ of the $m(\HH)$ contour will be enough for our purposes.) Also if only the density result $N(\sigma, T) = O(T^{B(1-\sigma)(\log T)^B})$ and the zero free region $\sigma \geq 1 - C_5(\log T)^{-1}$ are available then we end up with
\[
O\left(\log \log T + \sum_{n \leq \exp((\log T)^{3})} |a_n|^2 n^{-1}\right)
\]
for the error term and it is not hard to improve this to some extent. We now proceed to state our general result.

Consider the set $S_1$ of all abelian $L$-series of all algebraic number fields. We can define $\log L(s, \chi)$ in the half plane $\text{Re } s > 1$ by the series
\[
\sum_{m} \sum_{p} \chi(p^n)(mp^{ms})^{-1}
\]
where the sum is over all positive integers $m \geq 1$ and $p$ runs over all primes (in the case of algebraic number fields $p$ runs over the norm of all prime ideals). More generally, we can (by analytic continuation) define $\log L(s, \chi)$
in any simply connected domain containing \( \text{Re} s > 1 \) which does not contain any zero or pole of \( L(s, \chi) \). For any complex constant \( z \) we can define \( (L(s, \chi))^z \) as \( \exp(z \log L(s, \chi)) \). Let \( S_2 \) consist of the derivatives of \( L(s, \chi) \) for all \( L \)-series and let \( S_3 \) consist of the logarithms as defined above for all \( L \)-series.

Let \( P_1(s) \) be any finite power product (with complex exponents) of functions in \( S_1 \). Let \( P_2(s) \) be any finite power product (with non-negative integral exponents) of functions in \( S_2 \). Also let \( P_3(s) \) be any finite power product (with non-negative integral exponents) of functions in \( S_3 \). Let \( b_n (n = 1, 2, 3, \ldots) \) be complex numbers which are \( O(e^{\varepsilon \log n}) \) for every fixed \( \varepsilon > 0 \) and suppose that \( F_0(s) = \sum_{n=1}^{\infty} b_n n^{-s} \) is absolutely convergent in \( \text{Re} s \geq 1 - \delta \) where \( 0 < \delta < 1/10 \) is a positive constant.

Finally, put

\[
(12) \quad F(s) = P_1(s)P_2(s)P_3(s)F_0(s) = \sum_{n=1}^{\infty} a_n n^{-s}.
\]

Then we have

**Theorem 3.** We have

\[
(13) \quad \int_1^T |F(1 + it)|^2 \, dt = T \sum_{n=1}^{\infty} |a_n|^2 n^{-2} + O\left( \log \log T + \sum_{n \leq T^{C_6}} |a_n|^2 n^{-1} \right)
\]

where \( C_6 (> 0) \) is a large constant.

**Remark 1.** It is possible to have a more general result. For example we can replace \( F(s) \) in (12) and (13) by \( F(s) + \sum_{n=1}^{\infty} d_m(n)(n + \alpha)^{-s} \) where \( m \) is a positive integer constant and \( \alpha \) is any constant with \( 0 < \alpha < 1 \). Then the right hand side of (13) has to be replaced by

\[
T \sum_{n=1}^{\infty} |a_n|^2 n^{-2} + T \sum_{n=1}^{\infty} (d_m(n))^2 (n + \alpha)^{-2} + O(\log \log T)
\]

\[
+ O\left( \sum_{n \leq T^{C_6}} (|a_n|^2 + (d_m(n))^2)n^{-1} \right).
\]

**2. Proof of Theorem 3.** We form the \( m(HH) \) contour (associated with \( L \)-functions occurring in \( F(s) \)) as in [2]. But we select a small constant \( \delta (0 < \delta < 1) \) and treat the points \( 1 - \delta + iv (\nu = 0, \pm 1, \pm 2, \ldots) \) as though they were zeros associated with \( L \)-functions occurring in \( F(s) \). We recall
\[ R(w) = \exp((\sin(w/100))^2). \] Put \( s = 1 + it, \ T_0 = C_T \log \log T \leq t \leq T, \)

\[ A(s) = \sum_{n=1}^{\infty} a_n n^{-s} \Delta \left( \frac{X}{n} \right) \]

where \( \Delta(u) \) and \( X \) are as in (8). Then

\[ \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s + w) X^w R(w) \frac{dw}{w} = A(s). \]

We write \( w = u + iv \) and truncate the portion \( |v| \geq \frac{1}{2}T_0 \) and move the \( w \)-line of integration so that \( s + w \) lies in the portion of the \( m(HH) \) contour pertaining to \( |v| \leq \frac{1}{2}T_0 \). We obtain

\[ F(s) = A(s) + E(s) \]

where for fixed \( t \) in \( (T_0 \leq t \leq T) \),

\[ E(s) = -\frac{1}{2\pi i} \int_{P} F(s + w) X^w R(w) \frac{dw}{w} \]

where \( P \) is the path consisting of the \( m(HH) \) contour in \( (u \geq -\delta, \ |v| \leq \frac{1}{2}T_0) \) and the lines connecting it to \( \sigma = 1 \) by lines perpendicular to it at the ends. Notice that to the right of the \( m(HH) \) we have (by Lemma 5 of [2])

\[ |F(s + w)| \leq \exp((\log t)^{\psi}) \]

with a certain constant \( \psi \) (satisfying \( 0 < \psi < 1 \)) for \( s + w \) on \( M_{1,1} \) and \( M_{1,2} \) (we adopt the notation of [2]). Also

\[ |F(s + w)| \leq \exp((\log t)^{\psi'}) \]

with a small constant \( \psi' \) (\( 0 < \psi' < 1/5 \)) for \( s + w \) on \( M_{1,3} \). With these we have the following contributions to \( \int_{T_0/2}^{T + T_0/2} |E(s)| \, dt \) and \( \int_{T_0/2}^{T + T_0/2} |E(s)|^2 \, dt \).

We handle the first integral and the treatment of the second is similar. We have (denoting by \( P_1 \) the contour \( P \) with the horizontal lines connecting \( P \) to \( \sigma = 1 \) omitted)

\[ \int_{T_0}^{T} |E(s)| \, dt \leq (\log T)^2 \int_{T_0}^{T} |F(s + w)| X^w |dw| \, dt + T^{-10} \]

\[ \leq (\log T)^3 \int_{Q} |F(s)| X^{\sigma-1} |ds| + T^{-10} \]

where \( Q \) is the portion of the \( m(HH) \) in \( (\sigma \geq 1 - \delta, T_0/2 \leq t \leq T + T_0/2) \). (Note that \( s \) is used as a variable on the \( m(HH) \) in the integral in (20).)
(In the case of $\int_{T_0}^T |E(s)|^2 \, dt$ we majorise it by

$$\frac{(\log T)^4}{T_0} \left( \int_{P_1}^T \int |F(s + w)| X^u |dw| \right)^2 \, dt + T^{-10}$$

$$\leq \frac{(\log T)^5}{T_0} \int_{P_1}^T \int |F(s + w)|^2 X^{2u} |dw| \, dt + T^{-10}$$

by Hölder’s inequality.)

The contribution to (20) from $M_{1,1}$ is

$$O((\log T)^{20} \max_{1-\delta \leq \sigma \leq 1-\tau_1} (N(\sigma, T)X^{-(1-\sigma)}) \exp((\log T)^\psi))$$

and that from $M_{1,2}$ is

$$O((\log T)^{20} \max_{1-\tau_1 \leq \sigma \leq 1-\tau_2} (N(\sigma, T)X^{-(1-\sigma)}) \exp((\log T)^\psi'))$$

and that from $M_{1,3}$ is

$$O((\log T)^D \exp((\log T)^\psi') X^{-\tau_3})$$

where $\tau_1$ and $\tau_2$ are determined by $M_{1,1}$, $M_{1,2}$ and $M_{1,3}$ and $\tau_3 = C_3(\log T)^{-2/3}(\log \log T)^{-1/3}$. Here $D$ is some constant. (Note that $X$ is a large positive constant power of $T$.) Using the standard estimates (for some details which are very much similar to what we need, see equations (1)–(3) of [3]) we obtain

**Lemma 1.** Both $\int_{T_0}^T |E(s)| \, dt$ and $\int_{T_0}^T |E(s)|^2 \, dt$ are $O(\exp(-(\log T)^{0.1}))$.

**Lemma 2.** We have $A(s) = O(\exp((\log T)^{\varepsilon}))$.

**Proof.** Follows from the fact that

$$|A(s)| \leq \sum_{n=1}^\infty |a_n| n^{-1} \left| \Delta \left( \frac{X}{n} \right) \right|.$$ 

**Lemma 3.** The integral $\int_{T_0}^T |A(s)E(s)| \, dt$ is $O(\exp(-\frac{1}{2}(\log T)^{0.1}))$.

**Proof.** Follows from Lemmas 1 and 2.

**Lemma 4.** We have

$$\int_{T_0}^T |F(s)|^2 \, dt = \int_{T_0}^T |A(s)|^2 \, dt + O(\exp(-\frac{1}{2}(\log T)^{0.1})).$$

**Proof.** Follows from Lemmas 2 and 3. Now the integral on the right hand side of (21) is

$$\sum_{n=1}^\infty (T - T_0 + O(n)) |a_n|^2 n^{-2} \left| \Delta \left( \frac{X}{n} \right) \right|^2$$
by a well-known theorem of H. L. Montgomery and R. C. Vaughan, and so
Theorem 3 follows by a slight further work since $a_n = O(n^\varepsilon)$ for all $\varepsilon > 0$.

References

[1] R. Balasubramanian, A. Ivić and K. Ramachandra, The mean square of the
Riemann zeta-function on the line $\sigma = 1$, Enseign. Math. 38 (1992), 13–25.
(1976), 313–324.
[3] A. Sankaranarayanan and K. Srinivas, On the papers of Ramachandra and