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The asymptotic distribution of the number of summands in unrestricted Λ-partitions

by

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By an *unrestricted Λ-partition* of n we mean a solution in non-negative integers a_i of

$$\sum_{i=1}^{\infty} a_i \lambda_i = n \,, \tag{1}$$

where $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ is an infinite multiset of positive integers. Much work has been done, see for example [5], on statistical aspects of the restricted partitions with $a_i \le 1$, but little appears to be known about the distribution of the number of summands in the solutions of (1) in the unrestricted case beyond the theorem of Erdős and Lehner for the case $\Lambda = \mathbb{N}$, which shows in particular that if all solutions of (1) are equally likely, then the random variable

$$\pi \left(\frac{2}{3}\right)^{1/2} \sum_{i=1}^{\infty} a_i / n^{1/2} \log n$$

converges in distribution to unity as $n \to \infty$. For further details, see [4].

In the paper of Loxton and Yeung [7], there is the following question communicated to the authors by Erdős: "Does there exist $f(c)$ such that the number of partitions of n into squares in which the number of summands is less than $cn^{2/3} \log n$ is asymptotic to $f(c)p_2(n)$?" Here $p_2(n)$ is the total number of partitions of n into squares.

As we shall see, the answer to this problem is

$$f(c) = \begin{cases} 0\,, & c = 0\,, \\ 1\,, & c > 0\,, \end{cases}$$

and a more appropriate question is obtained by replacing $cn^{2/3} \log n$ by $cn^{2/3}$, when the corresponding $f(c)$ is continuous. We shall derive the equivalent of $f(c)$ for a wide range of sets Λ, including $\mathbb{N}$ and the set of squares.

We denote by $P_\Lambda(n)$ the number of solutions of (1), and let Λ_k be the multiset consisting of Λ together with k copies of unity. As a prerequisite for the proof of our theorem, we must have asymptotic formulae for $P_\Lambda(n)$ and $P_{\Lambda_k}(n)$ of a particular form; these are shown to exist, with certain restrictions on Λ, by Meinardus in [8] and by Ingham, Auluck and Haselgrove in [6] and [2]. The former result can also be found in Chapter 6 of [1]. We state a suitable version of the latter result.

THEOREM (Ingham, Auluck, Haselgrove). *Let Λ be a multiset as above. If*

$$N(n) = \sum_{\substack{i \\ \lambda_i \le n}} 1$$

satisfies

$$N(n) = A\alpha^{-1}n^\alpha + R(n)\,,$$

where α, $A > 0$ *and*, *as* $u \to \infty$,

$$\int_0^u \frac{R(v)}{v}\,dv = a\log u + b + o(1)\,,$$

and the elements of Λ have no non-trivial common factor, *then*

$$P_\Lambda(n) \sim \{2\pi(1+\alpha)\}^{-1/2} e^b M^{-(a-1/2)} n^{-1/2+(a-1/2)/(\alpha+1)} \times \exp\{(1+1/\alpha)Mn^{\alpha/(\alpha+1)}\} \tag{2}$$

as $n \to \infty$, *where*

$$M = \{A\Gamma(\alpha+1)\zeta(\alpha+1)\}^{1/(\alpha+1)}\,.$$

If Meinardus's theorem applies also, then the quantities A and α above are equal to Meinardus's A and α. Now we can state our theorem.

THEOREM. *Suppose* $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$, $\lambda_1 \le \lambda_2 \le \ldots$, *satisfies the conditions of the theorem of Ingham et al.* (*or the conditions of Meinardus's theorem*); *suppose also that* $1 \in \Lambda$ *or there is no* λ_i *such that* $\Lambda\backslash\{\lambda_i\}$ *has all its elements divisible by a non-trivial common factor. Then if* $\alpha = 1$ (*so that necessarily* $\sum \lambda_i^{-1}$ *diverges*), *we have*

$$\sqrt{\frac{A\pi^2}{6n}}\Big\{\sum_{\substack{i \\ \lambda_i \le n^{1/2}}} 1/\lambda_i\Big\}^{-1} \sum_{i=1}^\infty a_i$$

tends in distribution to one, *and if* $0 < \alpha < 1$ (*so that* $\sum \lambda_i^{-1}$ *converges*), *then*

$$M\sum_{i=1}^\infty a_i/n^{1/(1+\alpha)}$$

converges in distribution to the random variable with moment generating function

$$\prod_{i=1}^{\infty}(1-x/\lambda_i)^{-1}\,.$$

Hence if the λ_i *are distinct*, *the limit distribution has distribution function*

$$F(x)=1-\sum_{i=1}^{\infty}\Big\{\prod_{\substack{j=1\\ j\neq i}}^{\infty}(1-\lambda_i/\lambda_j)^{-1}\Big\}e^{-\lambda_i x},\qquad x>0\,, \tag{3}$$

provided the sum in (3) *converges for* $x>0$.

Proof. We consider the auxiliary generating function

$$g\ \ (x)=\prod_{i=1}^{\infty}(1-\alpha_i x^{\lambda_i})^{-1}\,.$$

The coefficient of x_n in $g\ \ (x)$ is

$$\sum_{\substack{\text{partitions}\\ \Sigma a_i\lambda_i=n}}\prod_{i=1}^{\infty}\alpha_i^{a_i}\,.$$

Let k be a fixed positive integer and let $l_1,l_2,\ldots$ be non-negative integers with $\sum_{i=1}^{\infty}l_i=k$. Let

$$r^{(l)}=r(r-1)\ldots(r-l+1),\quad l>0,\qquad r^{(0)}=1\,.$$

Then

$$\sum_{\substack{\text{partitions}\\ \Sigma a_i\lambda_i=n}}\prod_{i=1}^{\infty}a_i^{(l_i)}$$

is the coefficient of x^n in

$$h(x)=\prod_{i=1}^{\infty}\Big(\frac{\partial}{\partial\alpha_i}\Big)^{l_i}g\ \ (x)\Big|_{\alpha_1=\alpha_2=\ldots=1}=\prod_{i=1}^{\infty}\frac{l_i!x^{l_i\lambda_i}}{(1-x^{\lambda_i})^{l_i+1}}\,.$$

Therefore we have

$$\sum_{\substack{\text{partitions}\\ \Sigma a_i\lambda_i=n}}\prod_{i=1}^{\infty}a_i^{(l_i)}=\Big(\prod_{i=1}^{\infty}l_i!\Big)P_{\Lambda'}\Big(n-\sum_{i=1}^{\infty}l_i\lambda_i\Big)\,,$$

where Λ' is the multiset obtained from Λ by adjoining l_i copies of λ_i for each i. We cannot apply the theorems previously mentioned to obtain an asymptotic formula for $P_{\Lambda'}(n)$, since Λ' is not in general independent of n. In order to progress, we must restrict the choice of the l_i so that $l_i=0$ for all i such that

$$\lambda_i > n_0 = n^{1/(1+\alpha)}/\omega(n)\,,$$

where $\omega(n) \to \infty$ slowly; we shall determine the permissible rate of growth of $\omega(n)$ later. We now show that, under this restriction,

$$P_{\Lambda'}\Big(n - \sum_{i=1}^{\infty} l_i\lambda_i\Big) = \prod_{i=1}^{\infty} \lambda_i^{-l_i} P_{\Lambda_k}(n)(1+o(1))\,, \tag{4}$$

where $o(1)$ depends on the l_i only through k. Observe that if $\mu_1, \ldots, \mu_k$ are positive integers and $f(m)$ is a non-decreasing function of m, then

$$\begin{aligned}
\prod_{i=1}^{k} \mu_i^{-1} \sum_{b_1=0}^{\mu_1-1} \cdots \sum_{b_k=0}^{\mu_k-1} f\Big(m - \sum_{i=1}^{k} c_i\mu_i - \sum_{i=1}^{k} b_i\Big)& \\
\le f\Big(m - \sum_{i=1}^{k} c_i\mu_i\Big)& \\
\le \prod_{i=1}^{k} \mu_i^{-1} \sum_{b_1=0}^{\mu_1-1} \cdots \sum_{b_k=0}^{\mu_k-1} f\Big(m - \sum_{i=1}^{k} c_i\mu_i + \sum_{i=1}^{k} b_i\Big)&\,,
\end{aligned}$$

whence if $f(m) = 0$ for $m < 0$,

$$\begin{aligned}
\prod_{i=1}^{k} \mu_i^{-1} \sum_{\substack{b_1,\ldots,b_k \\ m-\Sigma b_i \ge 0}} f\Big(m - \sum_{i=1}^{k} b_i\Big)& \\
\le \sum_{\substack{c_1,\ldots,c_k \\ m-\Sigma c_i\mu_i \ge 0}} f\Big(m - \sum_{i=1}^{k} c_i\mu_i\Big)& \\
\le \prod_{i=1}^{k} \mu_i^{-1} \sum_{\substack{b_1,\ldots,b_k \\ m+\Sigma\mu_i-\Sigma b_i \ge 0}} f\Big(m + \sum_{i=1}^{k} \mu_i - \sum_{i=1}^{k} b_i\Big)&\,.
\end{aligned} \tag{5}$$

By hypothesis and the result of Bateman and Erdős [3], $P_\Lambda(m)$ is non-decreasing for m sufficiently large, say $m \ge d$. Thus by (5), for $m = n - \sum_{i=1}^{\infty} l_i\lambda_i$, which satisfies $m \ge n - kn_0 \sim n$, we have

$$\prod_{i=1}^{\infty} \lambda_i^{-l_i} \Big\{ \sum_{\substack{b_1,\ldots,b_k \\ m-\Sigma b_i \ge 0}} P_\Lambda\Big(m - \sum_{i=1}^{k} b_i\Big) - O(m^{k-1})\Big\}$$

$$\le \sum_{\substack{c_1,\ldots,c_k \\ m-\Sigma c_i\lambda_{j_i}\ge 0}} P_\Lambda\Big(m-\sum_{i=1}^k c_i\lambda_{j_i}\Big)$$

$$\le \prod \lambda_i^{-l_i}\Big\{\sum_{\substack{b_1,\ldots,b_k \\ m+\Sigma l_i\lambda_i-\Sigma b_i\ge 0}} P_\Lambda\Big(m+\sum_{i=1}^\infty l_i\lambda_i-\sum_{i=1}^k b_i\Big)+O(m^{k-1})\Big\},$$

since there are $O(m^{k-1})$ solutions of $0\le m-\sum_{i=1}^k b_i\le d$ in positive integers $b_1,\ldots,b_k$. Here $\lambda_{j_1},\ldots,\lambda_{j_k}$ are the elements λ_i with multiplicity l_i. Clearly,

$$\sum_{\substack{b_1,\ldots,b_k \\ m-\Sigma b_i\ge 0}} P_\Lambda\Big(m-\sum_{i=1}^k b_i\Big)=P_{\Lambda_k}(m)$$

and

$$\sum_{\substack{c_1,\ldots,c_k \\ m-\Sigma c_i\lambda_{j_i}\ge 0}} P_\Lambda\Big(m-\sum_{i=1}^k c_i\lambda_{j_i}\Big)=P_{\Lambda'}(m),$$

so we deduce that

$$\text{(6)}\quad \prod_{i=1}^\infty \lambda_i^{-l_i}P_{\Lambda_k}(m)(1+o(1))\le P_{\Lambda'}(m)$$

$$\le \prod_{i=1}^\infty \lambda_i^{-l_i}P_{\Lambda_k}\Big(m+\sum_{i=1}^k l_i\lambda_i\Big)(1+o(1)),$$

because $P_{\Lambda_k}(m)\ge P_\Lambda(m)$ and, by (2), $\log\{P_\Lambda(m)/m^{k-1}\}\sim(1+1/\alpha)\times Mm^{\alpha/(\alpha+1)}$. It is easily seen that the theorems of Ingham *et al.* and Meinardus apply to Λ_k; for the former, $N(n)$ corresponding to Λ becomes $N(n)+k$ for $n\ge 1$, and for the latter $D(s)$ and $g(\tau)$ corresponding to Λ become $D(s)+k$ and $g(\tau)+ke^{-\tau}$. We obtain in particular

$$\frac{P_{\Lambda_k}(m+\sum l_i\lambda_i)}{P_{\Lambda_k}(m)}=\frac{n^{-1/2+(a-1/2+k)/(\alpha+1)}}{(n-\sum l_i\lambda_i)^{-1/2+(a-1/2)/(\alpha+1)}}$$

$$\times\exp\Big\{(1+\alpha^{-1})M\Big[n^{\alpha/(\alpha+1)}-\Big(n-\sum l_i\lambda_i\Big)^{\alpha/(\alpha+1)}\Big]\Big\},$$

and since

$$n^{\alpha/(\alpha+1)}-(n-kn_0)^{\alpha/(\alpha+1)}\sim\frac{\alpha}{\alpha+1}\frac{k}{\omega(n)}\to 0,$$

we see that

$$P_{\Lambda_k}\Big(m+\sum_{i=1}^\infty l_i\lambda_i\Big)=P_{\Lambda_k}(m)(1+o(1)),$$

where $o(1)$ depends on the l_i only through k. Now (4) follows from (6). Hence

$$\sum_{\substack{\text{partitions}\\ \Sigma a_i\lambda_i=n}} \prod_{i=1}^{\infty} a_i^{(l_i)} = \prod_{i=1}^{\infty}\left(\frac{l_i!}{\lambda_i^{l_i}}\right) P_{\Lambda_k}(n)(1+o(1))\,,$$

and

$$(7)\qquad E\Big(\prod_{i=1}^{\infty} a_i^{(l_i)}\Big) = \prod_{i=1}^{\infty}\left(\frac{l_i!}{\lambda_i^{l_i}}\right)\frac{P_{\Lambda_k}(n)}{P_{\Lambda}(n)}(1+o(1))$$
$$= \prod_{i=1}^{\infty}\left(\frac{l_i!}{\lambda_i^{l_i}}\right) M^{-k} n^{k/(1+\alpha)}(1+o(1))$$

by the theorem of Ingham *et al.* or Meinardus, where $o(1)$ depends on the l_i only through k.

We now prove by induction on k that, provided $l_i = 0$ for i such that $\lambda_i > n_0$,

$$(8)\qquad E\Big(\prod_{i=1}^{\infty} a_i^{l_i}\Big) = E\Big(\prod_{i=1}^{\infty} a_i^{(l_i)}\Big)(1+o(1))$$

as $n\to\infty$. Let

$$R = E\Big(\prod_{i=1}^{\infty} a_i^{(l_i)}\Big) - E\Big(\prod_{i=1}^{\infty} a_i^{l_i}\Big)\,.$$

Then

$$(9)\qquad R = \sum_{d=1}^{k} \sum_{\substack{j_1,j_2,\ldots\\ 0\le j_i\le l_i\\ \Sigma(l_i-j_i)=d}} \Big(\prod_{i=1}^{\infty} c_{j_i}\Big) E\Big(\prod_{i=1}^{\infty} a_i^{j_i}\Big)\,,$$

where c_{j_i} is the coefficient of r^{j_i} in the polynomial $r^{(l_i)}$. Observe that $\prod_{i=1}^{\infty} c_{j_i}$ is bounded in terms of k, for

$$|c_{j_i}| \le \binom{l_i-1}{j_i-1}(l_i-1)^{j_i} \le l_i^{2j_i}$$

for $j_i \ge 1$, and so

$$\prod_{i=1}^{\infty} |c_{j_i}| \le (\max l_i)^{2k} \le k^{2k}\,.$$

The claim (8) is trivial if $k=0$, so we assume that $k>0$ and that (8) holds for all $j_1, j_2, \ldots$ with $\sum_{i=1}^{\infty} j_i < k$, $j_i \le l_i$, when l_i is replaced by j_i. Then by (7), (9) and the induction hypothesis,

$$\Big|R\Big\{E\Big(\prod a_i^{(l_i)}\Big)\Big\}^{-1}\Big|$$

$$\le A_1 \sum_{d=1}^{k} \sum_{\substack{j_1,j_2,\dots\\ j_i\le l_i\\ \Sigma(l_i-j_i)=d}} E\Big(\prod a_i^{j_i}\Big) M^k n^{-k/(1+\alpha)} \prod_{i=1}^{\infty}(\lambda_i^{l_i}/l_i!)$$

$$\le A_2 \sum_{d=1}^{k} M^d n^{-d/(1+\alpha)} \sum_{\substack{j_1,j_2,\dots\\ j_i\le l_i\\ \Sigma(l_i-j_i)=d}} \prod_{i=1}^{\infty}\Big\{\frac{j_i!}{l_i!}\lambda_i^{l_i-j_i}\Big\}$$

$$\le A_3 \sum_{d=1}^{k} n^{-d/(1+\alpha)}\Big\{\sum_{\substack{i\\ l_i>0}} \lambda_i\Big\}^d$$

$$\le A_3 \sum_{d=1}^{k} n^{-d/(1+\alpha)}\{kn^{1/(1+\alpha)}/\omega(n)\}^d = O(1/\omega(n))\,,$$

where A_1, A_2 and A_3 depend only on k.

Hence if $l_i = 0$ for i with $\lambda_i > n_0$,

$$E\Big(\prod_{i=1}^{\infty} a_i^{l_i}\Big) = \prod_{i=1}^{\infty}(l_i!/\lambda_i^{l_i})M^{-k}n^{k/(1+\alpha)}(1+o(1))\,, \tag{10}$$

where, again, $o(1)$ depends on the l_i only through k. We have

$$\begin{aligned} E\Big\{\Big(\sum_{i=1}^{\infty} a_i/n^{1/(1+\alpha)}\Big)^k\Big\} &= E\Big\{\sum_{\substack{l_1,l_2,\dots\\ \Sigma l_i=k}} \binom{k}{l_1,l_2,\dots}\prod_{i=1}^{\infty} a_i^{l_i}/n^{k/(1+\alpha)}\Big\} \\ &= E\Big\{\sum_{(0)} \binom{k}{l_1,l_2,\dots}\prod_{i=1}^{\infty} a_i^{l_i}/n^{k/(1+\alpha)}\Big\} \\ &\quad + \sum_{d=1}^{k} E\Big\{\sum_{(d)} \binom{k}{l_1,l_2,\dots}\prod_{i=1}^{\infty} a_i^{l_i}/n^{k/(1+\alpha)}\Big\} \\ &= E_1 + E_2\,, \end{aligned}$$

say, where $\sum_{(d)}$ denotes a sum over all $l_1, l_2, \dots$ such that $\sum l_i = k$ and

$$\sum_{\substack{i\\ \lambda_i>n_0}} l_i = d\,.$$

We have by (10) that

$$E_1 = k!M^{-k}\sum_{(0)}\prod_{i=1}^{\infty}\lambda_i^{-l_i}(1+o(1))\,, \tag{11}$$

and since

$$
\begin{aligned}
&E\Big\{\sum_{(0)}\prod_{i=1}^{\infty} a_i^{l_i}\Big\}\\
&\quad\le E\Big\{\sum_{\substack{l_1,l_2,\ldots\\ \Sigma l_i=k-d\\ l_i=0\text{ if }\lambda_i>n_0}}\prod_{i=1}^{\infty} a_i^{l_i}\Big\}\Big\{\max\sum_{\substack{i\\ \lambda_i>n_0}} a_i\Big\}^d\\
&\quad\le \sum_{\substack{l_1,l_2,\ldots\\ \Sigma l_i=k-d\\ l_i=0\text{ if }\lambda_i>n_0}}\prod_{i=1}^{\infty}(l_i!/\lambda_i^{l_i})M^{-(k-d)}n^{(k-d)/(1+\alpha)}(1+o(1))\{n^{\alpha/(\alpha+1)}\omega(n)\}^d\\
&\quad\le k!\Big(\sum_{\substack{i\\ \lambda_i\le 0}} 1/\lambda_i\Big)^{k-d} M^{-(k-d)}n^{(k-d(1-\alpha))/(1+\alpha)}\omega(n)^d(1+o(1))\,,
\end{aligned}
$$

we have the bound

$$
\begin{aligned}
(12)\qquad E_2\le (k!)^2\sum_{d=1}^{k}\Big(\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i\Big)^{k-d}&\\
\times M^{-(k-d)}n^{-d(1-\alpha)/(1+\alpha)}\omega(n)^d(1+o(1))\,.&
\end{aligned}
$$

If $\alpha<1$, and so $\sum 1/\lambda_i$ converges, we have $E_2\to 0$ as $n\to\infty$ provided we choose $\omega(n)$ so that $\omega(n)=o(n^{(1-\alpha)/(1+\alpha)})$. Hence by (11),

$$
\lim_{n\to\infty} E\Big\{\Big(M\sum_{i=1}^{\infty} a_i/n^{1/(1+\alpha)}\Big)^k\Big\}=k!\sum_{\substack{l_1,l_2,\ldots\\ \Sigma l_i=k}}\prod_{i=1}^{\infty}\lambda_i^{-l_i}=k!r_k\,,
$$

where r_k is the coefficient of x^k in the power-series

$$
g(x)=\prod_{i=1}^{\infty}(1-x/\lambda_i)^{-1}\,,
$$

and so if $M\sum a_i/n^{1/(1+\alpha)}$ converges in distribution, $g(x)$ is the moment generating function of the limit distribution. Since $(1-x/\lambda_i)^{-1}$ is the moment generating function associated with the distribution function

$$
(13)\qquad 1-e^{-\lambda_i x},\qquad x\ge 0\,,
$$

the product

$$
(14)\qquad \prod_{i=1}^{m}(1-x/\lambda_i)^{-1}
$$

is the moment generating function of the sum of m independent random variables with the distributions (13) for $1 \le i \le m$. Since the product converges to a function continuous at zero, $g(x)$ is a moment generating function for a proper distribution function and $M \sum a_i / n^{1/(1+\alpha)}$ converges to this distribution. The necessary probability theory can be found in [9], Chapter 6, especially Theorems 6.2 and 6.16. If the λ_i are distinct, then the partial fractions representation,

$$\prod_{i=1}^{m} (1 - x/\lambda_i)^{-1} = \sum_{i=1}^{m} \Big\{ \prod_{\substack{j=1 \\ j \ne i}}^{m} (1 - \lambda_i/\lambda_j)^{-1} \Big\} (1 - x/\lambda_i)^{-1} , \tag{15}$$

shows that the distribution associated with (14) is

$$1 - \sum_{i=1}^{m} \prod_{\substack{j=1 \\ j \ne i}}^{m} (1 - \lambda_i/\lambda_j)^{-1} e^{-\lambda_i x}, \qquad x \ge 0 .$$

This converges to

$$1 - \sum_{i=1}^{\infty} \prod_{\substack{j=1 \\ j \ne i}}^{\infty} (1 - \lambda_i/\lambda_j)^{-1} e^{-\lambda_i x}$$

for $x > 0$, provided

$$\limsup_{i \to \infty} \prod_{\substack{j=1 \\ j \ne i}}^{\infty} |1 - \lambda_i/\lambda_j|^{-1/\lambda_i} \le 1 ,$$

and so the second part of the theorem is established.

If $\alpha = 1$, then by (11) for $k = 1$,

$$\lim_{n \to \infty} M E_1 \Big\{ \sum_{\substack{i \\ \lambda_i \le n_0}} 1/\lambda_i \Big\}^{-1} = 1 ,$$

and by (12) for $k = 1$,

$$\lim_{n \to \infty} M E_2 \Big\{ \sum_{\substack{i \\ \lambda_i \le n_0}} 1/\lambda_i \Big\}^{-1} \le \lim_{n \to \infty} \Big(M \Big\{ \sum_{\substack{i \\ \lambda_i \le n_0}} 1/\lambda_i \Big\}^{-1} \omega(n) \Big) = 0 ,$$

provided $\omega(n)$ grows sufficiently slowly that

$$\lim_{n \to \infty} \Big\{ \omega(n) \Big/ \sum_{\substack{i \\ \lambda_i \le n^{1/2}/\omega(n)}} 1/\lambda_i \Big\} = 0 .$$

Similarly, for $k=2$,

$$\lim_{n\to\infty} M^2 E_1\Big\{\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i\Big\}^{-2}$$

$$= \lim_{n\to\infty}\Big\{2\Big[\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i^2 + \sum_{\substack{i<j\\ \lambda_i,\lambda_j\le n_0}} 1/\lambda_i\lambda_j\Big]\Big/\Big(\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i\Big)^2\Big\}$$

$$= \lim_{n\to\infty}\Big\{\Big[\Big(\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i\Big)^2 + \sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i^2\Big]\Big/\Big(\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i\Big)^2\Big\} = 1\,,$$

and

$$\lim_{n\to\infty} M^2 E_2\Big\{\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i\Big\}^{-2} = \lim_{n\to\infty} O\Big(\omega(n)\Big/\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i\Big) = 0\,.$$

Hence, for

$$Y_n = M\Big\{\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i\Big\}^{-1}\sum_{i=1}^{\infty} a_i/n^{1/2}\,, \tag{16}$$

we have

$$\lim_{n\to\infty} EY_n = 1 \quad\text{ and }\quad \lim_{n\to\infty}\operatorname{var} Y_n = 0\,,$$

and so Y_n converges in distribution to one. We have thus shown that

$$\sum_{\substack{i\\ \lambda_i\le n_0}} 1/\lambda_i$$

is asymptotically independent of $\omega(n)$ provided $\omega(n)$ grows sufficiently slowly. It follows that for any $f(n)$ increasing to infinity and any $\omega_1(n)$ and $\omega_2(n)$ tending to infinity sufficiently slowly compared to $f(n)$,

$$\sum_{\substack{i\\ \lambda_i\le f(n)/\omega_1(n)}} 1/\lambda_i \sim \sum_{\substack{i\\ \lambda_i\le f(n)/\omega_2(n)}} 1/\lambda_i\,.$$

If we take $f(n)=n^{1/2}\omega_1(n)$ and arrange that $\omega_2(n)/\omega_1(n)\to\infty$, we see that we may replace n_0 in (16) by $n^{1/2}$. When we note that

$$M = \{A\Gamma(2)\zeta(2)\}^{1/2} = \{A\pi^2/6\}^{1/2}\,,$$

we have the first part of the theorem. ∎

EXAMPLES. (a) $\Lambda=\{1,2,3,\ldots\}=\mathbb{N}$. Meinardus's theorem applies with $D(s)=\zeta(s)$ and $g(\tau)=e^{-\tau}/(1-e^{-\tau})$. Ingham's theorem applies with

$R(n) = [n] - n$. In either case, we have $\alpha = A = 1$ and

$$\sum_{\substack{i\\ \lambda_i \le n^{1/2}}} 1/\lambda_i \sim (1/2)\log n\,,$$

whence the first part of our theorem gives the result of Erdős and Lehner.

(b) *Λ a union of arithmetic progressions*, $\Lambda = \{m \in \mathbb{N} : m \equiv b_1, b_2, \ldots,$ or $b_l \bmod k\}$, $(b_1, b_2, \ldots, b_l, k) = 1$. Meinardus's theorem applies with

$$D(s) = \sum_{i=1}^{l} k^{-s}\zeta(s, b_i/k)$$

and

$$g(\tau) = \sum_{i=1}^{l} e^{-b_i\tau}/(1 - e^{-k\tau})\,.$$

Ingham's theorem applies with

$$R(n) = l\{[n/k] - n/k\} + c_n\,,$$

where c_n of the b_i belong to congruence classes modulo k with representatives in the interval $([n/k]k, n]$. In either case we have $\alpha = 1$, $A = l/k$ and

$$\sum_{\substack{i\\ \lambda_i \le n^{1/2}}} 1/\lambda_i \sim (l/2k)\log n\,,$$

whence by the first part of our theorem,

$$\pi(2k/3l)^{1/2}(n^{1/2}\log n)^{-1}\sum_{i=1}^{\infty} a_i$$

converges in distribution to one.

(c) *Λ the set of squares.* Meinardus's theorem applies with $D(s) = \zeta(2s)$ and $g(\tau) = \sum_{r=1}^{\infty} e^{-r^2\tau}$. One way to see that $g(\tau)$ satisfies the required condition is as follows. If, as Meinardus, we put $\tau = y + 2\pi i x$, where y and x are real, then for $y > 0$,

$$(17)\qquad g(y) - \operatorname{Re} g(\tau) = \sum_{n=1}^{\infty} e^{-n^2 y}(1 - \cos(2\pi n^2 x))$$

$$\ge \sum_{\substack{n=1\\ \cos(2\pi n^2 x)\le 0}}^{\infty} e^{-n^2 y} \ge \sum_{\substack{n=1\\ \cos(2\pi n^2 x)\le 0}}^{[\{\log 2/y\}^{1/2}]} 1/2\,.$$

As was first shown by Weyl [10], if x is irrational the sequence $(n^2x)_{n\ge1}$ is uniformly distributed modulo one, and so, given $\eta > 0$, the number of summands in (17) is at least $(\frac{1}{2}-\eta)\{\log 2/y\}^{1/2}$ for sufficiently small y. Hence for x irrational,

$$g(y)-\operatorname{Re} g(\tau)\geq \tfrac{1}{5}(\log 2)^{1/2}y^{-1/2} \tag{18}$$

for small enough y. Finally, $g(\tau)$ is continuous for $y>0$, since in any half-plane $y\geq a>0$, the sum in the definition of $g(\tau)$ converges uniformly. Therefore (18) holds also for rational x, and we may take Meinardus's ε to be 1/2.

Ingham's theorem applies also, since we have $R(n)=[n^{1/2}]-n^{1/2}$ and

$$\lim_{u\to\infty}\left\{\int_1^u \frac{v^{1/2}-[v^{1/2}]}{v}\,dv-\frac{1}{2}\log u\right\}$$

$$=\lim_{u\to\infty}\int_1^u \frac{v^{1/2}-[v^{1/2}]-\frac{1}{2}}{v}\,dv=\lim_{n\to\infty}\sum_{r=1}^{n}\int_{r^2}^{(r+1)^2}\frac{v^{1/2}-[v^{1/2}]-\frac{1}{2}}{v}\,dv$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\{2-(2r+1)\log(1+1/r)\}=\lim_{n\to\infty}\sum_{r=1}^{n}O(1/r^2)$$

exists. We have $A=\alpha=1/2$. Our theorem now says that

$$M\sum_{i=1}^{\infty}a_i/n^{2/3}$$

converges in distribution to the distribution

$$F(x)=1-\lim_{m\to\infty}\sum_{r=1}^{m}\Big\{\prod_{\substack{j=1\\ j\neq r}}^{m}(1-r^2/j^2)^{-1}\Big\}e^{-r^2x},$$

and since

$$\prod_{\substack{j=1\\ j\neq r}}^{\infty}(1-r^2/j^2)^{-1}=(-1)^{r-1}\prod_{j=1}^{r-1}\frac{j^2}{(r-j)(r+j)}\prod_{j=r+1}^{\infty}\frac{j^2}{(j-r)(j+r)}$$

$$=(-1)^{r-1}\frac{(r-1)!^2r}{(2r-1)!}\frac{(2r)!}{r!^2}=(-1)^{r-1}\cdot 2,$$

we have

$$F(x)=\sum_{r=-\infty}^{\infty}(-1)^r e^{-r^2x},\qquad x>0.$$

By the result of Jacobi and Gauss, given for example on page 23 of [1],

$$F(x)=\prod_{m=1}^{\infty}\frac{1-e^{-mx}}{1+e^{-mx}},\qquad x\geq 0.$$

Hence if the factor $\log n$ is removed from his original question, Erdős's distribution function $f(c)$ satisfies

$$f(c) = F(Mc)\,,$$

where, since $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$,

$$M = \{\tfrac{1}{4}\sqrt{\pi}\zeta(\tfrac{3}{2})\}^{2/3}\,.$$

In particular the distribution given by $f(c)$ has mean

$$\tfrac{1}{6}\pi^{5/3}\{4/\zeta(\tfrac{3}{2})\}^{2/3} = 1.49\ldots$$

and variance

$$\tfrac{1}{90}\pi^{10/3}\{4/\zeta(\tfrac{3}{2})\}^{4/3} = 0.89\ldots$$

(d) *Λ the set of k-th powers*, $k \geq 2$. By similar reasoning as in (c), the theorems of Meinardus and Ingham apply. Note that for $k > 2$,

$$-r^{-k}\sum_{\substack{j=1\\ j\neq r}}^{\infty}\log|1 - r^k/j^k| = r^{-k}\sum_{j=1}^{r-1}\log\left(\frac{j^k}{r^k - j^k}\right) + r^{-k}\sum_{j=r+1}^{\infty}\log\left(\frac{j^k}{j^k - r^k}\right)$$

$$\leq r^{-(k-1)}k\log r + r^{-k}\sum_{j=r+1}^{\infty}\left(\frac{r^k}{j^k - r^k}\right),$$

and for $j > r$, $j^{k-1}(j-1) > r^k$, whence

$$-r^{-k}\sum_{\substack{j=1\\ j\neq r}}^{\infty}\log|1 - r^k/j^k| \leq r^{-(k-1)}k\log r + \sum_{j=r+1}^{\infty} 1/j^{k-1} \to 0$$

as $r \to \infty$. Hence

$$M\sum_{i=1}^{\infty} a_i/n^{k/(k+1)}$$

has a limit distribution of

$$1 - \sum_{r=1}^{\infty}\Big\{\prod_{\substack{j=1\\ j\neq r}}^{\infty}(1 - r^k/j^k)^{-1}\Big\}e^{-r^k x}, \qquad x > 0\,,$$

where

$$M = \{k^{-1}\Gamma(1 + 1/k)\zeta(1 + 1/k)\}^{k/(k+1)}\,.$$

By the Weierstrass product form for the reciprocal of the gamma function,

$$\prod_{j=1}^{\infty}(1 - x^k/j^k) = \prod_{\substack{\varrho\\ \varrho^k=1}} 1/\Gamma(1 - \varrho x)\,,$$

and so

$$\prod_{\substack{j=1\\ j\neq r}}^{\infty}(1-r^k/j^k) = \lim_{x\to r}\Big\{\prod_{\substack{\varrho\\ \varrho^k=1}}\Gamma(1-\varrho x)(1-x^k/r^k)\Big\}$$

$$= k\prod_{\substack{\varrho\\ \varrho^k=1,\varrho\neq 1}}\Gamma(1-\varrho r)\lim_{x\to r}(1-x/r)\Gamma(1-x)$$

$$= k\prod_{\substack{\varrho\\ \varrho^k=1,\varrho\neq 1}}\Gamma(1-\varrho r)\lim_{x\to r}\frac{(1-x/r)\pi}{\Gamma(x)\sin(\pi x)}$$

$$= \frac{(-1)^{r-1}}{r!}k\prod_{\substack{\varrho\\ \varrho^k=1,\varrho\neq 1}}\Gamma(1-\varrho r)\,.$$

In particular, for $k=4$ we have the distribution function

$$1-\sum_{r=1}^{\infty}(-1)^{r-1}\frac{4\pi r}{\sinh(\pi r)}e^{-r^4x}\,.$$

We remark that (15) leads to the identity

$$\prod_{r=1}^{\infty}(1-x/r^4)^{-1} = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{4\pi r}{\sinh(\pi r)}(1-x/r^4)^{-1}$$

when $k=4$, and the identity

$$\prod_{r=1}^{\infty}(1-x/r^2)^{-1} = \lim_{z\to 1^-}2\sum_{r=1}^{\infty}(-1)^{r-1}z^r(1-x/r^2)^{-1}$$

when $k=2$. This last identity shows that $\zeta(2n)$ is a rational multiple of $\zeta(2)^n$. As an example, comparison of the coefficients of x^2 on either side gives

$$\sum_{r=1}^{\infty}1/r^4+\sum_{1\le r<s<\infty}1/r^2s^2 = 2\sum_{r=1}^{\infty}(-1)^{r-1}\cdot 1/r^4\,,$$

which is equivalent to $\frac{1}{2}\zeta(2)^2+\frac{1}{2}\zeta(4) = 2(1-2/2^4)\zeta(4)$, whence $\zeta(4) = \frac{2}{5}\zeta(2)^2$.

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