ACTA ARITHMETICA
LXIV.4 (1993)

Ramsey problems in additive number theory

by

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1. Introduction. In 1964 Erdős and Heilbronn [2] proved that if p is a prime and A is a set of at least $3\sqrt{6p}$ residues modulo p, then $\sum_{b\in B} b \equiv 0 \pmod{p}$ for a non-empty subset B of A. Subsequently Olson [5] proved the essentially best possible result that if A is a set of more than $\sqrt{4p-3}$ non-zero residues modulo p then for every integer n there is a non-empty subset B of A such that $\sum_{b\in B} b \equiv n \pmod{p}$.

In 1985, Burr and Erdős [1] studied infinite sequences A of natural numbers such that if $A = A_1 \cup A_2$ then every (or every sufficiently large) number n is a sum of distinct terms of some A_i, with i depending on n. This is a Ramsey-type question (see [3], [4]) for integers: when is it true that every partition is such that at least one of the parts has a certain property?

Our aim in this note is to study some new question related to the problems above.

Denote by $f_k(n)$ the minimal integer m such that no matter how we divide the integers from 1 to m into k classes, n is a sum of distinct terms of one of the classes. What can one say about $f_k(n)$? Also, let us denote by $g_k(n)$ the minimal integer m such that there is a subset A of $\{1, 2, \ldots, n-1\}$ with $g_k(n) = \sum_{a\in A} a$ such that if the integers in A are partitioned into k classes, then n is always a sum of some integers from the same class. What can one say about $g_k(n)$? In this paper we shall investigate these two questions for $k = 2$.

It is easily seen that if $\sum_{i=1}^{m} i \le 2n-2$ then $f_2(n) > m$; thus $f_2(n) \ge \lfloor 2\sqrt{n-1} + 1/2 \rfloor$. In Theorem 4 in the second section we shall show that this trivial lower bound is close to the true value of $f_2(n)$, namely that $f_2(n) \le 2\sqrt{n} + c_0 \log n$ for some constant c_0. It is rather surprising that it seems to be difficult to improve substantially the trivial lower bound above: all we shall show is that $f_2(n) \ge \lfloor 2\sqrt{n} \rfloor + 2$ if n is large enough (Theorem 5).

It is immediate from the definitions that $g_k(n) \le f_k(n)(f_k(n)+1)/2$ so $g_2(n) \le 2n + c_1\sqrt{n}\log n$. However, concerning a lower bound on $g_2(n)$, it is not even obvious that $g_2(n) \ge 2n$. In Theorem 7 in the third section we shall show that $g_2(n)$ is substantially larger than $2n$, in fact, $g_2(n) \ge 2n + \sqrt{2n}/8$ if n is sufficiently large.

2. Bounds for the function $f_2(n)$. In order to give an upper bound for $f_2(n)$ we need three easy lemmas. As usual, denote by $[b]$ and $[a,b]$ the sets of integers $\{i : 1 \le i \le b\}$ and $\{i : a \le i \le b\}$ respectively.

LEMMA 1. *Let $c < d$ and m be positive integers. Suppose that $A \subset [m]$ does not contain elements a and b with $c \le b - a \le d$. Then*

$$\sigma(A) \le \frac{c}{2(c+d)}(m+d+r+1)(m-r) + \tfrac{1}{2}(2r-e+1)e\,,$$

where $r = m - \lfloor m/(c+d)\rfloor(c+d)$ and $e = \min\{r, c\}$. In particular,

$$\sigma(A) \le \frac{c}{2(c+d)}(m+1)(m+d)\,. \ \blacksquare$$

For the sake of convenience, given a finite set $A \subset \mathbb{N}$ denote by $\sigma(A)$ the total sum of its elements, and let $\Sigma(A) = \{\sigma(B) : B \subset A\}$ be the set of integers that can be written as a sum of some elements of A. Our upper bound on $f_2(n)$ will be the function

$$m(n) = \lfloor 2\sqrt{n} + \log_{5/4} n + 8\rfloor\,. \qquad (1)$$

LEMMA 2. *If $m \ge 3$ and $A \subset [m]$ contains all odd numbers not greater than m then $\Sigma(A) \supset [3, \sigma(A) - 3]$. In particular*, *if $n \ge 2$ and $A \subset [m(n)]$ contains all odd numbers not greater than m then $n \in \Sigma(A)$.*

Proof. The assertion is easily checked for $3 \le m \le 7$; the rest follows by induction on m since $\sigma(A) - 5 \ge m + 1$ for $m \ge 7$. $\blacksquare$

LEMMA 3. *For $n \ge 4$, if $[m(n)] = A_1 \cup A_2$ and neither A_1 nor A_2 is the set of all odd numbers in $[m(n)]$ then for some i we have*

$$\sigma(A_i) \ge \frac{m(n)(m(n)+1)}{4} - \frac{m(n)-3}{4} = \frac{(m(n))^2 - 3}{4}$$

and in A_i there are two integers with difference 1.

Proof. The lemma is trivial if both A_1 and A_2 contain two integers with difference 1. Suppose that A_2 does not contain two integers with difference 1. Then A_1 must contain two integers with difference 1. Indeed, otherwise, one of A_1 and A_2 must be the set of all odd numbers in $[m(n)]$.

Since A_2 does not contain two integers with difference 1, we have

$$\begin{aligned}\sigma(A_2) &\le m(n) + (m(n)-2) + (m(n)-4) + \ldots \\ &\le \frac{m(n)(m(n)+1)}{4} + \frac{m(n)-3}{4}\,.\end{aligned}$$

As $A_1 \cup A_2 = [m(n)]$, this gives

$$\sigma(A_1) \ge \frac{m(n)(m(n)+1)}{4} - \frac{m(n)-3}{4} = \frac{(m(n))^2-3}{4},$$

completing the proof of the lemma. ■

THEOREM 4. *If n is sufficiently large then*

$$f_2(n) \le m(n) = \lfloor 2\sqrt{n} + \log_{5/4} n + 8 \rfloor .$$

Proof. Let $[m(n)] = A_1 \cup A_2$. To prove the theorem, we have to show that

$$n \in \Sigma(A_1) \cup \Sigma(A_2) .$$

By Lemmas 2 and 3, we may assume that

$$\sigma(A_1) \ge \frac{m(n)(m(n)+1)}{4} - \frac{m(n)-3}{4} = \frac{(m(n))^2-3}{4}, \tag{2}$$

and there are a and b in A_1 such that $b - a = 1$.

We claim that A_1 contains a set $F_l = \{a_1, b_1, \ldots, a_l, b_l\}$, where $l \le \lceil \frac{1}{2} \log_{5/4} n \rceil + 6$, such that the set

$$\Big\{ \sum_{i=1}^{l} c_i : c_i = a_i \text{ or } b_i \Big\}$$

contains the interval $[a, b]$, where $a = \sum_{i=1}^{l} a_i$ and $b = \sum_{i=1}^{l} b_i$, and this interval has length at least $m(n)$. Having proved the claim it is easy to see that $n \in \Sigma(A_1)$. Indeed, inequalities (1) and (2) imply that

$$\sigma(A_1 - F_l') \ge \frac{(m(n))^2-3}{4} - a > n ,$$

where $F_l' = \{a_1, \ldots, a_l\}$. Let D be a maximal subset of $A_1 - F_l$ such that

$$\sigma(D) \le n - a .$$

Then, rather crudely, $\sigma(D) + m(n) \ge n - a$, so

$$\sigma(D) + a \le n \le \sigma(D) + b .$$

Hence $n \in \Sigma(A_1)$, as asserted by our theorem.

Now we return to prove our claim. To construct the sequence F_l, pick elements $a_1, b_1 \in A_1$ with $b_1 - a_1 = 1$. Suppose we have constructed $\{a_1, b_1, \ldots, a_{i-1}, b_{i-1}\}$, where $2 \le i \le l$. Inequality (2) and Lemma 1 imply that there are two elements $a_i, b_i \in A_1 - \{a_1, b_1, \ldots, a_{i-1}, b_{i-1}\}$ such that

$$1 \le b_i - a_i \le \sum_{j=1}^{i-1} (b_j - a_j) + 1 \tag{3}$$

and $b_i - a_i$ is maximal subject to (3). We get a new set $\{a_1, b_1, \ldots, a_{i-1}, b_{i-1}, a_i, b_i\}$. This completes the construction of the set F_l.

To see the first property of the set $F_l = \{a_1, b_1, \ldots, a_i, b_i\}$, note that if the integers $1 \le c_1 \le c_2 \le \ldots \le c_k$ are such that $c_i \le \sum_{j=1}^{i-1} c_j + 1$ for every i, $i = 1, \ldots, k$, then every integer $m \le \sum_{i=1}^{k} c_i$ can be represented as $m = \sum_{i \in I} c_i$ for some $I \subset [k]$. Therefore, the first property holds.

Now we shall prove that the set F_l satisfies the second property. Lemma 1 and maximality of $b_i - a_i$ imply that

$$\frac{1}{4}\sum_{j=1}^{i-1}(b_j - a_j) \le b_i - a_i \le \sum_{j=1}^{i-1}(b_j - a_j) + 1$$

for every $i = 1, 2, \ldots, l$. This gives that

$$b - a = \sum_{j=1}^{l}(b_j - a_j) \ge \left(\frac{5}{4}\right)^{l-1} \ge m(n)\,,$$

completing the proof of the claim and so that of the theorem. ∎

As we remarked in the introduction, the above upper bound on $f_2(n)$ is close to being best possible. Indeed, if m is the maximal integer such that $\sum_{i=1}^{m} i \le 2n-2$ then $[m]$ has a subset $A_1 = \{m, m-1, \ldots, m-j\} \cup \{m-h\}$ with $\sigma(A_1) = n-1$, so with $A_2 = [m] - A_1$ we have $[m] = A_1 \cup A_2$, $\sigma(A_1) = n-1$ and $\sigma(A_2) \le n-1$. Hence $f_2(n) \ge \lfloor 2\sqrt{n-1} + 1/2 \rfloor$. It does not seem unreasonable to conjecture that if $\sum_{i=1}^{m} i \ge 2n$ then $f_2(n) \le m+1$. Our next aim is to show that this is not the case.

THEOREM 5. *If n is sufficiently large then*

$$f_2(n) \ge \lfloor 2\sqrt{n} \rfloor + 2\,.$$

Proof. Suppose that $n \ge 1100$ and $f_2(n) \le m$, where

$$m = \lfloor 2\sqrt{n} \rfloor + 1\,. \tag{4}$$

Then for all partitions $[m] = A_1 \cup A_2$, either $n \in \Sigma(A_1)$ or $n \in \Sigma(A_2)$. Note first that

$$\sum_{i=1}^{m} i = m(m+1)/2 \le 2n + 3\sqrt{n} + 1\,. \tag{5}$$

Let k be the integer such that

$$l - k < n \quad \text{and} \quad l \ge n$$

where $l = \sum_{i=k}^{m} i$. Then $k \le \sqrt{2n} + 2$, so $m - k \ge 3$ and $l < n + k < n + 3m - 3k + 3$. Let $A_1 = [k+1, m]$ and $A_2 = [k]$. Then $n \notin \Sigma(A_1)$ so $n \in \Sigma(A_2)$. Thus $k(k+1)/2 \ge n$. Therefore

$$k > \sqrt{2n} - 1\,. \tag{6}$$

To arrive at a contradiction, we shall partition $[m]$ into classes A_1 and A_2 such that $n \notin \Sigma(A_i)$ for any i. Depending on the value of l, $n \le l < n + 3m - 3k + 3$, we partition the integers from 1 to m into two classes A_1 and A_2 in the following way.

(i) If $n \le l < n + m - k + 1$ then let $A_1 = [k-4, m] - \{a\}$ and $A_2 = [k-5] \cup \{a\}$, where $a = l + k - n$.

(ii) If $n+m-k+1 \le l < n+2m-2k+2$ then let $A_1 = [k-5, m] - \{a, b\}$ and $A_2 = [k-6] \cup \{a, b\}$, where $a, b \in [k-1, m]$, $a \ne b$ and $l+2k-2-n = a+b$.

(iii) If $n + 2m - 2k + 2 \le l < n + 3m - 3k + 3$ then let $A_1 = [k-6, m] - \{a, b, c\}$ and $A_2 = [k-7] \cup \{a, b, c\}$, where $a, b, c \in [k, m]$, $a \ne b \ne c \ne a$ and $l + 3k - 5 - n = a + b + c$.

To complete the proof, we simply check that $n \notin \Sigma(A_1) \cup \Sigma(A_2)$ for the partitions above.

Case (i). By (6), we have

$$\sigma(A_1) = l-a+k-1+k-2+k-3+k-4 = n+3k-10 > n+3\sqrt{2n}-13\,. \tag{7}$$

Then, by (5), we have $\sigma(A_2) < n$, so $n \notin \Sigma(A_2)$.

Suppose that $n \in \Sigma(A_1)$. Let Q be a subset of A_1 such that $n = \sigma(Q)$. Since, by (7), $\sigma(A_1) = n + 3k - 10$ and the minimal integer of A_1 is $k - 4$, we have $|A_1 - Q| \le 2$. But by (4) and (7), $\sigma(A_1) > n + 3\sqrt{2n} - 13 > n+m+(m-1)$ so we have $|A_1 - Q| \ge 3$, contradicting the inequality above.

Case (ii). By (6), we have

$$\begin{aligned} \sigma(A_1) &= l + k - 1 + k - 2 + k - 3 + k - 4 + k - 5 - a - b \\ &= n + 3k - 13 > n + 3\sqrt{2n} - 16\,. \end{aligned} \tag{8}$$

Then, by (5), we have $\sigma(A_2) < n$, so $n \notin \Sigma(A_2)$.

Suppose that $n \in \Sigma(A_1)$. Let Q be a subset of A_1 such that $n = \sigma(Q)$. Since, by (8), $\sigma(A_1) = n + 3k - 13$ and the minimal integer of A_1 is $k - 5$, we have $|A_1 - Q| \le 2$. But by (4) and (8), $\sigma(A_1) > n + 3\sqrt{2n} - 16 > n + m + (m - 1)$ so we have $|A_1 - Q| \ge 3$, which is a contradiction.

Case (iii). By (6), we have

$$\begin{aligned} \sigma(A_1) &= l + k - 1 + k - 2 + k - 3 + k - 4 + k - 5 + k - 6 - a - b - c \\ &= n + 3k - 16 > n + 3\sqrt{2n} - 19\,. \end{aligned} \tag{9}$$

Then, by (5), we have $\sigma(A_2) < n$, so $n \notin \Sigma(A_2)$.

Suppose that $n \in \Sigma(A_1)$. Let Q be a subset of A_1 such that $n = \sigma(Q)$. Since, by (9), $\sigma(A_1) = n + 3k - 16$ and the minimal integer of A_1 is $k - 6$, we have $|A_1 - Q| \le 2$. But by (4) and (9), $\sigma(A_1) > n + 3\sqrt{2n} - 19 > n + m + (m - 1)$ so we have the contradiction that $|A_1 - Q| \ge 3$. ∎

It would be of interest to decide whether $f_2(n)-2\sqrt{n}$ is bounded or not. We are inclined to hazard the guess that it tends to infinity.

3. Bounds for $g_2(n)$. As Theorem 4 implies easily that $g_2(n)$ is close to $2n$, the real question is the order of $g_2(n)-2n$. Since, trivially, $g_2(n) \le f_2(n)(f_2(n)+1)/2$, Theorem 4 gives the following upper bound on $g_2(n)-2n$.

THEOREM 6. *If n is sufficiently large then*

$$g_2(n) - 2n \le 3\sqrt{n}\log_{5/4} n\,. \ \blacksquare$$

When trying to prove a good lower bound for $g_2(n)-2n$, we encounter considerably more serious difficulties than in giving a lower bound for $f_2(n)$. It is intuitively obvious that $g_2(n)-2n \ge 0$ but, somewhat surprisingly, this does not seem to be trivial to prove. Nevertheless, we shall show that the bound $5\sqrt{n}\log_2 n$ in Theorem 6 is not far from the truth.

THEOREM 7. *If $n \ge 3$ then*

$$g_2(n) - 2n \ge \sqrt{2n}/8\,.$$

Proof. As the proof is rather long, we shall put most of the work into five lemmas. Suppose that, contrary to the assertion, there is a set $A \subset [n-1]$ such that $\sigma(A) < 2n + \sqrt{2n}/8$ and $n \in \Sigma(A_1) \cup \Sigma(A_2)$ for all partitions $A = A_1 \cup A_2$. Let $A = \{a_1, a_2, \ldots, a_m\}$, where $n > a_1 > a_2 > \ldots > a_m > 0$.

Our first aim is to define an increasing sequence of indices $k_0, k_1, \ldots, k_t$. In order to make this definition somewhat more convenient, let us add an auxiliary term to the sequence $(a_i)_{i=1}^m$, namely the term $a_{m+1} = n$. Let k_0 be the minimal index such that $s(k_0) = \sum_{i=1}^{k_0} a_i \ge n$. Clearly $k_0 \ge 2$. If $s(k_0) > n$ then set $t = 0$, otherwise let k_1 be the minimal index such that

$$s(k_1) = \sum_{i=1}^{k_1} a_i - a_{k_0} \ge n\,.$$

If $s(k_1) > n$ then set $t = 1$, otherwise let k_2 be the minimal index such that

$$s(k_2) = \sum_{i=1}^{k_2} a_i - a_{k_0} - a_{k_1} \ge n\,.$$

As we have the auxiliary term $a_{m+1} = n$, continuing in this way we arrive at a sequence $k_0, k_1, \ldots, k_t$, where k_j is the minimal index such that $s(k_j) = \sum_{i=1}^{k_j} a_i - \sum_{l=0}^{j-1} a_{k_l} \ge n$. Thus $s(k_j) = n$ for $j = 0, 1, \ldots, t-1$, and $s(k_t) > n$.

Let us start with some easy observations concerning the sequence a_{k_0}, $a_{k_1}, \ldots, a_{k_t}$. As $s(k_0) = \sum_{i=0}^{k_0} a_i = n$, we have $k_0 \ge 2$ and

$$a_{k_0} < n/k_0 \le n/2\,. \tag{10}$$

Furthermore, as

$$a_{k_j} = \sum_{i=k_j+1}^{k_{j+1}} a_i$$

holds for $j = 0, 1, \ldots, t-2$, we have

$$a_{k_{j+1}} < a_{k_j}/(k_{j+1} - k_j) \le a_{k_j}/2\,, \tag{11}$$

implying

$$\sum_{j=0}^{t-1} a_{k_j} < a_{k_0} \sum_{j=0}^{t-1} 2^{-j} < 2a_{k_0} < n\,. \tag{12}$$

Our next aim is to prove a simple lemma claiming that, in fact, the auxiliary term a_{m+1} is not needed in the definition above.

LEMMA 8. *In the notation above, we have* $k_t \le m$.

Proof. Suppose that, contrary to the assertion, $k_t = m+1$. Then $t \ge 1$ and so

$$s(k_{t-1}) = \sum_{i=1}^{k_{t-1}} a_i - \sum_{j=0}^{t-2} a_{k_j} = n$$

and

$$a_{k_{t-1}} > a_{k_{t-1}+1} + a_{k_{t-1}+2} + \ldots + a_m\,.$$

Consequently,

$$\sum_{i=1}^{m} a_i - \sum_{j=0}^{t-1} a_{k_j} < n\,. \tag{13}$$

Inequalities (12) and (13) suggest a partition $A = A_1 \cup A_2$ contradicting our assumption on A: setting $A_1 = \{a_{k_j} : 0 \le j < t\}$ and $A_2 = A - A_1$, clearly $n \notin \Sigma(A_1) \cup \Sigma(A_2)$. ∎

Our next lemma is a considerable extension of Lemma 8: not only do we have $k_t \le m$ but also the sum $a_{k_t+1} + a_{k_t+2} + \ldots + a_m$ is quite large. In the proof of this lemma, and in the rest of the proof of Theorem 7, we shall make use of two sets, namely

$$K = \{a_{k_0}, a_{k_1}, \ldots, a_{k_{t-1}}\}, \qquad L = \{a_1, a_2, \ldots, a_{k_t}\} - K\,.$$

Note that $\sigma(L) = s(k_t) > n$ and $\sigma(K) < \sigma(L) - a_{k_t} < n$.

LEMMA 9. *We have* $a_{k_t} \ge \sqrt{2n}/4$ *and*

$$\sum_{i=k_t+1}^{m} a_i \ge n/16\,. \tag{14}$$

Proof. Clearly, inequality (14) implies that $a_{k_t} \ge a_{k_t+1} + 1 \ge \sqrt{2n}/4$, so it suffices to prove (14).

First we assume that $t \le 3$. Let $A_1 = L$ and $A_2 = A - L = K \cup \{a_{k_t+1}, \ldots, a_m\}$. By the definition of k_i, we have $n \notin \Sigma(A_1)$ so, by our assumption, n must be in the set $\Sigma(A_2)$. A fortiori,

$$\sigma(A_2) = \sum_{i=0}^{t-1} a_{k_i} + \sum_{i=k_t+1}^{m} a_i \ge n\,,$$

so, recalling (10) and (11), we find that

$$\sum_{i=k_t+1}^{m} a_i \ge n - \sum_{i=0}^{t-1} a_{k_i} > n - a_{k_0} \sum_{i=0}^{t-1} 2^{-i} > n - \frac{n}{2} \sum_{i=0}^{2} 2^{-i} = n/8\,. \tag{15}$$

Assume now that $t \ge 4$ and (14) is false. We claim that $k_0 = 2$, $k_1 = 4$ and $k_2 = 6$. Indeed, if $k_0 \ge 3$ then with $A_1 = L$ and $A_2 = A - A_1 = K \cup \{a_{k_t+1}, \ldots, a_m\}$ we have $n \notin \Sigma(A_1)$ so $n \in \Sigma(A_2)$. Consequently, analogously to (15), we have

$$\sum_{i=k_t+1}^{m} a_i > n - a_{k_0} \sum_{i=0}^{t-1} 2^{-i} > n - \frac{n}{k_0} \sum_{i=0}^{2} 2^{-i} > n - 2n/k_0 \ge n/3\,.$$

This shows that, contrary to our assumption, (14) does hold. The assertions $k_1 = 4$ and $k_2 = 6$ are proved in a similar manner, by making use of the inequality in (11) for $j = 0$ and $j = 1$.

As $n = a_1 + a_2$, $a_2 = a_3 + a_4$ and $a_4 = a_5 + a_6$, we have $a_2 < n/2$, $a_4 < n/4$ and $a_6 < n/8$; furthermore,

$$a_1 + a_3 + a_5 + a_6 = a_1 + a_3 + a_4 = a_1 + a_2 = n \tag{16}$$

and

$$a_2 + a_3 + a_5 + a_6 = a_2 + a_3 + a_4 = 2a_2 < a_1 + a_2 = n\,. \tag{17}$$

Our next aim is to show that

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 > n/16\,. \tag{18}$$

To this end, let $A_1 = \{a_2, a_3, a_4, a_5, a_6\}$ and $A_2 = \{a_1, a_7, a_8, \ldots, a_m\}$. By assumption, either $n \in \Sigma(A_1)$ or $n \in \Sigma(A_2)$.

In the first case, as $\sigma(A_1) - a_4 < n$ by inequality (17), we have

$$2a_2 + a_6 = a_2 + (a_3 + a_4) + a_6 = \sigma(A_1) - a_5 \le n\,. \tag{19}$$

As $a_4 + 3a_6/2 < 7n/16$, inequalities (16) and (19) imply that

$$\begin{aligned}a_1-a_2+a_3-a_4+a_5-a_6 \\ &= a_1+a_3+a_5+a_6-(a_2+a_6/2)-(a_4+3a_6/2) \\ &> n-n/2-7n/16=n/16\,.\end{aligned}$$

In the second case $n\le\sigma(A_2)$ so

$$\begin{aligned}2a_2+a_4 &= a_2+(a_3+a_4)+(a_5+a_6)=\sigma(A_1) \\ &= \sigma(A)-\sigma(A_2)<2n+\sqrt{2n}/8-n=n+\sqrt{2n}/8\,.\end{aligned}\tag{20}$$

As $a_4/2+2a_6<3n/8$, inequalities (16) and (20) imply that

$$\begin{aligned}a_1-a_2+a_3-a_4+a_5-a_6 \\ &= a_1+a_3+a_5+a_6-(a_2+a_4/2)-(a_4/2+2a_6) \\ &> n-n/2-\sqrt{2n}/16-3n/8=n/8-\sqrt{2n}/16\ge n/16\,,\end{aligned}$$

completing the proof of (18).

Armed with (18), the proof of our lemma is easily completed. Indeed, set $A_1=L-\{a_{k_t}\}$ and $A_2=A-A_1=K\cup\{a_{k_t},a_{k_t+1},\ldots,a_m\}$. Then $\sigma(A_1)<n$ so $\sigma(A_2)\ge n$. Hence, by inequality (18),

$$\begin{aligned}\sum_{i=k_t+1}^{m}a_i &= \sigma(A_2)-\sigma(K)-a_{k_t}\ge n-\sum_{j=0}^{t}a_{k_j} \\ &> \sigma(A_1)-\sum_{j=0}^{t}a_{k_j}\ge\sum_{j=0}^{t}(a_{k_j-1}-a_{k_j}) \\ &> a_1-a_2+a_3-a_4+a_5-a_6>n/16\,,\end{aligned}$$

as claimed. ∎

From Lemma 9 and the definitions of k_t and $s(k_t)$, we easily deduce two more lemmas.

LEMMA 10. *Let Q be a set of integers with $\sigma(Q)<n+a_{k_t}-s(k_t)$. Then $n\notin\Sigma(L\cup Q)$.*

P r o o f. Let us assume that $n\in\Sigma(L\cup Q)$. Then there is a set Q_1 such that $Q_1\subset L\cup Q$ and $\sigma(Q_1)=n$. As $\sigma(L)>n$, there is an a_j in L such that $a_j\notin Q_1$. Therefore

$$\begin{aligned}\sigma(Q_1) &\le \sigma(L\cup Q)-a_j\le s(k_t)-a_{k_t}+\sigma(Q) \\ &< s(k_t)-a_{k_t}+n+a_{k_t}-s(k_t)\le n\,. \ \blacksquare\end{aligned}$$

LEMMA 11. $n+a_{k_t}-s(k_t)>\sqrt{2n}/8$.

P r o o f. Let $A_1=L$ and $A_2=A-L$. Then $n\notin\Sigma(A_1)$, so $n\in\Sigma(A_2)$. As $\sigma(A)<2n+\sqrt{2n}/8$, we have

$$s(k_t)=\sigma(A_1)<2n+\sqrt{2n}/8-\sigma(A_2)\le n+\sqrt{2n}/8\,.$$

Therefore, by Lemma 9,

$$n + a_{k_t} - s(k_t) > n + \sqrt{2n}/4 - (n + \sqrt{2n}/8) = \sqrt{2n}/8 \,. \ \blacksquare$$

Before we can complete the proof of Theorem 7, we need one more lemma; this lemma is the heart of the entire proof. For the sake of convenience, let us extend the sequence $a_1 > a_2 > \ldots > a_m$ by the trivial term $a_{m+1} = 0$.

LEMMA 12. *There is an index h with $k_t + 1 \le h \le m + 1$ such that $n \notin \Sigma(L \cup \{a_h\})$ and*

$$\sigma(K) + \sum_{i=k_t+1}^{h-1} a_i < n \,. \tag{21}$$

Proof. We shall consider two cases.

Case 1. $k_t - k_{t-1} \ge 3$. We shall make use of the set

$$B = \{\, a_j + n - s(k_t) : k_{t-1} + 1 \le j \le k_t \,\} \,.$$

First assume that there is an index h with $k_t + 1 \le h \le k_t + (k_t - k_{t-1})$ such that $a_h \notin B$. It is easy to check that $n \notin \Sigma(L \cup \{a_h\})$. Indeed, if $n \in \Sigma(L \cup \{a_h\})$, i.e. there is a set $Q \subset L \cup \{a_h\}$ such that $n = \sigma(Q)$, then $Q = L \cup \{a_h\} - \{a_i\}$ for some a_i in L, so $n = \sigma(Q) = s(k_t) + a_h - a_i$. Since $a_h \notin B$, we have $i < k_{t-1}$, so $a_i > a_{t-1} > a_{k_{t-1}+1} + a_{k_{t-1}+2}$. Thus

$$\begin{aligned} n &= s(k_t) + a_h - a_i < s(k_t) + a_h - a_{k_{t-1}} \\ &< s(k_t) + a_h - a_{k_{t-1}+1} - a_{k_{t-1}+2} < s(k_t) - a_{k_t} < n \,, \end{aligned}$$

which is a contradiction. Therefore $n \notin \Sigma(L \cup \{a_h\})$, showing the first assertion of the lemma. To see the second assertion, note that as $j \le k_t + (k_t - k_{t-1})$, we have

$$\sigma(K) + \sum_{i=k_t+1}^{j-1} a_i < \sum_{i=1}^{k_{t-1}} a_i - \sigma(K) + \sum_{i=k_{t-1}+1}^{k_t-1} a_i = \sigma(L) - a_{k_t} < n \,.$$

Assume now that $a_j \in B$ for all j with $k_t + 1 \le j \le k_t + (k_t - k_{t-1})$ and so $B = \{a_{k_t+1}, a_{k_t+2}, \ldots, a_{2k_t-k_{t-1}}\}$. Then, in particular, $a_{2k_t-k_{t-1}} = a_{k_t} + n - s(k_t)$. We shall show that $h = 2k_t - k_t + 1$ will do. Clearly,

$$\sigma(L\cup\{a_h\}) - a_{k_t} = \sigma(L\cup\{a_{2k_t-k_{t-1}+1}\}) - a_{k_t} < \sigma(L) + a_{2k_t-k_{t-1}} - a_{k_t} = n \,.$$

As a_{k_t} is the smallest term in L and $\sigma(L) > n$, this implies that $n \notin \Sigma(L \cup \{a_h\})$. Furthermore,

$$\{a_{k_0-1}, a_{k_1-1}, \ldots, a_{k_{t-1}-1}\} \subset L - \{a_{k_{t-1}+1}, a_{k_{t-1}+2}, \ldots, a_{k_t}\} \,,$$

so

$$\sigma(K)=\sum_{i=0}^{t-1}a_{k_i}<\sum_{i=0}^{t-1}a_{k_i-1}\le\sigma(L)-\sum_{i=k_{t-1}+1}^{k_t}a_i$$

implying

$$\begin{aligned}\sigma(K)+\sum_{j=k_t+1}^{h-1}a_j&<\sigma(L)-\sum_{i=k_{t-1}+1}^{k_t}a_i+\sum_{j=k_t+1}^{h-1}a_j\\&=\sigma(L)-\sum_{i=1}^{k_t-k_{t-1}}(a_{k_{t-1}+i}-a_{k_t+i})\\&\le\sigma(L)+a_{h-1}-a_{k_t}=n\,.\end{aligned}$$

Therefore Lemma 12 holds if $k_t-k_{t-1}\ge 3$.

Case 2. $k_t-k_{t-1}=2$. This time we set

$$B=\{\,a_i+n-s(k_t):k_{t-2}+1\le i\le k_t,\ i\ne k_{t-1}\}$$

and $b=k_t-k_{t-2}-1=|B|$. Since $s(k_t)=\sum_{i=1}^{k_t}a_i-\sum_{i=0}^{t-1}a_{k_i}<n+\sqrt{2n}/8$, $n=s(k_{t-1})=\sum_{i=1}^{k_{t-1}}a_i-\sum_{i=0}^{t-2}a_{k_i}$, and, by Lemma 9, $a_{k_t}>\sqrt{2n}/4$, we have

$$\begin{aligned}(22)\qquad a_{k_{t-1}}&=s(k_{t-1})-\sum_{i=1}^{k_{t-1}-1}a_i+\sum_{i=0}^{t-2}a_{k_i}\\&=s(k_{t-1})-\sum_{i=1}^{k_{t-1}}a_i+\sum_{i=0}^{t-1}a_{k_i}=n-s(k_t)+a_{k_t-1}+a_{k_t}\\&>a_{k_t-1}+a_{k_t}-\sqrt{2n}/8>a_{k_t-1}+a_{k_t}/2\,.\end{aligned}$$

Let us assume first that $k_{t-2}-k_{t-1}\ge 4$ so that $b\ge 5$ and $a_{k_{t-2}}\ge a_{k_{t-2}+1}+a_{k_{t-2}+2}+a_{k_{t-2}+3}$. Let h be the first index in the interval $k_t+1\le h\le k_t+b+1$ such that $a_h\notin B$. Then $n\notin\Sigma(L\cup\{a_h\})$. Indeed, suppose that $n\in\Sigma(L\cup\{a_h\})$, i.e. there is a set $Q\subset L\cup\{a_h\}$ with $n=\sigma(Q)$. As

$$\sigma(L\cup\{a_h\}-\{a_{k_t},a_{k_t-1}\})=s(k_t)-a_{k_t}-a_{k_t-1}+a_h<s(k_t)-a_{k_t}<n\,,$$

the set Q is of the form $Q=L\cup\{a_h\}-\{a_i\}$ for some $a_i\in L$ so $n=\sigma(Q)=s(k_t)+a_h-a_i$. Therefore $a_h=a_i+n-s(k_t)$, and as $a_h\notin B$, we have $i\notin[k_{t-2}+1,\ k_t]$ so $i\le k_{t-2}$ and $a_i\ge a_{k_{t-2}}>a_{k_{t-2}+1}+a_{k_{t-2}+2}$. Thus

$$\begin{aligned}n&=s(k_t)+a_h-a_i<s(k_t)+a_h-a_{k_{t-2}+1}-a_{k_{t-2}+2}\\&<s(k_t)-a_{k_{t-2}+1}<s(k_t)-a_{k_t}<n\,,\end{aligned}$$

which is a contradiction. Therefore $n\notin\Sigma(L\cup\{a_h\})$, showing the first assertion of the lemma.

Let us turn to the proof of (21). If $h \le k_t + b - 1$, we can see (21) as follows:

$$\begin{aligned}\sigma(K) + \sum_{i=k_t+1}^{h-1} a_i &= \sum_{i=0}^{t-1} a_{k_i} + \sum_{i=k_t+1}^{h-1} a_i < \sum_{i=0}^{t-1} a_{k_i-1} + \sum_{i=k_t+1}^{h-1} a_i \\ &\le \Big(\sigma(L) - \sum_{j=k_{t-2}+1}^{k_{t-1}-2} a_j - \sum_{j=k_{t-1}+1}^{k_t} a_j\Big) + \sum_{i=k_t+1}^{h-1} a_i \\ &= \sigma(L) - \Big(\sum_{j=k_{t-2}+1}^{k_{t-1}-2} a_j + \sum_{j=k_{t-1}+1}^{k_t-1} a_j - \sum_{i=k_t+1}^{h-1} a_i\Big) - a_{k_t} \\ &< \sigma(L) - a_{k_t} = s(k_t) - a_{k_t} < n\,.\end{aligned}$$

Thus we may suppose that $h = k_t + b$ or $h = k_t + b + 1$. Then $a_{k_t+1}, a_{k_t+2}, \ldots, a_{h-1} \in B$ so $a_{h-1} = a_l + n - s(k_t)$ for some l with $l - k_{t-2} \ge h - k_t$. Hence $l \ge h - k_t + k_{t-2} \ge b + k_{t-2} = k_t - 1$. Therefore, arguing as above, by (22) we have

$$\begin{aligned}\sigma(K) + \sum_{i=k_t+1}^{h-1} a_i &< \sigma(L) - \sum_{j=k_{t-2}+1}^{k_{t-1}-2} a_j - \sum_{j=k_{t-1}+1}^{k_t} a_j + \sum_{i=k_t+1}^{h-1} a_i \\ &< s(k_t) - \sum_{j=k_{t-2}+1}^{k_{t-2}+2} a_j - \sum_{j=k_{t-2}+3}^{k_{t-1}-2} a_j - \sum_{j=k_{t-1}+1}^{k_t} a_j \\ &\quad + \sum_{i=k_t+1}^{k_t+3} a_i + \sum_{i=k_t+4}^{h-2} a_i + a_{h-1} \\ &= s(k_t) - \Big(\sum_{j=k_{t-2}+1}^{k_{t-2}+2} a_j - \sum_{i=k_t+1}^{k_t+3} a_i\Big) \\ &\quad - \Big(\sum_{j=k_{t-2}+3}^{k_{t-1}-2} a_j + \sum_{j=k_{t-1}+1}^{k_t} a_j - a_l - \sum_{i=k_t+4}^{h-2} a_i\Big) \\ &\quad + a_{h-1} - a_l\,.\end{aligned}$$

The sums in the parentheses are non-negative: the first by inequality (22), and the second as it has $(k_{t-1} - k_{t-2} - 4) + (k_t - k_{t-1})$ positive terms and $1 + (h - k_t - 5) \le (k_{t-1} - k_{t-2} - 4) + (k_t - k_{t-1})$ smaller negative terms. Thus,

$$\sigma(K) + \sum_{i=k_t+1}^{h-1} a_i < s(k_t) + a_{h-1} - a_l = n\,.$$

Hence the lemma holds if $k_{t-2} - k_{t-1} \ge 4$.

Let us assume then that $k_{t-2} - k_{t-1} \le 3$. Suppose first that there is an h such that $k_t + 1 \le h \le k_t + b - 1$ and $a_h \notin B$. Then, arguing as above, we find that $n \notin \Sigma(L \cup \{a_h\})$, and, as $h \le k_t + b - 1$,

$$\sigma(K) + \sum_{i=k_t+1}^{h-1} a_i < s(k_t) - a_t < n\,,$$

as required.

Suppose then that $a_j \in B$ for all j with $k_t + 1 \le j \le k_t + b - 1$. Then either (i) $a_{k_t+b-1} = a_{k_t-1} + n - s(k_t)$ or (ii) $a_{k_t+b-1} = a_{k_t} + n - s(k_t)$.

To complete the proof of our lemma, we shall show that the assertions of the lemma hold in these two cases.

(i) Assume that $a_{k_t+b-1} = a_{k_t-1} + n - s(k_t)$. Since $a_{k_t+b-2}, a_{k_t+b-3}, \ldots$ $\ldots, a_{k_t+1}$ are all in B, they are all of the form $a_i + n - s(k_t)$, where $a_i \in L$. We have $a_{k_t+b-2} \ge a_{k_t-3}$, $a_{k_t+b-3} \ge a_{k_t-4}$, $\ldots$ and

$$a_{k_t+1} \ge a_{k_{t-2}+1} + n - s(k_t)\,. \tag{23$'$}$$

In fact, as $d_{k_t+1} = a_i + n - s(k_t)$ for some $i \ge k_{t-2} + 1$, we have equality in (23′):

$$a_{k_t+1} = a_{k_{t-2}+1} + n - s(k_t)\,. \tag{23}$$

Similarly, if $k_{t-2} + 2 \ne k_{t-1}$ then

$$a_{k_t+2} = a_{k_{t-2}+2} + n - s(k_t)\,, \tag{24}$$

and if $k_{t-2} + 2 = k_{t-1}$ then

$$a_{k_t+2} = a_{k_{t-1}+1} + n - s(k_t)\,. \tag{25}$$

Inequalities (24) and (25) imply

$$s(k_t) + a_{k_t+2} - a_{k_t} > n\,, \tag{26}$$

and as, by Lemma 9, $s(k_t) < n + \sqrt{2n}/8$ and $a_{k_t} > \sqrt{2n}/4$ inequality (26) implies that

$$a_{k_t+1} > a_{k_t+2} > a_{k_t} + n - n - \sqrt{2n}/8 > \sqrt{2n}/8\,. \tag{27}$$

Let us partition A by setting $A_1 = L \cup \{a_{k_t+1}, a_{k_t+2}\} - \{a_{k_t}\}$ and $A_2 = A - A_1$. Then, by (26) and (27), we have $\sigma(A_1) > n + \sqrt{2n}/8$, so $\sigma(A_2) < n$ and thus $n \in \Sigma(A_1)$. Let Q be a subset of A_1 such that $n = \sigma(Q)$. Since $s(k_t) - a_{k_t} < 0$, inequality (26) implies that there is an $a_j \in L - \{a_{k_t}\}$ such that $Q = A_1 - \{a_j\}$. Therefore, by (23),

$$\begin{aligned} n = \sigma(Q) = \sigma(A_1) - a_j &= \sigma(L) + a_{k_t+1} + a_{k_t+2} - a_{k_t} - a_j \\ &= s(k_t) + a_{k_t+1} + a_{k_t+2} - a_{k_t} - a_j \\ &= s(k_t) + a_{k_{t-2}+1} + n - s(k_t) + a_{k_t+2} - a_{k_t} - a_j \\ &= n + (a_{k_{t-2}+1} - a_j + a_{k_t+2} - a_{k_t})\,. \end{aligned}$$

However, we claim that

$$a_{k_{t-2}+1} - a_j + a_{k_t+2} - a_{k_t} \neq 0\,. \tag{28}$$

It is trivial that (28) holds for $a_j = a_{k_{t-2}+1}$. If $a_j = a_{k_{t-2}+2}$, then $k_{t-2}+2 \neq k_{t-1}$, so, by (23) and (24), $a_{k_{t-2}+1} - a_j + a_{k_t+2} - a_{k_t} = a_{k_{t-2}+1} + n - s(k_t) - a_{k_t} > a_{k_{t-2}+1} + n - s(k_t) - a_{k_t+1} = 0$. If $a_j = a_{k_{t-1}+1}$, then, by (26) and (27), $a_{k_{t-2}+1} + a_{k_t+2} > a_{k_{t-1}} + \sqrt{2n}/8 > a_j + a_{k_t}$, so, inequality (28) holds again. In fact, these are all the cases since $k_{t-1} - k_{t-2} \le 3$ and $k_t - k_{t-1} = 2$. Thus (28) does hold, which is a contradiction.

(ii) Assume now that $a_{k_t+b-1} = a_{k_t} + n - s(k_t)$. We shall show that $h = k_t + b$ will do for the claim in the lemma. Clearly

$$\sigma(L \cup \{a_h\}) - a_{k_t} = s(k_t) + a_{k_t+b} - a_{k_t} < s(k_t) + a_{k_t+b-1} - a_{k_t} = n\,.$$

As $\sigma(L) > n$ and a_{k_t} is the smallest term in L, this implies $n \notin \Sigma(L \cup \{a_{k_t+b}\})$, showing the first assertion of the lemma. To see the second assertion, note that

$$\begin{aligned}\sigma(K) + \sum_{i=k_t+1}^{h-1} a_i &< s(k_t) - \sum_{i=k_{t-2}+1}^{k_{t-1}-2} a_i - \sum_{i=k_{t-1}+1}^{k_t} a_i + \sum_{i=k_t+1}^{k_t+b-1} a_i \\ &= s(k_t) - \Big(\sum_{i=k_{t-2}+1}^{k_{t-1}-2} a_i + \sum_{i=k_{t-1}+1}^{k_t-1} a_i - \sum_{i=k_t+1}^{k_t+b-2} a_i \Big) \\ &\quad - a_{k_t} + a_{k_t+b-1} \\ &< s(k_t) - a_{k_t} + a_{k_t+b-1} = n\,,\end{aligned}$$

since the sum in parentheses is positive as there are $b-2$ positive terms and $b-2$ smaller negative terms. This completes the proof of the lemma. ■

Armed with Lemma 12, the proof of Theorem 6 is easily completed. Let h be the index whose existence is guaranteed by Lemma 12.

Assume first that $a_h \ge \sqrt{2n}/8$. Let $A_1 = L \cup \{a_h\}$ and $A_2 = A - A_1 = K \cup \{a_{k_t+1}, \ldots, a_m\} - \{a_h\}$. Then $\sigma(A_1) > n + \sqrt{2n}/8$, so $\sigma(A_2) < n$, and so $n \notin \Sigma(A_2)$. However, by Lemma 11, $n \notin \Sigma(A_1)$. This contradicts our assumption on the set A.

Let us assume then that $a_h < \sqrt{2n}/8$. Then $\sum_{i=h}^{m} a_i \ge \sqrt{2n}/8$. Indeed, otherwise let $A_1 = L \cup \{a_h, \ldots, a_m\}$ and $A_2 = A - A_1 = K \cup \{a_{k_t+1}, \ldots, a_{h-1}\}$. Lemmas 10 and 11 imply that $n \notin \Sigma(A_1)$. However, by Lemma 12,

$$\sigma(A_2) = \sigma(K) + \sum_{i=k_t+1}^{h-1} a_i < n\,,$$

so $n \notin \Sigma(A_2)$, contradicting our assumption. Therefore $\sum_{i=h}^{m} a_i \geq \sqrt{2n}/8$, as claimed.

Since $a_h < \sqrt{2n}/8$, $0 < s(k_t) - n < \sqrt{2n}/8$ and $a_h > a_{h+1} > \ldots > a_m$, there exists an index l, $h \leq l \leq m$, such that

$$n + \sqrt{2n}/8 - s(k_t) \leq \sum_{i=h}^{l} a_i < n + \sqrt{2n}/4 - s(k_t) \leq n + a_{k_t} - s(k_t)\,.$$

Let $Q = \{a_h, a_{h+1}, \ldots, a_l\}$ so that $\sigma(Q) < n + a_{k_t} - s(k_t)$. Set $A_1 = L \cup Q$ and

$$A_2 = A - A_1 = K \cup \{a_{k_t+1}, \ldots, a_{h-1}\} \cup \{a_{l+1}, \ldots, a_m\}\,.$$

Then, by Lemma 10, $n \notin \Sigma(A_1)$. However, by the definition of l,

$$\sigma(A_1) = s(k_t) + \sum_{i=h}^{l} a_i \geq n + \sqrt{2n}/8\,,$$

so $\sigma(A_2) < n$ and hence $n \notin \Sigma(A_2)$, contradicting our assumption on A and completing the proof of the theorem. ■

It is tempting to conjecture that $g_2(n) = f_2(n)(f_2(n)+1)/2$ but, if true, this seems to be rather difficult. It may be easier to show that, as we suspect, $(g_2(n) - 2n)/\sqrt{n} \to \infty$.

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Received on 15.12.1992 (2353)