On Sidon sequences of even orders

by

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Let $h \geq 2$ be an integer. A set $A$ of positive integers is called a $B_h$-sequence if all sums $a_1 + \ldots + a_h$, where $a_i \in A$ ($i = 1, \ldots, h$), are distinct up to rearrangements of the summands. A $B_h$-sequence is also called a Sidon sequence of order $h$ [7]. Sidon was led to consider such sequences in connection with the theory of Fourier series. In [7] he raised the question of how many terms not exceeding $n$ a $B_h$-sequence may have. Some earlier important results on $B_h$-sequences may be found in [2, Ch. 2].

Let $A$ be a $B_h$-sequence. Denote by $A(n)$ the cardinality of $A \cap [0, n]$. For any positive integer $r$, denote by $rA$ the set of integers $\sum_{i=1}^{r} a_i$ where $a_i \in A$ ($i = 1, \ldots, r$). It follows from the definition of $B_h$-sequences that

$$hn \geq (hA)(hn) \geq \binom{A(n)}{h},$$

which implies $A(n) = O(h^{\sqrt{n}})$.

Erdős [2, 8] and Krückeberg [5] showed that there exists a $B_2$-sequence $A$ such that

$$\limsup_{n \to \infty} \frac{A(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{2}}.$$

On the other hand, Erdős [2] also proved that, for any $B_2$-sequence $A$,

$$\liminf_{n \to \infty} A(n) \sqrt{\frac{\log n}{n}} < \infty.$$

Nash extended the result to $B_4$-sequences. He showed [6] that, for any $B_4$-sequence $A$,

$$\liminf_{n \to \infty} A(n) \sqrt[4]{\frac{\log n}{n}} < \infty.$$

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A natural conjecture [4] is as follows.

**Conjecture.** If \( A \) is a \( B_h \)-sequence, then

\[
\liminf_{n \to \infty} A(n) \sqrt[n]{\frac{\log n}{n}} < \infty. 
\]

For \( h \) odd, no results of this kind have been proved as far as the author knows.

For \( h \) even (\( h = 2k > 4 \)), Jia [4] showed that, if \( A(n^2) \leq A(n)^2 \), then

\[
\liminf_{n \to \infty} A(n) \sqrt[2k]{\frac{\log n}{n}} < \infty. 
\]

As mentioned in [4], the condition \( A(n^2) \leq A(n)^2 \) does not hold for all \( B_{2k} \)-sequences.

The author [1] proved that a weaker result of a similar nature does hold, namely

\[
\liminf_{n \to \infty} \frac{A(n)}{2k} \sqrt[k]{\frac{\log n}{n}} < \infty, 
\]

which has been improved by Helm [3] to

\[
\liminf_{n \to \infty} \frac{A(n)}{2k} \sqrt[k]{\frac{\log n}{n}} < \infty. 
\]

In this paper, we give an affirmative partial answer to the Conjecture.

**Theorem.** For any \( B_{2k} \)-sequence \( A \),

\[
\liminf_{n \to \infty} A(n) \sqrt[2k]{\frac{\log n}{n}} < \infty. 
\]

**Corollary.** Let \( A = \{a_1 < a_2 < \ldots < a_n < \ldots\} \) be an infinite \( B_{2k} \)-sequence. Then

\[
\limsup_{n \to \infty} \frac{a_n}{n^{2k} \log n} > 0. 
\]

First some notations and lemmas.

Let \( A \) be a \( B_{2k} \)-sequence and \( r \) be a positive integer, \( 1 \leq r \leq 2k \). For any \( x = x_1 + \ldots + x_r \) where \( x_s \in A \) (\( s = 1, \ldots, r \)), denote by \( \tau \) the set \( \{x_1, \ldots, x_r\} \) (counting the multiplicities of the appearance of \( x_i \)'s) and

\[
r * A = \{x = x_1 + \ldots + x_r : x_s \in A, \ x_s \neq x_t, \ 1 \leq s, t \leq r\}.
\]

Note that, since \( A \) is a \( B_{2k} \)-sequence, \( x = y \in rA \) if and only if \( \tau = \gamma \).

Let \( A \) be a \( B_{2k} \)-sequence. Let \( n \) be a large integer fixed hereafter and \( u = \lceil n^{1/(2k)} \rceil \). For any sequence \( B \), denote by \( B_i(n) \), or simply \( B_i \), the set \( B \cap ((i-1)kn, ikn], \ i = 1, 2, \ldots \).
On \( B_{2k}\text{-sequences} \)

Set \( D = A \cap (0, ukn] \), \( C = k * D \), \( c_i = |C_i| \), and

\[
\tau(n) = \min_{n \leq m \leq un} \frac{A(m)}{2 \sqrt{m}}.
\]

**Lemma 1.**

\[
\tau(n)^{2k} n \log n = O\left( \sum_{i=1}^{u} c_i^2 \right).
\]

**Proof.** Note that

\[
\left( \sum_{i=1}^{u} \frac{c_i}{\sqrt{i}} \right)^2 \leq \left( \sum_{i=1}^{u} \frac{1}{i} \right) \left( \sum_{i=1}^{u} c_i^2 \right) \leq 2 \log u \sum_{i=1}^{u} c_i^2.
\]

On the other hand, for any positive integer \( i \) (\( 1 \leq i \leq u \)),

\[
C(ikn) \geq \binom{A(in)}{k} \geq cA(in)^k,
\]

where \( c > 0 \) is an absolute constant depending only on \( k \), and

\[
A(in)^k = \left( \frac{n}{2 \sqrt{m}} \right)^k \sqrt{m} \geq \tau(n)^{k} \sqrt{m}.
\]

Hence,

\[
\sum_{i=1}^{u} \frac{c_i}{\sqrt{i}} = \sum_{i=1}^{u} \left( \frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}} \right) \sum_{j=1}^{i} c_j + \frac{1}{\sqrt{u+1}} \sum_{j=1}^{u} c_j
\]

\[
\geq \frac{1}{8} \sum_{i=1}^{u} \frac{1}{i^{3/2}} C(ikn) \geq \frac{c}{8} \sum_{i=1}^{u} \frac{A(in)^k}{i^{3/2}}
\]

\[
\geq \frac{c}{8} \tau(n)^k \sqrt{n} \sum_{i=1}^{u} \frac{1}{i} \geq c \tau(n)^k \sqrt{n} \log u.
\]

So,

\[
\tau(n)^{2k} n \log u \leq O\left( \sum_{i=1}^{u} c_i^2 \right).
\]

As \( u = \lfloor n^{1/(2k)} \rfloor \), Lemma 1 follows. \( \blacksquare \)

**Lemma 2.**

\[
\sum_{i=1}^{u} c_i^2 = O(n).
\]

**Proof.** Set \( W = \{(x, y) \in (k * A) \times (k * A) : \pi \cap \overline{\eta} = \emptyset\} \). Define a map \( f : W \to (-\infty, \infty) \) by \( (x, y) \mapsto x - y \). If \( f(x', y') = f(x, y) \), i.e., \( x' - y' = x - y \), then \( x' + y = x + y' \). As \( A \) is a \( B_{2k}\text{-sequence} \), \( \overline{x'} \cup \overline{\eta} = \pi \cup \overline{\eta'} \).

Since \( \pi \cap \overline{\eta} = \overline{x'} \cap \overline{\eta'} = \emptyset \), we have \( x = x' \) and \( y = y' \). Thus \( f \) is one-to-one.
For any integer \( r, 1 \leq r \leq k \), let \( V(r, D) = \{(x, y) \in (r \ast D) \times (r \ast D) : -kn < x - y < kn\} \).

Then
\[
\sum_{i=1}^{u} c_i^2 \leq |V(k, D)|.
\]

Write \( V(k, D) = \bigcup_{j=0}^{k} V_j(k, D) \) (disjoint union) where
\[
V_j(k, D) = \{ (x, y) \in V(k, D) : |\overline{x} \cap \overline{y}| = j \}, \quad 0 \leq j \leq k.
\]

Then
\[
|V(k, D)| = \sum_{j=0}^{k} |V_j(k, D)|.
\]

For any \( B_{2k} \)-sequence \( B \) and any integers \( j, r \) \((0 \leq j, r \leq k)\), let \( W(r, D, j, B) \) be the set of 4-tuples \((x, b, b, y)\) such that
(a) \( x, y \in r \ast D \) with \(-kn < x - y < kn\),
(b) \( b \in j \ast B \), and
(c) \( b \cap x = b \cap y = x \cap y = \emptyset \).

If \( j = 0 \), we simply write \( W(r, D) \) instead.

Note that \( |V_j(k, D)| = O(|W(k - j, D, j, D)|) \). Combining this with (1) and (2), we have
\[
\sum_{i=1}^{u} c_i^2 = O\left( \sum_{j=0}^{k} |W(k - j, D, j, D)| \right).
\]

Hence it suffices to show that, for all \( j \) \((0 \leq j \leq k)\),
\[
|W(k - j, D, j, D)| = O(n).
\]

Case \( j = 0 \). Then \( W(k, D, 0, D) = W(k, D) \subseteq W \) and \( f(W(k, D)) \subseteq (-kn, kn) \). Hence,
\[
|W(k, D, 0, D)| \leq 2kn.
\]

Case \( j = k \). Then \( V(0, D, k, D) = \{(b, b) : b \in k \ast D\} \). Hence,
\[
|W(0, D, k, D)| = |k \ast D| \leq |D|^k = A(u^n k)^k = O((2\sqrt{u} kn)^k) = O(n).
\]

Case \( 1 \leq j \leq k - 1 \). Let \( I = \{i : 1 \leq i \leq u \text{ and } |D_i| \geq 2k\} \). Set \( B = \bigcup_{i \in I} D_i \) and \( B' = D \setminus B \). We divide our proof into two subcases.

Subcase I: \(|B| \leq |B'|\). Note \( B' = \sum_{1 \leq i \leq u; i \notin I} D_i \). So \(|B'| \leq u(2k - 1)\). Hence,
\[
|D| = |B'| + |B| \leq 2|B'| \leq 4ku.
\]
Therefore,

\[ |W(k - j, D, j, D)| \leq |(2k - j) \ast D| \leq |D|^{2k-j} \leq (4ku)^{2k-j} = O(n). \]

Subcase II: \(|B'| \leq |B|\). Note \(B\) is also an \(B_{2k}\)-sequence. We claim

\[ |W(k - j, D, j, B)| = O(n). \]

To show (7), define a map \(v : W(k - j, D, j, B) \rightarrow W\) as follows. Let \((x, b, b, y) \in W(k - j, D, j, B)\). As \(b \in j \ast B\), \(b = \sum_{s=1}^{j} b_s\) where \(b_s \in D_{i_s}\) and \(i_s \in I\) \((s = 1, \ldots, j)\). Since \(|D_{i_s}| \geq 2k\), we can choose \(b_s' \in D_{i_s}\) \((s = 1, \ldots, j)\) so that

(i) all \(b_s'\)'s are distinct and
(ii) \(b_s' \not\in \pi \cup \nu \cup \bar{\pi}, \ s = 1, \ldots, j.\)

Let \(b' = \sum_{s=1}^{j} b_s'\). Then \((x, b, b', y) \in W\). Define \(v : W(k - j, D, j, D) \rightarrow W\) by \((x, b, b, y) \mapsto (x, b, b', y)\). Clearly, \(v\) is well defined and one-to-one. Furthermore, by the choice of \(b_s'\), \(-kn \leq b_s - b_s' \leq kn\) for all \(1 \leq s \leq j\). So, \(-jkn \leq b - b' \leq jkn\). Hence,

\[ -(j+1)kn < (x + b) - (b' + y) < jkn + kn. \]

Thus,

\[ |W(k - j, D, j, B)| = |f(v(W(k - j, D, j, B)))| \leq 2(j + 1)kn \leq 2k^2n, \]

which is (7).

Note

\[ W(k - j, D, j, D) \]

\[ = \bigcup_{(x,y) \in W(k-j,D)} \{(x,b,b,y) : b \in j \ast D, \pi \cap (\pi \cup \nu) = \emptyset\} \]

\[ = \bigcup_{(x,y) \in W(k-j,D)} \{(x,b,b,y) : b \in j \ast (D \setminus (\pi \cup \nu))\}, \]

and similarly,

\[ W(k - j, D, j, B) = \bigcup_{(x,y) \in W(k-j,D)} \{(x,b,b,y) : b \in j \ast (B \setminus (\pi \cup \nu))\}. \]

On the other hand, for any \(z = (x, y) \in W(k - j, D)\), it follows from the assumption that \(|B'| \leq |B|\), and

\[ |D \setminus \pi| = |(B \setminus \pi)| + |B' \setminus \pi| \leq 2|B \setminus \pi| + |z| \leq 2|B \setminus \pi| + 2k. \]

Therefore,

\[ |j \ast (D \setminus \pi)| = \binom{|D \setminus \pi|}{j} \leq \binom{2|B \setminus \pi| + 2k}{j} \leq c(j \ast (B \setminus \pi)) + 1, \]

where \(c\) is an absolute constant depending only on \(j\), hence only on \(k\).
Combining (7)–(10), we have
\[
|W(k - j, D, j, D)| = \sum_{z \in W(k - j, D)} |j * (D \setminus z)|
\]
\[
\leq \sum_{z \in W(k - j, D)} c(|j * (B \setminus z)| + 1)
\]
\[
\leq c\left( \sum_{z \in W(k - j, D)} |j * (B \setminus z)| + \sum_{z \in W(k - j, D)} 1 \right)
\]
\[
= c(|W(k - j, D, j, B)| + |W(k - j, D)|) = O(n).
\]

Now Lemma 2 follows from (3)–(5) and (11).

**Proof of the Theorem.** It follows from Lemmas 1 and 2 that \(\tau(n)2^k \log n = O(1)\). Hence,
\[
\liminf_{n \to \infty} A(n) \frac{2k \sqrt{\log n}}{n} = \liminf_{n \to \infty} \inf_{m \leq un} \left( A(m) \frac{2k \sqrt{\log m}}{m} \right)
\]
\[
\leq \liminf_{n \to \infty} \inf_{m \leq un} \left( A(m) \frac{2k \sqrt{\log(un)}}{2k \sqrt{m}} \right)
\]
\[
\leq 2 \liminf_{n \to \infty} \tau(n) \frac{2k}{\log n} < \infty.
\]

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**References**