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On Sidon sequences of even orders

by

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Let $h \geq 2$ be an integer. A set A of positive integers is called a B_h-*sequence* if all sums $a_1 + \ldots + a_h$, where $a_i \in A$ $(i = 1, \ldots, h)$, are distinct up to rearrangements of the summands. A B_h-sequence is also called a *Sidon sequence of order* h [7]. Sidon was led to consider such sequences in connection with the theory of Fourier series. In [7] he raised the question of how many terms not exceeding n a B_h-sequence may have. Some earlier important results on B_h-sequences may be found in [2, Ch. 2].

Let A be a B_h-sequence. Denote by $A(n)$ the cardinality of $A \cap [0, n]$. For any positive integer r, denote by rA the set of integers $\sum_{i=1}^{r} a_i$ where $a_i \in A$ $(i = 1, \ldots, r)$. It follows from the definition of B_h-sequences that

$$hn \geq (hA)(hn) \geq \binom{A(n)}{h},$$

which implies

$$A(n) = O(\sqrt[h]{n}).$$

Erdős [2, 8] and Krückeberg [5] showed that there exists a B_2-sequence A such that

$$\limsup_{n\to\infty} \frac{A(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{2}}.$$

On the other hand, Erdős [2] also proved that, for any B_2-sequence A,

$$\liminf_{n\to\infty} A(n)\sqrt{\frac{\log n}{n}} < \infty.$$

Nash extended the result to B_4-sequences. He showed [6] that, for any B_4-sequence A,

$$\liminf_{n\to\infty} A(n)\sqrt[4]{\frac{\log n}{n}} < \infty.$$

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A natural conjecture [4] is as follows.

Conjecture. *If A is a B_h-sequence, then*

$$\liminf_{n\to\infty} A(n)\sqrt[h]{\frac{\log n}{n}} < \infty\,.$$

For h odd, no results of this kind have been proved as far as the author knows.

For h even ($h = 2k > 4$), Jia [4] showed that, if $A(n^2) \le A(n)^2$, then

$$\liminf_{n\to\infty} A(n)\sqrt[2k]{\frac{\log n}{n}} < \infty\,.$$

As mentioned in [4], the condition $A(n^2) \le A(n)^2$ does not hold for all B_{2k}-sequences.

The author [1] proved that a weaker result of a similar nature does hold, namely

$$\liminf_{n\to\infty} \frac{A(n)}{\sqrt[2k]{n}}\sqrt[4k-4]{\log n} < \infty\,,$$

which has been improved by Helm [3] to

$$\liminf_{n\to\infty} \frac{A(n)}{\sqrt[2k]{n}}\sqrt[3k-1]{\log n} < \infty\,.$$

In this paper, we give an affirmative partial answer to the Conjecture.

Theorem. *For any B_{2k}-sequence A,*

$$\liminf_{n\to\infty} A(n)\sqrt[2k]{\frac{\log n}{n}} < \infty\,.$$

Corollary. *Let $A = \{a_1 < a_2 < \ldots < a_n < \ldots\}$ be an infinite B_{2k}-sequence. Then*

$$\limsup_{n\to\infty} \frac{a_n}{n^{2k}\log n} > 0\,.$$

First some notations and lemmas.

Let A be a B_{2k}-sequence and r be a positive integer, $1 \le r \le 2k$. For any $x = x_1 + \ldots + x_r$ where $x_s \in A$ ($s = 1,\ldots,r$), denote by $\overline{x}$ the set $\{x_1,\ldots,x_r\}$ (counting the multiplicities of the appearance of x_i's) and

$$r * A = \{x = x_1 + \ldots + x_r : x_s \in A,\ x_s \ne x_t,\ 1 \le s,t \le r\}\,.$$

Note that, since A is a B_{2k}-sequence, $x = y \in rA$ if and only if $\overline{x} = \overline{y}$.

Let A be a B_{2k}-sequence. Let n be a large integer fixed hereafter and $u = \lfloor n^{1/(2k)}\rfloor$. For any sequence B, denote by $B_i(n)$, or simply B_i, the set $B \cap ((i-1)kn, ikn]$, $i = 1,2,\ldots$

Set $D = A \cap (0, ukn]$, $C = k * D$, $c_i = |C_i|$, and

$$\tau(n) = \min_{n \le m \le un} \frac{A(m)}{\sqrt[2k]{m}}\,.$$

LEMMA 1.

$$\tau(n)^{2k} n \log n = O\Big(\sum_{i=1}^{u} c_i^2\Big)\,.$$

Proof. Note that

$$\Big(\sum_{i=1}^{u} \frac{c_i}{\sqrt{i}}\Big)^2 \le \Big(\sum_{i=1}^{u} \frac{1}{i}\Big)\Big(\sum_{i=1}^{u} c_i^2\Big) \le 2\log u \sum_{i=1}^{u} c_i^2\,.$$

On the other hand, for any positive integer i ($1 \le i \le u$),

$$C(ikn) \ge \binom{A(in)}{k} \ge cA(in)^k\,,$$

where $c > 0$ is an absolute constant depending only on k, and

$$A(in)^k = \Big(\frac{A(in)}{\sqrt[2k]{in}}\Big)^k \sqrt{in} \ge \tau(n)^k \sqrt{in}\,.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{u} \frac{c_i}{\sqrt{i}} &= \sum_{i=1}^{u} \Big(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}}\Big) \sum_{j=1}^{i} c_j + \frac{1}{\sqrt{u+1}} \sum_{j=1}^{u} c_j \\ &\ge \frac{1}{8} \sum_{i=1}^{u} \frac{1}{i^{3/2}} C(ikn) \ge \frac{c}{8} \sum_{i=1}^{u} \frac{A(in)^k}{i^{3/2}} \\ &\ge \frac{c}{8} \tau(n)^k \sqrt{n} \sum_{i=1}^{u} \frac{1}{i} \ge c\tau(n)^k \sqrt{n} \log u\,.\end{aligned}$$

So,

$$\tau(n)^{2k} n \log u \le O\Big(\sum_{i=1}^{u} c_i^2\Big)\,.$$

As $u = \lfloor n^{1/(2k)} \rfloor$, Lemma 1 follows. ∎

LEMMA 2.

$$\sum_{i=1}^{u} c_i^2 = O(n)\,.$$

Proof. Set $W = \{(x, y) \in (k * A) \times (k * A) : \overline{x} \cap \overline{y} = \emptyset\}$. Define a map $f : W \to (-\infty, \infty)$ by $(x, y) \mapsto x - y$. If $f(x', y') = f(x, y)$, i.e., $x' - y' = x - y$, then $x' + y = x + y'$. As A is a B_{2k}-sequence, $\overline{x'} \cup \overline{y} = \overline{x} \cup \overline{y'}$. Since $\overline{x} \cap \overline{y} = \overline{x'} \cap \overline{y'} = \emptyset$, we have $x = x'$ and $y = y'$. Thus f is one-to-one.

For any integer r, $1 \le r \le k$, let

$$V(r,D) = \{(x,y) \in (r*D) \times (r*D) : -kn < x-y < kn\}\,.$$

Then

(1) $$\sum_{i=1}^{u} c_i^2 \le |V(k,D)|\,.$$

Write $V(k,D) = \bigcup_{j=0}^{k} V_j(k,D)$ (disjoint union) where

$$V_j(k,D) = \{\,(x,y) \in V(k,D) : |\overline{x} \cap \overline{y}| = j\}, \qquad 0 \le j \le k\,.$$

Then

(2) $$|V(k,D)| = \sum_{j=0}^{k} |V_j(k,D)|\,.$$

For any B_{2k}-sequence B and any integers j,r $(0 \le j,r \le k)$, let $W(r,D,j,B)$ be the set of 4-tuples (x,b,b,y) such that

(a) $x,y \in r*D$ with $-kn < x-y < kn$,
(b) $b \in j*B$, and
(c) $\overline{b} \cap \overline{x} = \overline{b} \cap \overline{y} = \overline{x} \cap \overline{y} = \emptyset$.

If $j=0$, we simply write $W(r,D)$ instead.

Note that $|V_j(k,D)| = O(|W(k-j,D,j,D)|)$. Combining this with (1) and (2), we have

(3) $$\sum_{i=1}^{u} c_i^2 = O\Big(\sum_{j=0}^{k} |W(k-j,D,j,D)|\Big)\,.$$

Hence it suffices to show that, for all j $(0 \le j \le k)$,

$$|W(k-j,D,j,D)| = O(n)\,.$$

Case $j=0$. Then $W(k,D,0,D) = W(k,D) \subseteq W$ and $f(W(k,D)) \subseteq (-kn,kn)$. Hence,

(4) $$|W(k,D,0,D)| \le 2kn\,.$$

Case $j=k$. Then $V(0,D,k,D) = \{(b,b) : b \in k*D\}$. Hence,

(5) $$|W(0,D,k,D)| = |k*D| \le |D|^k = A(ukn)^k = O((\sqrt[2k]{ukn})^k) = O(n)\,.$$

Case $1 \le j \le k-1$. Let $I = \{i : 1 \le i \le u$ and $|D_i| \ge 2k\}$. Set $B = \bigcup_{i\in I} D_i$, and $B' = D \setminus B$. We divide our proof into two subcases.

Subcase I: $|B| \le |B'|$. Note $B' = \sum_{1 \le i \le u; i \notin I} D_i$. So $|B'| \le u(2k-1)$. Hence,

$$|D| = |B'| + |B| \le 2|B'| \le 4ku\,.$$

Therefore,

$$(6)\quad |W(k-j,D,j,D)| \le |(2k-j)*D| \le |D|^{2k-j} \le (4ku)^{2k-j} = O(n)\,.$$

Subcase II: $|B'| \le |B|$. Note B is also an B_{2k}-sequence. We claim

$$(7)\qquad |W(k-j,D,j,B)| = O(n)\,.$$

To show (7), define a map $v : W(k-j,D,j,B) \to W$ as follows. Let $(x,b,b,y) \in W(k-j,D,j,B)$. As $b \in j*B$, $b = \sum_{s=1}^{j} b_s$ where $b_s \in D_{i_s}$ and $i_s \in I$ $(s = 1,\dots j)$. Since $|D_{i_s}| \ge 2k$, we can choose $b'_s \in D_{i_s}$ $(s = 1,\dots,j)$ so that

(i) all b'_s's are distinct and
(ii) $b'_s \notin \overline{x} \cup \overline{y} \cup \overline{b}$, $s = 1,\dots,j$.

Let $b' = \sum_{s=1}^{j} b'_s$. Then $(x,b,b',y) \in W$. Define $v :\ W(k-j,D,j,D) \to W$ by $(x,b,b,y) \mapsto (x,b,b',y)$. Clearly, v is well defined and one-to-one. Furthermore, by the choice of b'_s, $-kn \le b_s - b'_s \le kn$ for all $1 \le s \le j$. So, $-jkn \le b - b' \le jkn$. Hence,

$$-(j+1)kn < (x+b) - (b'+y) < jkn + kn\,.$$

Thus,

$$|W(k-j,D,j,B)| = |f(v(W(k-j,D,j,B)))| \le 2(j+1)kn \le 2k^2 n\,,$$

which is (7).

Note

$$\begin{aligned}(8)\quad W(k-j,&D,j,D)\\ &= \bigcup_{(x,y)\in W(k-j,D)} \{(x,b,b,y) : b \in j*D,\ \overline{b} \cap (\overline{x} \cup \overline{y}) = \emptyset\}\\ &= \bigcup_{(x,y)\in W(k-j,D)} \{(x,b,b,y) : b \in j*(D \setminus (\overline{x} \cup \overline{y}))\}\,,\end{aligned}$$

and similarly,

$$(9)\quad W(k-j,D,j,B) = \bigcup_{(x,y)\in W(k-j,D)} \{(x,b,b,y) : b \in j*(B \setminus (\overline{x} \cup \overline{y}))\}\,.$$

On the other hand, for any $z = (x,y) \in W(k-j,D)$, it follows from the assumption that $|B'| \le |B|$, and

$$|D \setminus \overline{z}| = |(B \setminus \overline{z})| + |B' \setminus \overline{z}| \le 2|B \setminus \overline{z}| + |z| \le 2|B \setminus \overline{z}| + 2k\,.$$

Therefore,

$$(10)\quad |j*(D \setminus \overline{z})| = \binom{|D \setminus \overline{z}|}{j} \le \binom{2|B \setminus \overline{z}| + 2k}{j} \le c(|j*(B \setminus \overline{z})| + 1)\,,$$

where c is an absolute constant depending only on j, hence only on k.

Combining (7)–(10), we have

$$
\begin{aligned}
(11)\ |W(k-j,D,j,D)| &= \sum_{z\in W(k-j,D)} |j*(D\setminus \overline{z})| \\
&\le \sum_{z\in W(k-j,D)} c(|j*(B\setminus \overline{z})|+1) \\
&\le c\Big(\sum_{z\in W(k-j,D)} |j*(B\setminus \overline{z})| + \sum_{z\in W(k-j,D)} 1\Big) \\
&= c(|W(k-j,D,j,B)|+|W(k-j,D)|) = O(n)\,.
\end{aligned}
$$

Now Lemma 2 follows from (3)–(5) and (11). ■

P r o o f o f t h e T h e o r e m. It follows from Lemmas 1 and 2 that $\tau(n)^{2k}\log n = O(1)$. Hence,

$$
\begin{aligned}
\liminf_{n\to\infty} A(n)\sqrt[2k]{\frac{\log n}{n}} &= \liminf_{n\to\infty}\inf_{n\le m\le un}\bigg(A(m)\sqrt[2k]{\frac{\log m}{m}}\bigg) \\
&\le \liminf_{n\to\infty}\inf_{n\le m\le un}\bigg(\frac{A(m)}{\sqrt[2k]{m}}\sqrt[2k]{\log(un)}\bigg) \\
&\le 2\liminf_{n\to\infty}\tau(n)\sqrt[2k]{\log n} < \infty\,. \ ■
\end{aligned}
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