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On the number of abelian groups of a given order (supplement)

by

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To the days I lived in Manhattan, New York (90.9–91.6)

1. Introduction. The aim of this paper is to supply a still better result for the problem considered in [2]. Let $A(x)$ denote the number of distinct abelian groups (up to isomorphism) of orders not exceeding x. We shall prove

THEOREM 1. *For any* $\varepsilon > 0$,

$$A(x) = C_1x + C_2x^{1/2} + C_3x^{1/3} + O(x^{50/199+\varepsilon}),$$

where C_1, C_2 *and* C_3 *are constants given on page* 261 *of* [2].

Note that $50/199 = 0.25125\ldots$, thus improving our previous exponent $40/159 = 0.25157\ldots$ obtained in [2].

To prove Theorem 1, we shall proceed along the line of approach presented in [2]. The new tool here is an improved version of a result about enumerating certain lattice points due to E. Fouvry and H. Iwaniec (Proposition 2 of [1], which was listed as Lemma 6 in [2]).

2. A result about enumerating certain lattice points. In this section we prove the following improved version of Proposition 2 of [1].

THEOREM 2. *Let* $Q \ge 1$, $m \sim M$, $q \sim Q$, *let* α $(\ne 0, 1)$ *be a real number*, $t(m,q) = (m+q)^\alpha - (m-q)^\alpha$, $T = M^{\alpha-1}Q$, *and let* $B(M,Q,\Delta)$ *be the number of lattice points* (m, m_1, q, q_1) *such that*

$$|t(m,q) - t(m_1,q_1)| < \Delta T.$$

If $Q < \varepsilon M^{3/4}$, *where* ε *is a sufficiently small positive number*, *we have*

$$B(M,Q,\Delta) \ll (MQ + \Delta M^2Q^2 + Q^{8/3})(\log 2M)^4,$$

where the $\ll$ *constant depends at most on* α *and* ε.

It is obvious that Theorem 2 follows from the next two lemmas.

LEMMA 1. *Let $B_1(M,Q,\Delta_1)$ be the number of lattice points (m_1,q,q_1) such that $m_1\sim M$, $q,q_1\sim Q$ and*

$$\left\|\left(\frac{q_1}{q}\right)^{\beta} m_1 + d_1 m_1^{-1} f(q,q_1) + m_1^{-3} g(q,q_1)\right\| \le \varepsilon^{-1}\Delta_1,$$

where

$$\|x\| = \min_{n\in\mathbb{Z}} |n-x|, \qquad \Delta_1 = \Delta M + Q^6 M^{-5}, \qquad \beta = \frac{1}{\alpha-1},$$

$$f(q,q_1) = q^2\left(\frac{q}{q_1}\right)^{\beta} - q_1^2\left(\frac{q_1}{q}\right)^{\beta},$$

$$g(q,q_1) = d_2\left(q^4\left(\frac{q}{q_1}\right)^{3\beta} - q_1^4\left(\frac{q_1}{q}\right)^{\beta}\right) - d_1^2 q^2\left(q_1^2\left(\frac{q}{q_1}\right)^{\beta} - q^2\left(\frac{q}{q_1}\right)^{3\beta}\right),$$

and d_1, d_2 are the constants given by the Taylor expansion

$$\left(\frac{(1+u)^{\alpha}-(1-u)^{\alpha}}{2\alpha u}\right)^{\beta} = 1 + d_1 u^2 + d_2 u^4 + \ldots, \qquad 0<u<1.$$

Then, for $Q < M^{5/6-\varepsilon}$,

$$B(M,Q,\Delta) \ll B_1(M,Q,\Delta_1).$$

Proof. We assume that ΔM is small, for otherwise Theorem 2 follows immediately from the inequality

$$|t(m,q) - t(m_1,q_1)| < \Delta T. \tag{1}$$

From (1) it is easy to see that the Taylor expansion implies

$$m\left(1 + d_1\left(\frac{q}{m}\right)^2 + d_2\left(\frac{q}{m}\right)^4\right) - \left(\frac{q_1}{q}\right)^{\beta} m_1\left(1 + d_1\left(\frac{q_1}{m_1}\right)^2 + d_2\left(\frac{q_1}{m_1}\right)^4\right) \ll \Delta_1. \tag{2}$$

From (2) we get

$$m = \left(\frac{q_1}{q}\right)^{\beta} m_1(1 + O(\Delta + Q^2 M^{-2})), \tag{3}$$

and

$$m - \left(\frac{q_1}{q}\right)^{\beta} m_1 + d_1\left(q^2 m^{-1} - q_1^2 m_1^{-1}\left(\frac{q_1}{q}\right)^{\beta}\right) = O(\Delta M + Q^4 M^{-3}). \tag{4}$$

By substituting (3) into (4), we get a more precise expansion

$$m = m_1\left(\frac{q_1}{q}\right)^{\beta} + d_1 m_1^{-1}\left(q_1^2\left(\frac{q_1}{q}\right)^{\beta} - q^2\left(\frac{q}{q_1}\right)^{\beta}\right) + O(\Delta M + Q^4 M^{-3}). \tag{5}$$

We now use (3) to expand $d_2 q^4 m^{-3}$ and use (5) to expand $d_1 q^2 m^{-1}$, thereby obtaining, in view of (2), the estimate

$$m - \left(\frac{q_1}{q}\right)^{\beta} m_1 + d_1 m_1^{-1} f(q, q_1) + m_1^{-3} g(q, q_1) \ll \Delta_1. \tag{6}$$

Lemma 1 follows from (6) and the fact that Δ_1 is small.

LEMMA 2. *Let $B_1(M, Q, \Delta_1)$ be defined in Lemma* 1 *and $Q < \varepsilon M^{3/4}$. Then*

$$B_1(M, Q, \Delta_1) \ll (MQ + \Delta M^2 Q^2 + Q^{8/3})(\log 2M)^4.$$

Proof. Let $\Delta_2 = \Delta M + M^{-1} Q^{2/3}$. Clearly,

$$B_1(M, Q, \Delta_1) \le B_1(M, Q, \Delta_2).$$

For fixed (q, q_1), the number of lattice points counted in $B_1(M, Q, \Delta_2)$ is (with $S = \varepsilon(4\Delta_2)^{-1}$)

$$\ll S^{-1} \sum_{1 \le s \le S} \Big| \sum_{m \sim M} e(Asm + Bsm^{-1} + Csm^{-3}) \Big| + \Delta_2 M, \tag{7}$$

by virtue of the identity

$$\sum_{|s|<S} \left(1 - \frac{|s|}{S}\right) e(sx) = \frac{1 - \{S\}}{S}\left(\frac{\sin \pi x [S]}{\sin \pi x}\right)^2 + \frac{\{S\}}{S}\left(\frac{\sin \pi x [S+1]}{\sin \pi x}\right)^2;$$

in (7), A, B and C are given by

$$A = \left(\frac{q_1}{q}\right)^{\beta}, \qquad B = d_1 f(q, q_1), \qquad C = g(q, q_1).$$

Under our assumption, the innermost sum in (7) is

$$\int_M^{2M} e(\pm\|As\|\xi + Bs\xi^{-1} + Cs\xi^{-3})\, d\xi + O(1) = I + O(1), \quad \text{say}, \tag{8}$$

by using the truncated Poisson's summation formula.

If $\|As\| \ge 3s|B|M^{-2}$, then by partial integration,

$$I \ll \|sA\|^{-1}; \tag{9}$$

and if $\|As\| < 3s|B|M^{-2}$, then we apply the well-known second derivative estimate to get

$$I \ll (s|B|)^{-1/2} M^{3/2} \qquad \text{for } B \ne 0, \tag{10}$$

where we have used the fact that $|C| \ll |B|Q^2$. From (7)–(10) we conclude that

$$B_1(M,Q,\Delta_2) \ll \Delta_2 MQ^2 + E_1(M,Q,\Delta_2) + E_2(M,Q,\Delta_2), \tag{11}$$

where

$$E_1(M,Q,\Delta_2) = \Delta_2 \sum_{1\le s\le S} \sum_{q,q_1\sim Q} \min(M, 1/\|As\|),$$

$$E_2(M,Q,\Delta_2) = \Delta_2 \sum_{\substack{1\le s\le S\\ \|As\|<3s|B|M^{-2}}} \sum_{q,q_1\sim Q} \min(M, (s|B|)^{-1/2}M^{3/2}).$$

$E_i(M,Q,\Delta_2)$ $(i=1,2)$ can be estimated just as $D_i(M,Q,\Delta)$ on page 320 of [1], and we have

$$E_1(M,Q,\Delta_2) \ll MQ(\log 2M)^3, \tag{12}$$

$$E_2(M,Q,\Delta_2) \ll (MQ + (\Delta_2 M)^{-1/2}Q^3)(\log 2M)^4. \tag{13}$$

Lemma 2 follows from (11)–(13).

3. A bound for a kind of triple exponential sums. By means of Theorem 2, we can sharpen Lemma A of [2] as follows. We have

THEOREM 3. *Let $H \ge 1$, $X \ge 1$, $Y \ge 1000$; let α, β and γ be real numbers such that $\alpha\gamma(\gamma-1)(\beta-1) \ne 0$, and $A > C(\alpha,\beta,\gamma) > 0$, $f(h,x,y) = Ah^\alpha x^\beta y^\gamma$. Define*

$$S(H,X,Y) = \sum_{(h,x,y)\in D} C_1(h,x)C_2(y)e(f(h,x,y)),$$

where D is a region contained in the rectangle

$$\{(h,x,y) \mid h\sim H,\ x\sim X,\ y\sim Y\}$$

such that for any fixed pair (h_0,x_0), the intersection $D\cap\{(h_0,x_0,y)\mid y\sim Y\}$ has at most $O(1)$ segments. Also, suppose $|C_1(h,x)| \le 1$, $|C_2(y)| \le 1$, $F = AH^\alpha X^\beta Y^\gamma \gg Y$. Then

$$\begin{aligned} L^{-3}S(H,X,Y) &\ll \sqrt[22]{(HX)^{19}Y^{13}F^3} + HXY^{5/8}(1+Y^7F^{-4})^{1/16} \\ &\quad + \sqrt[32]{(HX)^{29}Y^{28}F^{-2}M^5} + \sqrt[4]{(HX)^3Y^4M} \\ &\equiv E_1, \end{aligned} \tag{14}$$

where $L = \log(AHXY+2)$, $M = \max(1, FY^{-2})$.

Proof. We have

$$S(H,X,Y) \ll \sum_{h\sim H}\sum_{x\sim X}\Big|\sum_{y\in I(h,x)} C_2(y)e(f(h,x,y))\Big|,$$

where $I(h,x)$ is some subinterval of $(Y,2Y]$. From Lemma 1 of [2], we get

$$L^{-1}S(H,X,Y) \ll \sum_{h\sim H}\sum_{x\sim X}\Big|\sum_{y\sim Y}C(y,\theta)e(f(h,x,y))\Big|,$$

where $C(y,\theta)=C_2(y)e(\theta y)$ for some real number θ (θ is independent of h, x, and y). We consider the expression

$$(15)\quad R(q)=(HXY)^2q^{-1}+(HX)^2(Y^5F^{-1}Mq^{-1})^{1/2} + \sqrt[6]{(HX)^9Y^3F^3q^5}+(HX)^2Yq^{1/3}+\sqrt{(HX)^3Y^4M}.$$

By Lemma 2 of [2], we can choose a $Q\in(0,\varepsilon Y^{3/4}]$ such that

$$(16)\quad R(Q)\ll \sqrt[11]{(HX)^{19}Y^{13}F^3}+(HX)^2Y^{5/4}+(HX)^2(F^{-4}M^4Y^{17})^{1/8} + (HX)^2(Y^8F^{-1}M)^{1/5}+\sqrt[16]{(HX)^{29}Y^{28}F^{-2}M^5} + \sqrt{(HX)^3Y^4M}\ll E_1^2$$

(see (14)). If $Q\le 100$, then we trivially have

$$L^{-1}S(H,X,Y)\ll HXYQ^{-1/2}\ll\sqrt{R(Q)}\ll E_1.$$

Now we assume that $Q>100$. By Cauchy's inequality and Lemma 3 of [2], we get

$$(17)\qquad L^{-3}|S(H,X,Y)|^2\ll (HXY)^2Q^{-1}+(HXY)Q^{-1}|S_1|,$$

where

$$S_1=\sum_{(q,y,h,x)\in D_1}C(y+q,\theta)\overline{C(y-q,\theta)}e(Ah^\alpha x^\beta t(y,q)),$$

$$t(y,q)=(y+q)^\gamma-(y-q)^\gamma,$$

$$D_1=D_1(Q_1)=\{(q,y,h,x)\mid y+q,y-q\sim Y,\ q\sim Q_1,\ h\sim H,\ x\sim X\}$$

for some Q_1 with $1\le 2Q_1\le Q/2$. By Lemma 4 of [2] we have (note that $F\gg Y$ by our assumption)

$$(18)\qquad |S_1|^2\ll FY^{-1}Q_1A_1A_2,$$

where A_1 is the number of lattice points (h,x,h_1,x_1) such that

$$|h^\alpha x^\beta-h_1^\alpha x_1^\beta|\ll A^{-1}Q_1^{-1}Y^{1-\gamma}$$

with $h,h_1\sim H$, $x,x_1\sim X$, which is estimated by Lemma 5 of [2] as

$$(19)\qquad A_1\ll (HX+H^2X^2YQ_1^{-1}F^{-1})L^2;$$

and A_2 stands for the number of lattice points (q,y,q_1,y_1) such that

$$|t(y,q)-t(y_1,q_1)|\ll (AH^\alpha X^\beta)^{-1}$$

with $Y/2 < y, y_1 < 3Y$, $q, q_1 \sim Q_1$. Recall that $Q_1 \le Q/4 < \varepsilon Y^{3/4}$. Theorem 2 gives (with $\Delta = Q_1^{-1} Y F^{-1}$)

$$A_2 \ll (Q_1 Y + Q_1 Y^3 F^{-1} + Q_1^{8/3}) L^4. \tag{20}$$

From (17)–(20), we deduce that (see (15))

$$L^{-6}|S(H,X,Y)|^2 \ll (HXY)^2 Q^{-1} \tag{21}$$
$$+ HXYQ^{-1}(FHXQ(Q + HXYF^{-1})(1 + Y^2F^{-1} + Q^{5/3}Y^{-1}))^{1/2} \ll R(Q).$$

Theorem 3 follows from (21) and (16).

4. The proof of Theorem 1. Put

$$\theta = 50/199, \qquad S_{1,2,3} = \sum_{\substack{mn \le x^{1/3} \\ m > n}} \Psi(xm^{-2}n^{-3}), \qquad \Psi(u) = u - [u] - 1/2.$$

By Lemmas 7, 8 and Theorems 1, 2 of [2], to prove Theorem 1 it is sufficient to establish the following lemma.

LEMMA B.

$$S_{1,2,3} \ll x^{\theta + \varepsilon}.$$

Obviously, we have

$$S_{1,2,3} = \sum_{(M,N)} S_{1,2,3}(M,N) + O(x^{\theta+\varepsilon}), \tag{22}$$

where M and N run through the sequences $\{2^{-j}x^{1/3} \mid j = 0, 1, \ldots\}$ and $\{2^{-k}x^{1/3} \mid k = 0, 1, \ldots\}$ respectively, such that

$$MN \ge x^\theta, \qquad 2M \ge N, \qquad MN \le x^{1/3}, \tag{23}$$

and

$$S_{1,2,3}(M,N) = \sum_{(m,n) \in D} \Psi(xm^{-2}n^{-3}),$$

$$D = D(M,N) = \{(m,n) \mid m \sim M,\ n \sim N,\ mn \le x^{1/3},\ m > n\}. \tag{24}$$

By means of the standard expansion for the function $\Psi(\cdot)$, we get, for any parameter K, $K \in [100, MN]$, the inequality

$$(\log K)^{-1} S_{1,2,3}(M,N)$$
$$\ll MNK^{-1} + \sum_{1 \le h \le K^2} \min\left(\frac{1}{h}, \frac{K}{h^2}\right) \Big| \sum_{(m,n) \in D} e(f(h,m,n)) \Big|,$$

where $f(h,m,n) = hxm^{-2}n^{-3}$. Thus, for some $H \in [1, K^2]$, we have

$$x^{-\varepsilon} S_{1,2,3}(M,N) \ll MNK^{-1} + \min(1, K/H) \Phi_{1,2,3}(H,M,N), \tag{25}$$

where

$$\Phi_{1,2,3}(H,M,N)=H^{-1}\sum_{h\sim H}\Big|\sum_{(m,n)\in D}e(f(h,m,n))\Big| \tag{26}$$

(we have adopted the notations on pp. 266–267 of [2]). We now use our Theorem 3 three times to estimate the sum $S_{1,2,3}(M,N)$. Lemma B will then be proved by invoking (49) of [2].

LEMMA 3.

$$x^{-\varepsilon}S_{1,2,3}(M,N)\ll \sqrt[30]{x^{11}M^{-11}N^{-12}}+\sqrt[12]{x^4M^{-4}N^{-3}} + \sqrt[45]{x^{16}M^{-16}N^{-17}}+\sqrt[5]{x^2M^{-2}N^{-3}}+x^{1/4}\equiv E_2.$$

Proof. We use Lemma 10 of [2] to the summation over m, and obtain, in view of (23),

$$\sum_{(m,n)\in D}e(f(h,m,n)) = c_1(hx)^{1/6}\sum_{(n,u)\in D_1}(n^3u^4)^{-1/6}e(g(h,n,u))+O(x^{1/4}), \tag{27}$$

where

$$g(h,n,u)=c_2(xhn^{-3}u^2)^{1/3},$$

$$D_1=\{(n,u)\mid un^6\le c_3hx,\ h\le c_4u,\ n\sim N,\ c_5\le hx/(n^3uM^3)\le c_6\},$$

with c_i $(1\le i\le 6)$ being some absolute constants. From (26) and (27), we find that

$$x^{-\varepsilon/2}\Phi_{1,2,3}(H,M,N) \ll M(H^3G)^{-1/2}\sum_{h\sim H}\Big|\sum_{(n,u)\in D_1}C(n)\widetilde{C}(u)e(g(h,n,u))\Big|+x^{1/4}, \tag{28}$$

where $|C(n)|\le 1$, $|\widetilde{C}(u)|\le 1$, and $G=xM^{-2}N^{-3}$. We apply Theorem 3 with $(H,X,Y)\simeq(H,GH/M,N)$ to get (note that $(n,u)\in D_1$ implies $u\simeq GH/M$)

$$x^{-\varepsilon/2}\sum_{h\sim H}\Big|\sum_{(n,u)\in D_1}C(n)\widetilde{C}(u)e(g(h,n,u))\Big| \ll \sqrt[22]{H^{41}G^{22}M^{-19}N^{13}}+H^2GM^{-1}N^{5/8}+\sqrt[16]{H^{28}G^{12}M^{-16}N^{11}} + \sqrt[32]{H^{56}G^{27}M^{-29}N^{28}}+\sqrt[32]{H^{61}G^{32}M^{-29}N^{18}} + \sqrt[4]{H^6G^3M^{-3}N^4}+\sqrt[4]{H^7G^4M^{-3}N^2}. \tag{29}$$

From (25), (26), (28) and (29), we obtain

$$(30)\quad x^{-\varepsilon}S_{1,2,3}(M,N) \ll MNK^{-1} + \sqrt[22]{K^8x^{11}M^{-19}N^{-20}} + \sqrt[8]{K^4x^4M^{-8}N^{-7}} + \sqrt[16]{K^4x^4M^{-8}N^{-1}} + \sqrt[32]{K^8x^{11}M^{-19}N^{-5}} + \sqrt[32]{K^{13}x^{16}M^{-29}N^{-30}} + \sqrt[4]{Kx^2M^{-3}N^{-4}} + x^{1/4} = E_2(K) + x^{1/4}, \quad \text{say.}$$

By Lemma 2 of [2], there exists a $K_0 \in [0, MN]$ such that

$$(31)\qquad E_2(K_0) \ll E_2.$$

If $K_0 \ge 100$, we put $K = K_0$ in (30), and Lemma 3 follows from (30) and (31); if $K_0 < 100$, we trivially get

$$(32)\qquad S_{1,2,3}(M,N) \ll MNK_0^{-1} \ll E_2(K_0),$$

and Lemma 3 follows from (32) and (31).

Lemma 4. *For $K = MNx^{-\theta}$, $1 \le H \le K^2$, we have*

$$x^{-\varepsilon}\Phi_{1,2,3}(H,M,N) \ll \sqrt[22]{x^3M^7N^{10}} + NM^{5/8} + \sqrt[16]{H^{-4}x^{-4}M^{25}N^{28}} + \sqrt[32]{H^{-5}x^{-2}M^{32}N^{35}} + \sqrt[32]{x^3M^{12}N^{20}} + \sqrt[4]{H^{-1}M^4N^3} + x^{1/4}.$$

Proof. Applying Theorem 3 to the sum $H\Phi_{1,2,3}(H,M,N)$ directly, with $(H,X,Y) \simeq (H,N,M)$, we get the required estimate.

Lemma 5. *For $K = MNx^{-\theta}$, $1 \le H \le K^2$, we have*

$$x^{-\varepsilon}\min(1, K/H)\Phi_{1,2,3}(H,M,N) \ll \sqrt[22]{x^{5-2\theta}MN^6} + \sqrt[8]{x^{1-\theta}M^2N^6} + \sqrt[32]{x^{5-2\theta}M^6N^{16}} + \sqrt[32]{x^4M^9N^{16}} + \sqrt[52]{x^8M^{12}N^{20}} + \min\left(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}},\ \sqrt[22]{x^3M^7N^{10}}\right) + x^{\theta}.$$

Proof. Applying Theorem 3 to the triple exponential sum of (28), with $(H,X,Y) \simeq (H,N,GH/M)$, we get

$$(33)\quad x^{-\varepsilon/2}\sum_{h\sim H}\Big|\sum_{(n,u)\in D_1} C(n)\widetilde{C}(u)e(g(h,n,u))\Big| \ll \sqrt[22]{H^{35}G^{16}M^{-13}N^{19}} + \sqrt[8]{H^{13}G^5M^{-5}N^8} + \sqrt[16]{H^{29}G^{13}M^{-17}N^{16}} + \sqrt[32]{H^{55}G^{26}M^{-28}N^{29}} + \sqrt[32]{H^{50}G^{21}M^{-18}N^{29}} + \sqrt[4]{H^7G^4M^{-4}N^3} + \sqrt[4]{H^6G^3M^{-2}N^{-3}} + x^{1/4}.$$

From (28) and (33), we obtain

$$x^{-\varepsilon}\Phi_{1,2,3}(H,M,N) \ll \sqrt[22]{H^2x^5M^{-1}N^4} + \sqrt[8]{HxMN^5} + \sqrt[16]{H^5x^5M^{-11}N} + \sqrt[32]{H^7x^{10}M^{-16}N^{-1}} + \sqrt[32]{H^2x^5M^4N^{14}} + \sqrt[4]{Hx^2M^{-4}N^{-3}} + x^{1/4},$$

which, in conjunction with Lemma 4 and (23), gives

$$\begin{aligned}(34)\qquad & x^{-\varepsilon}\min(1,K/H)\Phi_{1,2,3}(H,M,N) \\ & \ll \sqrt[22]{x^{5-2\theta}MN^6} + \sqrt[8]{x^{1-\theta}M^2N^6} + \sqrt[32]{x^{5-2\theta}M^6N^{16}} \\ & + \min(\sqrt[4]{x^2HM^{-4}N^{-3}},\ \sqrt[16]{H^{-4}x^{-4}M^{25}N^{28}}) \\ & + \min(\sqrt[4]{x^2HM^{-4}N^{-3}},\ \sqrt[32]{H^{-5}x^{-2}M^{32}N^{35}}) \\ & + \min(\sqrt[4]{x^2HM^{-4}N^{-3}},\ \sqrt[4]{H^{-1}M^4N^3}) \\ & + \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}},\ \sqrt[22]{x^3M^7N^{10}}) \\ & + \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}}, NM^{5/8}) \\ & + \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}},\ \sqrt[32]{x^3M^{12}N^{20}}) + x^{\theta}.\end{aligned}$$

Obviously,

$$(35)\qquad \min(\sqrt[4]{x^2HM^{-4}N^{-3}},\ \sqrt[16]{H^{-4}x^{-4}M^{25}N^{28}}) \le \sqrt[32]{x^4M^9N^{16}},$$

$$(36)\qquad \min(\sqrt[4]{x^2HM^{-4}N^{-3}},\ \sqrt[32]{H^{-5}x^{-2}M^{32}N^{35}}) \le \sqrt[52]{x^8M^{12}N^{20}},$$

$$(37)\qquad \min(\sqrt[4]{x^2HM^{-4}N^{-3}},\ \sqrt[4]{H^{-1}M^4N^3}) \le x^{1/4};$$

and, in view of (23),

$$\begin{aligned}(38)\qquad & \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}},\ NM^{5/8}) \\ & \ll \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}},\ (M^3N^2)^{13/40}) \le x^{(26-13\theta)/92} < x^{\theta},\end{aligned}$$

$$\begin{aligned}(39)\qquad & \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}},\ \sqrt[32]{x^3M^{12}N^{20}}) \\ & \ll \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}},\ \sqrt[32]{x^3(M^3N^2)^{32/5}}) \le x^{(79-32\theta)/288} < x^{\theta}.\end{aligned}$$

From (34) to (39), Lemma 5 follows.

P r o o f o f L e m m a B. By (49) of [2], we have

$$(40)\qquad x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sqrt[8]{x^2MN^{-1}}.$$

By (25), Lemma 5 and (40), we get

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sqrt[22]{x^{5-2\theta}MN^6} + \sqrt[8]{x^{1-\theta}M^2N^6} + \sqrt[32]{x^{5-2\theta}M^6N^{16}} + \sqrt[32]{x^4M^9N^{16}} + \sqrt[52]{x^8M^{12}N^{20}} + R_1(M,N) + x^\theta,$$

where

$$\begin{aligned} R_1(M,N) &= \min(\sqrt[22]{x^3M^7N^{10}},\ \sqrt[4]{x^{2-\theta}M^{-3}N^{-2}},\ \sqrt[8]{x^2MN^{-1}}) \\ &\le (\sqrt[22]{x^3M^7N^{10}})^{\alpha_1}(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}})^{\beta_1}(\sqrt[8]{x^2MN^{-1}})^{\gamma_1} \\ &= x^{(81-17\theta)/306} < x^\theta, \end{aligned}$$

with $(\alpha_1,\beta_1,\gamma_1) = (110/306, 68/306, 128/306)$; thus

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sqrt[22]{x^{5-2\theta}MN^6} + \sqrt[8]{x^{1-\theta}M^2N^6} + \sqrt[32]{x^{5-2\theta}M^6N^{16}} + \sqrt[32]{x^4M^9N^{16}} + \sqrt[52]{x^8M^{12}N^{20}} + x^\theta. \tag{41}$$

If $MN \le x^{0.3}$, then (41) gives

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sqrt[22]{x^{5-2\theta}MN^6} + x^\theta. \tag{42}$$

From Lemma 3, (40) and (42), we deduce that

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sum_{i=2}^{5} R_i(M,N) + x^\theta, \tag{43}$$

where

$$\begin{aligned} R_2(M,N) &= \min(\sqrt[30]{x^{11}M^{-11}N^{-12}},\ \sqrt[22]{x^{5-2\theta}MN^6},\ \sqrt[8]{x^2MN^{-1}}) \\ &\le (\sqrt[30]{x^{11}M^{-11}N^{-12}})^{\alpha_2}(\sqrt[22]{x^{5-2\theta}MN^6})^{\beta_2}(\sqrt[8]{x^2MN^{-1}})^{\gamma_2} \\ &= x^{(150-23\theta)/574} = x^\theta, \end{aligned} \tag{44}$$

with $(\alpha_2,\beta_2,\gamma_2) = (105/574, 253/574, 216/574)$;

$$\begin{aligned} R_3(M,N) &= \min(\sqrt[12]{x^4M^{-4}N^{-3}},\ \sqrt[22]{x^{5-2\theta}MN^6}) \\ &\ll (\sqrt[12]{x^4M^{-4}N^{-3}})^{12/34}(\sqrt[22]{x^{5-2\theta}M^4N^3})^{22/34} \\ &= x^{(9-2\theta)/34} < x^\theta; \end{aligned} \tag{45}$$

$$\begin{aligned} R_4(M,N) &= \min(\sqrt[45]{x^{16}M^{-16}N^{-17}},\ \sqrt[22]{x^{5-2\theta}MN^6}) \\ &\ll (\sqrt[45]{x^{16}M^{-16}N^{-17}})^{105/347}(\sqrt[22]{x^{5-2\theta}(M^{16}N^{17})^{7/33}})^{242/347} \\ &= x^{(277-66\theta)/1041} < x^\theta; \end{aligned} \tag{46}$$

$$\begin{aligned} &R_5(M,N) \\ &= \min(\sqrt[5]{x^2M^{-2}N^{-3}},\ \sqrt[22]{x^{5-2\theta}MN^6},\ \sqrt[8]{x^2MN^{-1}}) \end{aligned} \tag{47}$$

$$\le (\sqrt[5]{x^2M^{-2}N^{-3}})^{35/217}(\sqrt[22]{x^{5-2\theta}MN^6})^{110/217}(\sqrt[8]{x^2MN^{-1}})^{72/217}$$
$$= x^{(57-10\theta)/217} < x^\theta.$$

From (43) to (47), we have

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll x^\theta. \tag{48}$$

If $MN > x^{0.3}$, from Lemma 3 we find

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sqrt[30]{x^{11}M^{-11}N^{-12}} + \sqrt[5]{x^2M^{-2}N^{-3}} + x^\theta. \tag{49}$$

From (40), (41) and (49), we deduce that

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sum_{i=6}^{15} R_i(M,N) + x^\theta, \tag{50}$$

where, by (44) and (47),

$$R_6(M,N) = R_2(M,N) \le x^\theta, \qquad R_7(M,N) = R_5(M,N) < x^\theta, \tag{51}$$

$$\begin{aligned} R_8(M,N) &= \min(\sqrt[30]{x^{11}M^{-11}N^{-12}}, \sqrt[8]{x^{1-\theta}M^2N^6}) \\ &\ll (\sqrt[30]{x^{11}M^{-11}N^{-12}})^{30/53}(\sqrt[8]{x^{1-\theta}(M^{11}N^{12})^{8/23}})^{23/53} \\ &= x^{(111-23\theta)/424} < x^\theta; \end{aligned} \tag{52}$$

$$\begin{aligned} R_9(M,N) &= \min(\sqrt[5]{x^2M^{-2}N^{-3}}, \sqrt[8]{x^{1-\theta}M^2N^6}) \\ &\ll (\sqrt[5]{x^2M^{-2}N^{-3}})^{1/2}(\sqrt[8]{x^{1-\theta}(M^2N^3)^{8/5}})^{1/2} \\ &= x^{(21-5\theta)/80} < x^\theta; \end{aligned} \tag{53}$$

$$\begin{aligned} &R_{10}(M,N) \\ &= \min(\sqrt[30]{x^{11}M^{-11}N^{-12}}, \sqrt[32]{x^{5-2\theta}M^6N^{16}}) \\ &\ll (\sqrt[30]{x^{11}M^{-11}N^{-12}})^{165/349}(\sqrt[32]{x^{5-2\theta}(M^{11}N^{12})^{22/23}})^{184/349} \\ &= x^{(357-46\theta)/1396} < x^\theta; \end{aligned} \tag{54}$$

$$\begin{aligned} R_{11}(M,N) &= \min(\sqrt[5]{x^2M^{-2}N^{-3}}, \sqrt[32]{x^{5-2\theta}M^6N^{16}}) \\ &\ll (\sqrt[5]{x^2M^{-2}N^{-3}})^{22/54}(\sqrt[32]{x^{5-2\theta}(M^2N^3)^{22/5}})^{32/54} \\ &= x^{(69-10\theta)/270} < x^\theta; \end{aligned} \tag{55}$$

$$\begin{aligned} R_{12}(M,N) &= \min(\sqrt[30]{x^{11}M^{-11}N^{-12}}, \sqrt[32]{x^4M^9N^{16}}) \\ &\ll ((\sqrt[30]{x^{11}M^{-11}N^{-12}})^{750}(\sqrt[32]{x^4(M^{11}N^{12})^{25/23}})^{736})^{1/1486} \\ &= x^{367/1486}; \end{aligned} \tag{56}$$

(57) $$R_{13}(M,N)=\min(\sqrt[5]{x^2M^{-2}N^{-3}},\ \sqrt[32]{x^4M^9N^{16}}) \ll ((\sqrt[5]{x^2M^{-2}N^{-3}})^{25}(\sqrt[32]{x^4(M^2N^3)^5})^{32})^{1/57}=x^{14/57};$$

(58) $$R_{14}(M,N)=\min(\sqrt[30]{x^{11}M^{-11}N^{-12}},\ \sqrt[26]{x^4M^6N^{10}}) \ll ((\sqrt[30]{x^{11}M^{-11}N^{-12}})^{240}(\sqrt[13]{x^2(M^{11}N^{12})^{8/23}})^{299})^{1/539} = x^{134/539};$$

(59) $$R_{15}(M,N)=\min(\sqrt[5]{x^2M^{-2}N^{-3}},\ \sqrt[13]{x^2M^3N^5}) \ll ((\sqrt[5]{x^2M^{-2}N^{-3}})^{40}(\sqrt[13]{x^2(M^2N^3)^{8/5}})^{65})^{1/105}=x^{26/105}.$$

From (50) to (59), we have

(60) $$x^{-\varepsilon}S_{1,2,3}(M,N)\ll x^{\theta}.$$

Lemma B follows from (48) and (60).

5. Concluding remarks. It is clear that our result 50/199 is closely connected with the term $Q^{8/3}$ in Theorem 2. This term actually comes from the method given in Lemmas 3 and 4 of [1]. The fraction 50/199 can be reduced whenever $Q^{8/3}$ can be reduced in our Theorem 2. If, for example, $Q^{8/3}$ could be "omitted", then one may attain the expected exponent 1/4, in place of 50/199.

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