On the number of abelian groups of a given order (supplement)

by

Hong-Quan Liu (Harbin)

To the days I lived in Manhattan, New York (90.9–91.6)

1. Introduction. The aim of this paper is to supply a still better result for the problem considered in [2]. Let $A(x)$ denote the number of distinct abelian groups (up to isomorphism) of orders not exceeding $x$. We shall prove

**Theorem 1.** For any $\varepsilon > 0$,

$$A(x) = C_1 x + C_2 x^{1/2} + C_3 x^{1/3} + O(x^{50/199 + \varepsilon}),$$

where $C_1$, $C_2$ and $C_3$ are constants given on page 261 of [2].

Note that $50/199 = 0.25125\ldots$, thus improving our previous exponent $40/159 = 0.25157\ldots$ obtained in [2].

To prove Theorem 1, we shall proceed along the line of approach presented in [2]. The new tool here is an improved version of a result about enumerating certain lattice points due to E. Fouvry and H. Iwaniec (Proposition 2 of [1], which was listed as Lemma 6 in [2]).

2. A result about enumerating certain lattice points. In this section we prove the following improved version of Proposition 2 of [1].

**Theorem 2.** Let $Q \geq 1$, $m \sim M$, $q \sim Q$, let $\alpha (\neq 0, 1)$ be a real number, $t(m, q) = (m + q)^\alpha - (m - q)^\alpha$, $T = M^{\alpha-1}Q$, and let $B(M, Q, \Delta)$ be the number of lattice points $(m, m_1, q, q_1)$ such that

$$|t(m, q) - t(m_1, q_1)| < \Delta T.$$

If $Q < \varepsilon M^{3/4}$, where $\varepsilon$ is a sufficiently small positive number, we have

$$B(M, Q, \Delta) \ll (MQ + \Delta M^2 Q^2 + Q^{8/3})(\log 2M)^4,$$

where the $\ll$ constant depends at most on $\alpha$ and $\varepsilon$. 
It is obvious that Theorem 2 follows from the next two lemmas.

**Lemma 1.** Let \( B_1(M, Q, \Delta_1) \) be the number of lattice points \((m_1, q, q_1)\) such that \( m_1 \sim M, q, q_1 \sim Q \) and

\[
\left\| \left( \frac{q_1}{q} \right)^\beta m_1 + d_1 m_1^{-1} f(q, q_1) + m_1^{-3} g(q, q_1) \right\| \leq \varepsilon^{-1} \Delta_1,
\]

where

\[
\|x\| = \min_{n \in \mathbb{Z}} |n - x|, \quad \Delta_1 = \Delta M + Q^6 M^{-5}, \quad \beta = \frac{1}{\alpha - 1},
\]

\[
f(q, q_1) = q^2 \left( \frac{q}{q_1} \right)^\beta - q_1^2 \left( \frac{q_1}{q} \right)^\beta,
\]

\[
g(q, q_1) = d_2 \left( q^4 \left( \frac{q}{q_1} \right)^{3\beta} - q_1^4 \left( \frac{q_1}{q} \right)^{3\beta} \right) - d_1^2 q^2 \left( \frac{q}{q_1} \right)^\beta - 2 \left( \frac{q}{q_1} \right)^\beta,
\]

and \( d_1, d_2 \) are the constants given by the Taylor expansion

\[
\left( \frac{(1 + u)^\alpha - (1 - u)^\alpha}{2 \alpha u} \right)^\beta = 1 + d_1 u^2 + d_2 u^4 + \ldots, \quad 0 < u < 1.
\]

Then, for \( Q < M^{5/6 - \varepsilon} \),

\[B(M, Q, \Delta) \ll B_1(M, Q, \Delta_1).\]

**Proof.** We assume that \( \Delta M \) is small, for otherwise Theorem 2 follows immediately from the inequality

\[(1)\]

\[|t(m, q) - t(m_1, q_1)| < \Delta T.\]

From (1) it is easy to see that the Taylor expansion implies

\[(2)\]

\[m \left( 1 + d_1 \left( \frac{q}{m} \right)^2 + d_2 \left( \frac{q}{m} \right)^4 \right) - \left( \frac{q_1}{q} \right)^\beta m_1 \left( 1 + d_1 \left( \frac{q_1}{m_1} \right)^2 + d_2 \left( \frac{q_1}{m_1} \right)^4 \right) \ll \Delta_1.
\]

From (2) we get

\[(3)\]

\[m = \left( \frac{q_1}{q} \right)^\beta m_1 (1 + O(\Delta + Q^2 M^{-2})),
\]

and

\[(4)\]

\[m - \left( \frac{q_1}{q} \right)^\beta m_1 + d_1 \left( q^2 m^{-1} - q_1^2 m_1^{-1} \left( \frac{q_1}{q} \right)^\beta \right) = O(\Delta M + Q^4 M^{-3}).
\]
By substituting (3) into (4), we get a more precise expansion

\[ m = m_1 \left( \frac{q_1}{q} \right)^\beta + d_1 m_1^{-1} \left( q_1 \right)^2 \left( \frac{q_1}{q} \right)^\beta - q_1^2 \left( \frac{q_1}{q} \right)^\beta + O(\Delta M + Q^4 M^{-3}). \]

We now use (3) to expand \( d_2 q^4 m^{-3} \) and use (5) to expand \( d_1 q^2 m^{-1} \), thereby obtaining, in view of (2), the estimate

\[ m - \left( \frac{q_1}{q} \right)^\beta m_1 + d_1 m_1^{-1} f(q,q_1) + m_1^{-3} g(q,q_1) \ll \Delta_1. \]

Lemma 1 follows from (6) and the fact that \( \Delta_1 \) is small.

**Lemma 2.** Let \( B_1(M, Q, \Delta_1) \) be defined in Lemma 1 and \( Q < \varepsilon M^{3/4} \). Then

\[ B_1(M, Q, \Delta_1) \ll (MQ + \Delta M^2 Q^2 + Q^{8/3})(\log 2M)^4. \]

**Proof.** Let \( \Delta_2 = \Delta M + M^{-1} Q^{2/3} \). Clearly,

\[ B_1(M, Q, \Delta_1) \leq B_1(M, Q, \Delta_2). \]

For fixed \( (q,q_1) \), the number of lattice points counted in \( B_1(M, Q, \Delta_2) \) is (with \( S = \varepsilon(4\Delta_2)^{-1} \))

\[ \ll S^{-1} \sum_{1 \leq s \leq S} \left| \sum_{m=M} e(Asm + Bsm^{-1} + Csm^{-3}) \right| + \Delta_2 M, \]

by virtue of the identity

\[ \sum_{|s|<S} \left( 1 - \frac{|s|}{S} \right) e(sx) = 1 - \left\{ \frac{S}{S} \left( \frac{\sin \pi x[S]}{\sin \pi x} \right)^2 + \left\{ \frac{S}{S} \left( \frac{\sin \pi x[S + 1]}{\sin \pi x} \right) \right)^2 ; \]

in (7), \( A, B \) and \( C \) are given by

\[ A = \left( \frac{q_1}{q} \right)^\beta, \quad B = d_1 f(q,q_1), \quad C = g(q,q_1). \]

Under our assumption, the innermost sum in (7) is

\[ \int_M^{2M} e(\pm \|As\| \xi + Bs \xi^{-1} + C s \xi^{-3}) d\xi + O(1) = I + O(1), \quad \text{say,} \]

by using the truncated Poisson’s summation formula.

If \( \|As\| \geq 3s|B|M^{-2} \), then by partial integration,

\[ I \ll \|sA\|^{-1}; \]

and if \( \|As\| < 3s|B|M^{-2} \), then we apply the well-known second derivative estimate to get

\[ I \ll (s|B|)^{-1/2} M^{3/2} \quad \text{for } B \neq 0, \]
where we have used the fact that $|C| \ll |B|Q^2$. From (7)–(10) we conclude that
\begin{equation}
(11) \quad B_1(M, Q, \Delta_2) \ll \Delta_2 M Q^2 + E_1(M, Q, \Delta_2) + E_2(M, Q, \Delta_2),
\end{equation}
where
\begin{align*}
E_1(M, Q, \Delta_2) &= \Delta_2 \sum_{1 \leq s \leq S} \sum_{q, q_1 \sim Q} \min(M, 1/\|As\|), \\
E_2(M, Q, \Delta_2) &= \Delta_2 \sum_{1 \leq s \leq S} \sum_{q, q_1 \sim Q} \min(M, (s|B|)^{-1/2}M^{3/2}).
\end{align*}

$E_i(M, Q, \Delta_2)$ $(i = 1, 2)$ can be estimated just as $D_i(M, Q, \Delta)$ on page 320 of [1], and we have
\begin{align*}
(12) & \quad E_1(M, Q, \Delta_2) \ll MQ (\log 2M)^3, \\
(13) & \quad E_2(M, Q, \Delta_2) \ll (MQ + (\Delta_2 M)^{-1/2}Q^3)(\log 2M)^4.
\end{align*}
Lemma 2 follows from (11)–(13).

3. A bound for a kind of triple exponential sums. By means of Theorem 2, we can sharpen Lemma A of [2] as follows. We have

**Theorem 3.** Let $H \geq 1$, $X \geq 1$, $Y \geq 1000$; let $\alpha$, $\beta$ and $\gamma$ be real numbers such that $\alpha \gamma (\gamma - 1)(\beta - 1) \neq 0$, and $A > C(\alpha, \beta, \gamma) > 0$, $f(h, x, y) = Ah^\alpha x^\beta y^\gamma$. Define
\begin{equation}
S(H, X, Y) = \sum_{(h, x, y) \in D} C_1(h, x)C_2(y)e(f(h, x, y)),
\end{equation}
where $D$ is a region contained in the rectangle
\begin{equation}
\{(h, x, y) \mid h \sim H, x \sim X, y \sim Y\}
\end{equation}
such that for any fixed pair $(h_0, x_0)$, the intersection $D \cap \{(h_0, x_0, y) \mid y \sim Y\}$ has at most $O(1)$ segments. Also, suppose $|C_1(h, x)| \leq 1$, $|C_2(y)| \leq 1$, $F = AH^{\alpha}X^\beta Y^\gamma \gg Y$. Then
\begin{equation}
L^{-3} S(H, X, Y) \ll 2^{3/8} (HX)^{19/13} F^{3} + HXY^{5/8}(1 + Y^7 F^{-4})^{1/16} + 3^{3/2} (HX)^{29/28} F^{-2} M^5 + 4 (HX)^{3} Y^{4} M \equiv E_1,
\end{equation}
where $L = \log(AHXY + 2)$, $M = \max(1, FY^{-2})$.

**Proof.** We have
\begin{equation}
S(H, X, Y) = \sum_{h \sim H} \sum_{x \sim X} \sum_{y \in I(h, x)} C_2(y)e(f(h, x, y)).
\end{equation}
where $I(h, x)$ is some subinterval of $(Y, 2Y]$. From Lemma 1 of [2], we get
\[
L^{-1} S(H, X, Y) \ll \sum_{h \sim H} \sum_{x \sim X} \left| \sum_{y \sim Y} C(y, \theta)e(f(h, x, y)) \right|,
\]
where $C(y, \theta) = C_2(y)e(\theta y)$ for some real number $\theta$ ($\theta$ is independent of $h$, $x$, and $y$). We consider the expression
\[
Q \succ (14). \text{ If } Q \succ 100, \text{ then we trivially have \[L^{-1} S(H, X, Y) \ll HXYQ^{-1/2} \ll \sqrt{R(Q)} \ll E_1.\]}
\]
Now we assume that $Q \leq 100$. By Cauchy’s inequality and Lemma 3 of [2], we get
\[
L^{-3} |S(H, X, Y)|^2 \ll (HXY)^2 Q^{-1} + (HXY)Q^{-1}|S_1|,
\]
where
\[
S_1 = \sum_{(q, y, h, x) \in D_1} C(q, \theta)\overline{C(y, \theta)}e(Ah^\alpha y^\beta t(y, q)),
\]
\[
t(y, q) = (y + q)^\gamma - (y - q)^\gamma,
\]
\[
D_1 = D_1(Q_1) = \{(q, y, h, x) \mid y + q, y - q \sim Y, q \sim Q_1, h \sim H, x \sim X\}
\]
for some $Q_1$ with $1 \leq 2Q_1 \leq Q/2$. By Lemma 4 of [2] we have (note that $F \gg Y$ by our assumption)
\[
|S_1|^2 \ll FY^{-1}Q_1 A_1 A_2,
\]
where $A_1$ is the number of lattice points $(h, x, h_1, x_1)$ such that
\[
|h^\alpha x^\beta - h_1^\alpha x_1^\beta| \ll A^{-1}Q_1^{-1}Y^{-1-\gamma}
\]
with $h, h_1 \sim H, x, x_1 \sim X$, which is estimated by Lemma 5 of [2] as
\[
A_1 \ll (HX + H^2X^3YQ_1^{-1}F^{-1})L^2;
\]
and $A_2$ stands for the number of lattice points $(q, y, q_1, y_1)$ such that
\[
|t(y, q) - t(y_1, q_1)| \ll (AH^\alpha X^\beta)^{-1}
\]
with $Y/2 < y, y_1 < 3Y$, $q, q_1 \sim Q_1$. Recall that $Q_1 \leq Q/4 < \varepsilon Y^{3/4}$.

Theorem 2 gives (with $\Delta = Q_1^{-1}YF^{-1}$)

(20) \[ A_2 \ll (Q_1Y + Q_1Y^3F^{-1} + Q_1^{8/3})L^4. \]

From (17)–(20), we deduce that (see (15))

(21) \[ L^{-6}|S(H, X, Y)|^2 \ll (HXY)^2Q^{-1} + HXYQ^{-1}(FHXQ(Q + HXYF^{-1})(1 + Y^2F^{-1} + Q^{5/3}Y^{-1}))^{1/2} \ll R(Q). \]

Theorem 3 follows from (21) and (16).

4. The proof of Theorem 1. Put

$\theta = 50/199$, \[ S_{1,2,3} = \sum_{m, n \leq x^{1/3}, m > n} \Psi(xm^{-2}n^{-3}), \quad \Psi(u) = u - \lfloor u \rfloor - 1/2. \]

By Lemmas 7, 8 and Theorems 1, 2 of [2], to prove Theorem 1 it is sufficient to establish the following lemma.

**Lemma B.** \[ S_{1,2,3} \ll x^{\theta + \varepsilon}. \]

Obviously, we have

(22) \[ S_{1,2,3} = \sum_{(M, N)} S_{1,2,3}(M, N) + O(x^{\theta + \varepsilon}), \]

where $M$ and $N$ run through the sequences $\{2^{-j}x^{1/3} \mid j = 0, 1, \ldots\}$ and $\{2^{-k}x^{1/3} \mid k = 0, 1, \ldots\}$ respectively, such that

(23) \[ MN \geq x^{\theta}, \quad 2M \geq N, \quad MN \leq x^{1/3}, \]

and

(24) \[ D = D(M, N) = \{(m, n) \mid m \sim M, \ n \sim N, \ mn \leq x^{1/3}, \ m > n\}. \]

By means of the standard expansion for the function $\Psi(\cdot)$, we get, for any parameter $K, K \in [100, MN]$, the inequality

$(\log K)^{-1}S_{1,2,3}(M, N)$

\[ \ll MNK^{-1} + \sum_{1 \leq h \leq K^2} \min \left( \frac{1}{h}, \frac{K}{h^2} \right) \left| \sum_{(m, n) \in D} e(f(h, m, n)) \right|, \]

where $f(h, m, n) = hxm^{-2}n^{-3}$. Thus, for some $H \in [1, K^2]$, we have

(25) \[ x^{-\varepsilon}S_{1,2,3}(M, N) \ll MNK^{-1} + \min(1, K/H)\Phi_{1,2,3}(H, M, N), \]
where

\[ \Phi_{1,2,3}(H, M, N) = H^{-1} \sum_{h \sim H} \left| \sum_{(m,n) \in D} e(f(h, m, n)) \right| \]  

(we have adopted the notations on pp. 266–267 of [2]). We now use our Theorem 3 three times to estimate the sum \( S_{1,2,3}(M, N) \). Lemma B will then be proved by invoking (49) of [2].

**Lemma 3.**

\[ x^{-\varepsilon} S_{1,2,3}(M, N) \ll 30 x^{11} M^{-11} N^{-12} + \frac{12}{\sqrt{x}} M^{-4} N^{-3} \]
\[ + \frac{45}{\sqrt{x}} M^{-16} N^{-17} + \frac{5}{\sqrt{x}} M^{-2} N^{-3} + x^{1/4} \equiv E_2. \]

**Proof.** We use Lemma 10 of [2] to the summation over \( m \), and obtain, in view of (23),

\[ \sum_{(m,n) \in D} e(f(h, m, n)) = c_1 (hx)^{1/6} \sum_{(n,u) \in D} (n^3 u^4)^{-1/6} e(g(h, n, u)) + O(x^{1/4}), \]

where

\[ g(h, n, u) = c_2 (xhn^{-3}u^2)^{1/3}, \]

\[ D_1 = \{(n, u) \mid u n^6 \leq c_3 h x, h \leq c_4 u, n \sim N, c_5 \leq h x/(n^3 u M^3) \leq c_6 \}, \]

with \( c_i (1 \leq i \leq 6) \) being some absolute constants. From (26) and (27), we find that

\[ x^{-\varepsilon/2} \Phi_{1,2,3}(H, M, N) \ll M(H^3 G)^{-1/2} \sum_{h \sim H} \left| \sum_{(n,u) \in D_1} C(n) \tilde{C}(u) e(g(h, n, u)) \right| + x^{1/4}, \]

where \( |C(n)| \leq 1, |\tilde{C}(u)| \leq 1, \) and \( G = x M^{-2} N^{-3} \). We apply Theorem 3 with \((H, X, Y) \simeq (H, GH/M, N)\) to get (note that \((n,u) \in D_1\) implies \( u \simeq GH/M\))

\[ x^{-\varepsilon/2} \sum_{h \sim H} \left| \sum_{(n,u) \in D_1} C(n) \tilde{C}(u) e(g(h, n, u)) \right| \ll \]
\[ \ll 22 H^{41} G^{22} M^{-19} N^{13} + H^2 G M^{-1} N^{5/8} + \frac{16}{\sqrt{H}} H^{28} G^{12} M^{-16} N^{11} \]
\[ + \frac{32}{H^{56} G^{27} M^{-29} N^{28}} + \frac{32}{H^{61} G^{32} M^{-29} N^{18}} \]
\[ + \sqrt{H} G^3 M^{-3} N^4 + \sqrt{H} G^4 M^{-3} N^2. \]
From (25), (26), (28) and (29), we obtain

\begin{align*}
(30) \quad x^{-\varepsilon}S_{1,2,3}(M, N) &\ll MNK^{-1} + 22\sqrt{K}x^{11}M^{-19}N^{-20} \\
&+ 3\sqrt{K}x^4M^{-8}N^{-7} + 2\sqrt{K}x^4M^{-8}N^{-1} \\
&+ 4\sqrt{K}x^{11}M^{-19}N^{-5} + 3\sqrt{K}x^{16}M^{-29}N^{-30} \\
&+ \sqrt{K}x^2M^{-3}N^{-4} + x^{1/4} \\
&= E_2(K) + x^{1/4}, \quad \text{say.}
\end{align*}

By Lemma 2 of [2], there exists a $K_0 \in [0, MN]$ such that

\begin{align*}
(31) \quad E_2(K_0) &\ll E_2.
\end{align*}

If $K_0 \geq 100$, we put $K = K_0$ in (30), and Lemma 3 follows from (30) and (31); if $K_0 < 100$, we trivially get

\begin{align*}
(32) \quad S_{1,2,3}(M, N) &\ll MNK_0^{-1} \ll E_2(K_0),
\end{align*}

and Lemma 3 follows from (32) and (31).

**Lemma 4.** For $K = MNx^{-\theta}$, $1 \leq H \leq K^2$, we have

\begin{align*}
x^{-\varepsilon}\Phi_{1,2,3}(H, M, N) &\ll \frac{22}{ \sqrt{K}x^3M^7N^{10}} + \frac{NM^{5/8} + \sqrt{H^{-4}x^{-4}M^{25}N^{28}}}{ \sqrt{K}x^4} \\
&+ \frac{3\sqrt{H^{-5}x^{-2}M^{32}N^{35}}}{ \sqrt{K}x^4} + \frac{2\sqrt{H^{-1}M^{12}N^{20}}}{ \sqrt{K}x^4} \\
&+ \frac{\sqrt{H^{-1}M^{12}N^{33}}}{ \sqrt{K}x^4} + x^{1/4}.
\end{align*}

**Proof.** Applying Theorem 3 to the sum $H\Phi_{1,2,3}(H, M, N)$ directly, with $(H, X, Y) \simeq (H, N, M)$, we get the required estimate.

**Lemma 5.** For $K = MNx^{-\theta}$, $1 \leq H \leq K^2$, we have

\begin{align*}
x^{-\varepsilon} \min(1, K/H)\Phi_{1,2,3}(H, M, N) &\ll \frac{22}{ \sqrt{K}x^5MN^6} + \frac{8\sqrt{x^{-1-\theta}M^2N^6}}{ \sqrt{K}x^4} + \frac{3\sqrt{Kx^{-2}M^6N^{16}}}{ \sqrt{K}x^4} \\
&+ \frac{5\sqrt{Kx^8M^{12}N^{20}}}{ \sqrt{K}x^4} + \min(\sqrt{Kx^{-2-\theta}M^{-3}N^{-2}}, \sqrt{Kx^{-3}M^7N^{10}}) + x^{\theta}.
\end{align*}

**Proof.** Applying Theorem 3 to the triple exponential sum of (28), with $(H, X, Y) \simeq (H, N, GH/M)$, we get

\begin{align*}
(33) \quad x^{-\varepsilon/2} \sum_{h \sim H} \left| \sum_{(n, u) \in D_1} C(n)\overline{C}(u)e(g(h, n, u)) \right| &\ll \frac{22H^{35}G^{16}M^{-13}N^{15}}{ \sqrt{K}x^7G^4M^{-2}N^3} + \frac{8H^{13}G^5M^{-5}N^8}{ \sqrt{K}x^7G^4M^{-2}N^3} \\
&+ \frac{16H^{29}G^{13}M^{-17}N^{16}}{ \sqrt{K}x^7G^4M^{-2}N^3} + \frac{32H^{55}G^{20}M^{-28}N^{20}}{ \sqrt{K}x^7G^4M^{-2}N^3} \\
&+ \frac{32H^{50}G^{21}M^{-18}N^{20}}{ \sqrt{K}x^7G^4M^{-2}N^3} + \frac{32H^{50}G^{21}M^{-18}N^{20}}{ \sqrt{K}x^7G^4M^{-2}N^3} \\
&+ \sqrt{K}x^2M^{-3}N^{-4} + x^{1/4}.
\end{align*}
From (28) and (33), we obtain
\[
x^{-\varepsilon} \Phi_{1,2,3}(H, M, N) \ll 2^2 H^2 M^{-1} N^4 + 8 H x M N^{\frac{5}{4}} + \frac{16}{\sqrt{H}} x^5 M^{-11} N
\]
\[
+ \frac{32}{\sqrt{H}} x^{10} M^{-16} N^{-1} + \frac{32}{H^2} x^5 M^4 N^{14}
\]
\[
+ \frac{4}{\sqrt{H}} x^2 M^{-4} N^{-3} + x^{1/4},
\]
which, in conjunction with Lemma 4 and (23), gives
\[
(34) \quad x^{-\varepsilon} \min(1, K/H) \Phi_{1,2,3}(H, M, N)
\]
\[
\ll 2^2 x^{5-2\theta} M N^6 + \sqrt[4]{x^{1-\theta} M^{2} N^6} + \frac{32}{\sqrt{H}} x^{5-2\theta} M^6 N^{16}
\]
\[
+ \min(\sqrt[4]{x^2 H M^{-4} N^{-3}}, \frac{16}{\sqrt{H}} x^{-4} M^{25} N^{28})
\]
\[
+ \min(\sqrt[4]{x^2 H M^{-4} N^{-3}}, \frac{32}{\sqrt{H}} x^{-5} M^{32} N^{35})
\]
\[
+ \min(\sqrt[4]{x^2 H M^{-4} N^{-3}}, \sqrt[4]{H^{-1}} M^{4} N^{3})
\]
\[
+ \min(\sqrt[4]{x^2 M^{-3} N^{-2}}, \sqrt[4]{x M^{7} N^{10}})
\]
\[
+ \min(\sqrt[4]{x^2 M^{-3} N^{-2}}, \sqrt[4]{x^3 M^{12} N^{20}}) + x^\theta.
\]
Obviously,
\[
\min(\sqrt[4]{x^2 H M^{-4} N^{-3}}, \frac{16}{\sqrt{H}} x^{-4} M^{25} N^{28}) \leq \frac{32}{\sqrt{H}} x M^{9} N^{16},
\]
\[
\min(\sqrt[4]{x^2 H M^{-4} N^{-3}}, \frac{32}{\sqrt{H}} x^{-5} M^{32} N^{35}) \leq \frac{32}{\sqrt{H}} x M^{12} N^{20},
\]
\[
\min(\sqrt[4]{x^2 H M^{-4} N^{-3}}, \sqrt[4]{H^{-1}} M^{4} N^{3}) \leq x^{1/4};
\]
and, in view of (23),
\[
\min(\sqrt[4]{x^2 H M^{-4} N^{-3}}, NM^{5/8})
\]
\[
\ll \min(\sqrt[4]{x^2 M^{-3} N^{-2}}, (M^3 N^2)^{13/40}) \leq x^{(26-13\theta)/92} < x^\theta,
\]
\[
\min(\sqrt[4]{x^2 H M^{-4} N^{-3}}, \sqrt[4]{x^3 M^{12} N^{20}})
\]
\[
\ll \min(\sqrt[4]{x^2 M^{-3} N^{-2}}, \sqrt[4]{x^3 (M^3 N^2)^{32/5}}) \leq x^{(79-32\theta)/288} < x^\theta.
\]
From (34) to (39), Lemma 5 follows.

Proof of Lemma B. By (49) of [2], we have
\[
(40) \quad x^{-\varepsilon} S_{1,2,3}(M, N) \ll \sqrt{x^2 M N^{-1}}.
\]
By (25), Lemma 5 and (40), we get

\[ x^{-\varepsilon} S_{1,2,3}(M, N) \ll \frac{2\varepsilon}{\sqrt{M}N^6} + \sqrt{x^{1-\theta}M^2N^6} + \frac{3\varepsilon}{\sqrt{x^{5-29}MN^6}N^{16}} + \frac{\varepsilon}{x^4M^9N^{16}} + \frac{\varepsilon}{x^8M^{12}N^{20}} + R_1(M, N) + x^\theta, \]

where

\[ R_1(M, N) = \min(\frac{\varepsilon}{\sqrt{x^3M^7N^{10}}}, \frac{\varepsilon}{\sqrt{x^2M^{-3}N^{-2}}}, \frac{\varepsilon}{\sqrt{x^4MN^{-1}}}) \]

\[ \leq (\frac{2\varepsilon}{\sqrt{x^3M^7N^{10}}} \alpha_1) (\sqrt{x^2M^{-3}N^{-2}} \beta_1) (\sqrt{x^4MN^{-1}} \gamma_1) \]

\[ = x^{(81-17\theta)/306} < x^\theta, \]

with \( (\alpha_1, \beta_1, \gamma_1) = (110/306, 68/306, 128/306) \); thus

\[ x^{-\varepsilon} S_{1,2,3}(M, N) \ll \frac{2\varepsilon}{\sqrt{x^{5-29}MN^6}} + \sqrt{x^{1-\theta}M^2N^6} + \frac{3\varepsilon}{\sqrt{x^{5-29}MN^6}N^{16}} + \frac{\varepsilon}{x^4M^9N^{16}} + \frac{\varepsilon}{x^8M^{12}N^{20}} + x^\theta. \]

If \( MN \leq x^{0.3} \), then (41) gives

\[ x^{-\varepsilon} S_{1,2,3}(M, N) \ll \frac{2\varepsilon}{\sqrt{x^{5-29}MN^6}} + x^\theta. \]

From Lemma 3, (40) and (42), we deduce that

\[ x^{-\varepsilon} S_{1,2,3}(M, N) \ll \sum_{i=2}^{5} R_i(M, N) + x^\theta, \]

where

\[ R_2(M, N) = \min(\frac{3\varepsilon}{\sqrt{x^{11}M^{-11}N^{-12}}}, \frac{\varepsilon}{\sqrt{x^{5-29}MN^6}}, \frac{\varepsilon}{\sqrt{x^2MN^{-1}}}) \]

\[ \leq (\frac{3\varepsilon}{\sqrt{x^{11}M^{-11}N^{-12}}} \alpha_2) (\frac{\varepsilon}{\sqrt{x^5-29MN^6}} \beta_2) (\frac{\varepsilon}{\sqrt{x^2MN^{-1}}} \gamma_2) \]

\[ = x^{(150-239)/574} = x^\theta, \]

with \( (\alpha_2, \beta_2, \gamma_2) = (105/574, 253/574, 216/574) \);

\[ R_3(M, N) = \min(\frac{12\varepsilon}{\sqrt{x^4M^{-4}N^{-3}}}, \frac{\varepsilon}{\sqrt{x^5-29MN^6}}) \]

\[ \ll (\frac{12\varepsilon}{\sqrt{x^4M^{-4}N^{-3}}} \alpha_3) (\frac{\varepsilon}{\sqrt{x^5-29MN^6}} \beta_3) \]

\[ \leq x^{(9-2\theta)/34} < x^\theta; \]

\[ R_4(M, N) = \min(\frac{45\varepsilon}{\sqrt{x^{16}M^{-16}N^{-17}}}, \frac{\varepsilon}{\sqrt{x^5-29MN^6}}) \]

\[ \ll (\frac{45\varepsilon}{\sqrt{x^{16}M^{-16}N^{-17}}} \alpha_4) (\frac{\varepsilon}{\sqrt{x^5-29MN^6}} \beta_4) \]

\[ \leq x^{(277-669)/1041} < x^\theta; \]

\[ R_5(M, N) \]

\[ = \min(\frac{\varepsilon}{\sqrt{x^2M^{-2}N^{-3}}}, \frac{\varepsilon}{\sqrt{x^5-29MN^6}}, \frac{\varepsilon}{\sqrt{x^2MN^{-1}}}) \]
\[
\leq (\sqrt[5]{x^2 M^{-2} N^{-3}})^{35/217} (\sqrt[22]{x^{5-2\theta} M N^6})^{110/217} (\sqrt[8]{x^2 M N^{-1}})^{72/217} = x^{(57-100)/217} < x^\theta.
\]

From (43) to (47), we have
\[
x^{-\varepsilon} S_{1,2,3}(M,N) \ll x^\theta.
\]

If \(MN > x^{0.3}\), from Lemma 3 we find
\[
x^{-\varepsilon} S_{1,2,3}(M,N) \ll 30 \sqrt[30]{x^{11} M^{-11} N^{-12}} + \sqrt[30]{x^2 M^{-2} N^{-3}} + x^\theta.
\]

From (40), (41) and (49), we deduce that
\[
x^{-\varepsilon} S_{1,2,3}(M,N) \ll \sum_{i=6}^{15} R_i(M,N) + x^\theta,
\]

where, by (44) and (47),
\[
R_6(M,N) = R_2(M,N) \leq x^\theta, \quad R_7(M,N) = R_5(M,N) < x^\theta,
\]

(51)

\[
R_8(M,N) = \min(30 \sqrt[30]{x^{11} M^{-11} N^{-12}}, \sqrt[30]{x^1 M^2 N^6}) 
\ll (30 \sqrt[30]{x^{11} M^{-11} N^{-12}})^{30/53} (\sqrt[30]{x^1 M^2 N^6})^{8/23} 23/53
\]

\[
= x^{(111-230)/424} < x^\theta;
\]

(52)

\[
R_9(M,N) = \min(\sqrt[6]{x^2 M^{-2} N^{-3}}, \sqrt[6]{x^1 M^2 N^6}) 
\ll (\sqrt[6]{x^2 M^{-2} N^{-3}})^{1/2} (\sqrt[6]{x^1 M^2 N^6})^{8/12} 12/23
\]

\[
= x^{(21-59)/80} < x^\theta;
\]

(53)

\[
R_{10}(M,N)
= \min(30 \sqrt[30]{x^{11} M^{-11} N^{-12}}, 32 \sqrt[32]{x^{5-2\theta} M^6 N^{16}})
\ll (30 \sqrt[30]{x^{11} M^{-11} N^{-12}})^{165/349} (32 \sqrt[32]{x^{5-2\theta} M^6 N^{16}})^{22/23} 23/349
\]

\[
= x^{(357-460)/1396} < x^\theta;
\]

(54)

\[
R_{11}(M,N) = \min(\sqrt[6]{x^2 M^{-2} N^{-3}}, 32 \sqrt[32]{x^{5-2\theta} M^6 N^{16}})
\ll (\sqrt[6]{x^2 M^{-2} N^{-3}})^{22/54} (32 \sqrt[32]{x^{5-2\theta} M^6 N^{16}})^{22/23} 23/54
\]

\[
= x^{(69-109)/270} < x^\theta;
\]

(55)

\[
R_{12}(M,N) = \min(30 \sqrt[30]{x^{11} M^{-11} N^{-12}}, 32 \sqrt[32]{x^1 M^9 N^{16}})
\ll ((30 \sqrt[30]{x^{11} M^{-11} N^{-12}})^{750} (32 \sqrt[32]{x^1 M^9 N^{16}})^{25/23} 23/750
\]

\[
= x^{367/1486};
\]

(56)
(57) \( R_{13}(M, N) = \min(\sqrt[5]{x^2 M^{-2} N^{-3}}, \sqrt[32]{x^4 M^2 N^4}) \approx (\sqrt[5]{x^2 M^{-2} N^{-3}}^{25}(\sqrt[32]{x^4 M^2 N^4})^{5})^{32} 1^{1/57} = x^{14/57}; \)

(58) \( R_{14}(M, N) = \min(\sqrt[30]{x^{11} M^{-11} N^{-12}}, \sqrt[26]{x^4 M^6 N^{10}}) \approx (\sqrt[30]{x^{11} M^{-11} N^{-12}}^{240}(\sqrt[13]{x^2 M^{11} N^{12}}^{8/23})^{299})^{1/539} = x^{134/539}; \)

(59) \( R_{15}(M, N) = \min(\sqrt[5]{x^2 M^{-2} N^{-3}}, \sqrt[13]{x^2 M^3 N^5}) \approx (\sqrt[5]{x^2 M^{-2} N^{-3}}^{40}(\sqrt[13]{x^2 M^3 N^5})^{8/5})^{65} 1^{1/105} = x^{26/105}. \)

From (50) to (59), we have

\( x^{-\varepsilon} S_{1,2,3}(M, N) \ll x^\theta. \)

Lemma B follows from (48) and (60).

5. Concluding remarks. It is clear that our result 50/199 is closely connected with the term \( Q^{8/3} \) in Theorem 2. This term actually comes from the method given in Lemmas 3 and 4 of [1]. The fraction 50/199 can be reduced whenever \( Q^{8/3} \) can be reduced in our Theorem 2. If, for example, \( Q^{8/3} \) could be “omitted”, then one may attain the expected exponent 1/4, in place of 50/199.

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