

Functions without residue and a bilinear differential equation

by

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*Dedicated to my colleague Jens Mennicke
in friendship on the occasion of his 60-th birthday*

1. To motivate the following investigation let us assume that we wish to have examples, as elementary as possible, of arc-length integrals for a calculus course. "As elementary as possible" might mean that f and $g = \int \sqrt{1 + f'^2}$ both should be rational functions. Putting $g - f =: u$, $g + f =: v$ turns the original $g'^2 = 1 + f'^2$ into $u'v' = 1$, u and v both rational. Phrasing it differently again, we ask for the w for which

(1) *all residues of w and w^{-1} vanish.*

DEFINITION 1. A (rational) function w with the property (1) will be called *special*. Extending this notion in a natural way we may call w *super-special* if all residues of all w^n , $n \in \mathbb{Z}$, vanish.

Obviously, if $w(x)$ is special then so is $c_1 w(c_2(x - x_0))$ for $c_1 c_2 \neq 0$. Trivial examples are all

(2) $w = c(x - x_0)^n$ with $n \in \mathbb{Z}$, $n \neq \pm 1$, $c \neq 0$.

For a rational function

(3)
$$w = c \frac{\prod_{i=1}^k (x - a_i)^{\mu_i}}{\prod_{j=1}^l (x - b_j)^{\nu_j}},$$

where $\mu_i, \nu_j \in \mathbb{N}$, $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$ and $a_i \neq b_j$, to be special it is necessary that all $\mu_i, \nu_j \geq 2$.

DEFINITION 2. Those (rational) functions which, in the representation (3), have all μ_i and $\nu_j = 2$ will be called *simple*.

As a shorthand for *without residue* we write w.res. Thus " w is special" means " w and w^{-1} are w.res."

If w , written as in (3), is w.res. then it has a primary function

$$f = \frac{\text{polynomial}}{\prod_{j=1}^l (x - b_j)^{\nu_j - 1}},$$

whose order at infinity, $\sum \mu_i - \sum \nu_j + 1$, is at least $-\sum(\nu_j - 1)$. Hence

$$l \leq 1 + \deg(\text{numerator}) \quad \text{if } w \text{ is w.res.}$$

In particular, the $c(x - x_0)^n$ quoted above are the only special polynomials.

G. Szekeres was the first to find a nontrivial special function. In essence his example is

$$w = \left(\frac{x^3 + 1}{x} \right)^2 = x^4 + 2x + \frac{1}{x^2},$$

$$\frac{1}{w} = \frac{x^2}{(x^3 + 1)^2} = \frac{-1}{3} \left(\frac{1}{x^3 + 1} \right)'.$$

Similar are

$$w = \frac{(x^m + 1)^n}{x^{m-1}} \quad \text{with } m \geq 3, n \geq 2$$

and with a little calculation one can see that

$$(4) \quad w = \frac{(x^m + 1)^n}{x^k}, \quad \text{where } k, m, n \in \mathbb{N},$$

is special exactly if

$$(5) \quad m \geq 3, n \geq 2 \quad \text{and} \quad k \in \{m - 1, 2m - 1, \dots, (n - 1)m - 1\}.$$

2.1. It seems to be very difficult to find all special functions, but we can give rather satisfactory results about the simple special functions and a somewhat wider class, to be defined later, that we call semisimple special functions. There are special rational functions that are not semisimple. Examples are all instances of (4), (5), with odd n or k , since semisimple special functions are squares. In 3.3.2 we also give an example of a special function that is a square but not semisimple.

Concerning superspecial functions the situation is easier. As a meromorphic solution we have \tan^2 , but the only rational ones are given by (2) (see Theorem 11). The proof is independent of the rest of the paper and will, therefore, be postponed to the end.

Our first observation on special functions is that among the simple $w = (q/p)^2$ the special ones can be characterized (Theorem 1) by the bilinear differential equation

$$(6) \quad B(p, q) := p''q - 2p'q' + pq'' = 0.$$

There is, to my knowledge, no theory of polynomial solutions of bilinear differential equations. Therefore Theorem 1 does not solve our problem,

but it provides an entry to its solution. A first consequence of (6) is that necessarily $\deg p, \deg q$ are $\binom{n}{2}, \binom{n+1}{2}$ with some $n \in \mathbb{Z}$ and that, therefore, $\deg(pq)$ is a square number (see Theorem 2). More important is the observation that the second solution of the differential equation in $y : B(y, q) = 0$, linearly independent of p ,

$$(7) \quad r := p \int \left(\frac{q}{p}\right)^2$$

happens to be a polynomial. Thus from $w = (q/p)^2$ another special function $(r/q)^2$ is derived. With care concerning the condition of simplicity we may iterate this procedure and obtain infinite sequences of polynomials

$$(8) \quad p_0 := p, \quad p_1 := q, \quad p_{n+1} := p_{n-1} \int \left(\frac{p_n}{p_{n-1}}\right)^2 \quad \text{for } n \geq 1,$$

for which all $(p_{i+1}/p_i)^2$ are special (Theorem 4 and Corollary 4.1).

At this point it becomes necessary to specify that *in all of this paper K is assumed to be a field with $\text{char } K = 0$* , and that the words “polynomials”, “rational functions” refer to elements of $K[x], K(x)$ respectively. This assumption guarantees that every rational function w.res. can be integrated within the field $K(x)$. In this respect it is irrelevant whether the poles of the function lie in K or in some extension of K .

Next we should mention a forward-backward symmetry in (8). In fact, (7) with any constant of integration is equivalent to

$$p = -r \int \left(\frac{q}{r}\right)^2$$

with a suitable constant of integration, since both relations express in different ways that

$$(9) \quad r'p - rp' = q^2.$$

Consequently, sequences like (8) can also be extended to negative indices n .

It should further be observed that putting $v_n := p_{n+1}/p_n$ from (8) a binary recursion is derived:

$$v_{n+1} = \frac{1}{v_n} \int v_n^2.$$

DEFINITION 3. An operation of the type that we met here,

$$u = c \frac{1}{v} \int v^2, \quad c = \text{const} \neq 0,$$

will be called a *squid* (SQUare, Integrate, Divide).

The constant c will usually and without loss of generality be ± 1 , but because of the constant of integration there is still a whole family of squids.

The above forward-backward symmetry reappears as

$$(10) \quad u = \frac{1}{v} \int v^2 \Leftrightarrow \frac{1}{v} = -u \int \frac{1}{u^2}.$$

Since every step in (8) generates a further constant of integration one should think of a tree rather than a sequence. The condition of simplicity cannot simply be dropped from Theorem 4. Unfortunately, it has the effect to prune seemingly sound branches off the tree. This is particularly irritating as it applies similarly to operations in the backward direction and becomes an obstacle to the quest for a simple “root” of our tree.

2.2. A much more harmonious picture appears after extending the set of simple special to what will be called *semisimple special*, in short sss-functions (see 4.1 and Definition 7). This is an astonishing set of functions v^2 :

All v^2 are special, the set is closed under all squids and under inversion and it contains all those v for which v^2 is simple and special (Theorem 7).

In fact, the sss-functions form the closure of the constant 1 under all these operations (Theorem 9). As a consequence we can construct a parametric solution (Theorem 10) $V_n(\Gamma_1, \dots, \Gamma_n, x)$ with indeterminates $\Gamma_1, \Gamma_2, \dots$,

$$V_0 = 1, \quad V_{n+1} = \frac{1}{V_n} \int V_n^2,$$

such that any sss-function v or its inverse v^{-1} is obtained by substituting field elements γ_i for the Γ_i . In particular, $n = 1$ corresponds to (2) and $n = 2$ to Szekeres’s example. The next step gives, with a bit of cosmetics, the special functions $w = (q/p)^2$, where

$$p = x^3 + \gamma, \quad q = x^6 + 5\gamma x^3 - 5\gamma^2 + \delta x, \quad \gamma, \delta \in K.$$

If p, q have a common zero $\omega \neq 0$ then $\gamma = -\omega^3$, $\delta = 9\omega^5$. Since ω is a simple zero of p Theorem 2(i) implies that ω is a triple zero of q ; in fact,

$$q = (x^3 + 3\omega x^2 + 6\omega^2 x + 5\omega^3)(x - \omega)^3.$$

Thus for any $\omega \in K$

$$w = \frac{(x^3 + 3\omega x^2 + 6\omega^2 x + 5\omega^3)^2 (x - \omega)^4}{(x^2 + \omega x + \omega^2)^2}$$

is special but not simple.

The concept of sss-functions is basically a local one. Thus, on the one hand, the proper setting for the proof of Theorem 7 is the field $K[[x]]$ of formal power series rather than that of rational functions. On the other hand, these ideas equally apply to meromorphic functions. Specific for rational functions is mainly the statement (Theorem 9) that all sss-functions are generated by squids from the constants. In this proof again it will be important that v has a representation $v = q/p$ with (6).

We use the occasion to introduce the following notion:

DEFINITION 4. Let us say that v has a B -representation if there are p, q such that

$$v = \frac{q}{p} \quad \text{and} \quad B(p, q) = 0.$$

Depending on the context v, p, q are understood to be in $K(x), K[[x]]$ or in $\text{Mer}(\mathcal{D})$, the set of functions meromorphic on a region \mathcal{D} .

3.1. A constantly repeated pattern of notation will be

$$w = v^2, \quad v = q/p,$$

or

$$w_n = v_n^2, \quad v_n = p_{n+1}/p_n.$$

To phrase some statements conveniently we further need

DEFINITION 5. A pair (a, b) of integers will be called *magic* if

$$(a - b)^2 = a + b.$$

The magic pairs are easily parametrized: if $i := a - b$ then $a = \binom{i+1}{2}$, $b = \binom{i}{2}$.

3.2. Global considerations in $K(x)$

THEOREM 1. Suppose that $p, q \in K[x]$ are coprime. Then

$$\left(\frac{q}{p}\right)^2 \text{ is special and simple if and only if } B(p, q) = 0.$$

The proof depends on

LEMMA 1. Suppose $p, f \in K[x]$, $(p, p') = 1$. Then

$$f/p^2 \text{ is w.res. if and only if } f'p' \equiv fp'' \pmod{p}.$$

Proof. Let ω be any zero of p . By assumption $p'(\omega) \neq 0$. Therefore

$$\begin{aligned} \frac{f}{p^2}(\omega + x) &= \frac{f(\omega) + xf'(\omega) + \dots}{(xp'(\omega) + \frac{1}{2}x^2p''(\omega) + \dots)^2} \\ &= \frac{1}{p'^2(\omega)} \cdot \frac{1}{x^2}(f(\omega) + xf'(\omega) + \dots) \left(1 - \frac{x}{2} \cdot \frac{p''(\omega)}{p'(\omega)} + \dots\right)^2 \\ &= \frac{f(\omega)}{p'^2(\omega)x^2} + \frac{1}{p'^3(\omega)}(f'(\omega)p'(\omega) - f(\omega)p''(\omega))\frac{1}{x} + \dots \end{aligned}$$

The residue at ω vanishes if and only if $(f'p' - fp'')|_{\omega} = 0$, and this condition for all zeros of p means $p \mid (f'p' - fp'')$.

Proof of Theorem 1. Suppose first that $w = (q/p)^2$ is special and simple. Then it is w.res. and $(p, p') = 1$. Apply Lemma 1 with $f = q^2$ and

note that q may be cancelled since $(p, q) = 1$. Hence

$$2p'q' \equiv qp'' \pmod{p}.$$

Symmetrically

$$2p'q' \equiv pq'' \pmod{q}.$$

Since $(p, q) = 1$ the two congruences can be combined into

$$p''q - 2p'q' + pq'' \equiv 0 \pmod{pq},$$

in other words

$$pq \mid B(p, q).$$

Since $\deg(B(p, q)) < \deg(pq)$ we have $B(p, q) = 0$. Now assume $B(p, q) = 0$. If q , say, had a zero ω of order $k \geq 2$ then $2p'q' - p''q$ would vanish with order $\geq k - 1$ but q'' only with order $k - 2$. Then

$$pq'' = 2p'q' - p''q$$

implies $p(\omega) = 0$, contradicting $(p, q) = 1$. (This is actually a special case of Theorem 2(i) below.) Thus q and similarly p are squarefree, and w is simple. We can now apply Lemma 1 the other way and find that $(q/p)^2$ and $(p/q)^2$ are w.res.

THEOREM 2. *Let $p, q \in K(x)$ and $B(p, q) = 0$. Then*

- (i) *for every place $x_0 \in K$ the pair $(\text{ord}_{x_0} p, \text{ord}_{x_0} q)$ is magic,*
- (ii) *p, q are polynomials,*
- (iii) *$(\deg p, \deg q)$ is magic.*

In particular, polynomials p, q with $B(p, q) = 0$ are coprime if and only if they are both squarefree.

Proof. Concerning (i) we take, without loss of generality, $x_0 = 0$. Write

$$p = \sum_{\kappa \geq k} a_{\kappa} x^{\kappa}, \quad a_{\kappa} \neq 0 \quad (k = \text{ord } p), \quad q = \sum_{\lambda \geq l} b_{\lambda} x^{\lambda}, \quad b_{\lambda} \neq 0 \quad (l = \text{ord } q).$$

Comparing coefficients translates $B(p, q) = 0$ into

$$\sum_{\kappa + \lambda = n} (\kappa(\kappa - 1) - 2\kappa\lambda + \lambda(\lambda - 1)) a_{\kappa} b_{\lambda} = 0$$

for all n . For $n = k + l$ this sum contains only one nonzero term:

$$(k(k - 1) - 2kl + l(l - 1)) a_k b_l = 0,$$

hence

$$(k - l)^2 - k - l = 0,$$

so (k, l) is magic, $k, l \geq 0$, and as this holds for all x_0 , p and q are polynomials.

For (iii) one argues as for (i), expanding at ∞ this time. The concluding statement follows from (i).

An obvious consequence is

COROLLARY 2.1. *If $v \in K(x)$ has a B -representation q/p then p and q are polynomials.*

Parts of Theorems 1 and 2 have an interesting generalization.

THEOREM 3. *Let $q = \prod(x - x_i)^{a_i}$, $p = \prod(x - x_i)^{b_i}$, $x_i \in K$, $a_i, b_i \in \mathbb{N}_0$, $(a_i, b_i) \neq (0, 0)$. Then $B(p, q) = 0$ if and only if*

- (i) *all (a_i, b_i) are magic*
- and
- (ii) *for all i*

$$\left(\frac{q}{p}\right)^{2/(b_i - a_i)}$$

is w.res. at x_i .

The proof can be given along the lines of the above theorems, or more conveniently by using Theorem 8 below. We shall not need Theorem 3 in the following and therefore omit the proof, but want to mention an identity that is helpful with such questions.

LEMMA 2. *For $p, q \neq 0$*

$$\frac{B(p, q)}{pq} = \left(\frac{p'}{p} - \frac{q'}{q}\right)^2 + \left(\frac{p'}{p} + \frac{q'}{q}\right)'$$

THEOREM 4. *Let $p_0, p_1 \in K[x], \neq 0$, $B(p_0, p_1) = 0$ and $(p_0, p_1) = 1$. Then with every choice of the constant of integration*

$$p_2 := p_0 \int \left(\frac{p_1}{p_0}\right)^2$$

is a polynomial, linearly independent of p_0 and such that $B(p_1, p_2) = 0$. Except for at most $\deg p_1$ values of the constant of integration we have $(p_1, p_2) = 1$.

PROOF. The expression for p_2 is just what the standard methods (variation of constant, Wronski determinant) produce for the other linearly independent solution of $B(y, p_1) = 0$. This can, of course, easily be verified. By Theorem 1, $(p_1/p_0)^2$ is w.res. Its poles, being all of order two, give simple poles for the integral, which are cancelled upon multiplying with p_0 . Thus $p_2 \in K[x]$.

If \tilde{p}_2 is any particular solution then all are of the form $p_2 = \tilde{p}_2 + \gamma p_0$, $\gamma \in K$. If furthermore ω is a zero of p_1 then $p_0(\omega) \neq 0$ since $(p_0, p_1) = 1$. Therefore $p_2(\omega) = 0$ only for

$$\gamma = \gamma_\omega := -\frac{\tilde{p}_2(\omega)}{p_0(\omega)}.$$

If $\gamma \neq \gamma_\omega$ for all zeros of p_1 then $(p_1, p_2) = 1$.

COROLLARY 4.1. *Let $w_0 = v_0^2 = (p_1/p_0)^2$ be special and simple, $(p_0, p_1) = 1$. If in each step finitely many exceptions, as described in Theorem 4, are avoided then the squid iteration*

$$(11) \quad v_{n+1} = \frac{1}{v_n} \int v_n^2$$

continues indefinitely and all v_n^2 are special and simple. (Remember that the field K is infinite.)

PROOF. By Theorem 1, $B(p_0, p_1) = 0$. For p_2 as in Theorem 4 and $v_1 = p_2/p_1$ we have (11). Apart from the $\gamma = \gamma_\omega$ we have $(p_1, p_2) = 1$ and $B(p_1, p_2) = 0$, so by Theorem 1, v_1 is special and simple and the iteration continues.

For an exceptional value $\gamma = \gamma_\omega$ there actually is a common zero ω of p_1 and p_2 which by Theorem 2(i) is a triple zero of p_2 . So v_1 definitely is not simple. But what about specialty? Intuitively, we should expect the residue of v_1^2 to depend continuously on the parameter γ , that is, to vanish for all γ . Since $v_1^{-2} = (p_1/p_2)^2 = -(p_0/p_2)'$ is w.res. anyway this means v_1^2 should always be special. It is in fact easy to give a rigorous argument concerning v_1 but v_2, v_3, \dots are more difficult. Our next theorem solves this problem in the case $\deg p_0 \leq \deg p_1$. Instead of continuity we use the simpler mechanism of specializing indeterminates.

THEOREM 5. *Let $w_0 = (p_1/p_0)^2$ be special and simple, $(p_0, p_1) = 1$, $\deg p_0 \leq \deg p_1$. Let $\Gamma_1, \Gamma_2, \dots$ be independent indeterminates over K . Then there are $P_i \in K[\Gamma_1, \dots, \Gamma_{i-1}, x]$ for $i \geq 2$ such that with $P_0 = p_0$, $P_1 = p_1$*

$$(12) \quad P'_{i+1}P_{i-1} - P_{i+1}P'_{i-1} = P_i^2 \quad \text{for all } i \in \mathbb{N}.$$

All $W_i = (P_{i+1}/P_i)^2$ are special and simple functions of x .

PROOF. Specialty is, of course, an immediate consequence of (12):

$$\left(\frac{P_i}{P_{i+1}}\right)^2 = -\left(\frac{P_{i-1}}{P_{i+1}}\right)^2, \quad \left(\frac{P_i}{P_{i-1}}\right)^2 = \left(\frac{P_{i+1}}{P_{i-1}}\right)'$$

The (inductive) proof depends on two further properties of the P_i . For all $i \in \mathbb{N}$

$$(P_i, P_{i-1}) = 1$$

and the main coefficients m_i of P_i (with respect to x) are in K .

In the first instance we are content to locate the P_i in $K(\Gamma_1, \dots, \Gamma_{i-1})[x]$. For this purpose we need only apply Corollary 4.1. Some

$$(13) \quad \tilde{P}_{i+1} = P_{i-1} \int \left(\frac{P_i}{P_{i-1}}\right)^2$$

can be found in $K(\Gamma_1, \dots, \Gamma_{i-1})[x]$. If then

$$(14) \quad P_{i+1} := \tilde{P}_{i+1} + \Gamma_i P_{i-1}$$

is set, P_i and P_{i+1} are coprime. A hypothetic common zero ω of P_i and P_{i+1} would give $P_{i-1}(\omega) \neq 0$ and would, by (14), create an algebraic dependence of $\Gamma_1, \dots, \Gamma_i$. So the exceptions, mentioned in the corollary, cannot occur. Another simple induction shows

$$\deg P_0 \leq \deg P_1 < \deg P_2 < \dots$$

Therefore, by (14), m_{i+1} is the main coefficient of \tilde{P}_{i+1} , which from (13) is seen to be constant, since m_i and m_{i-1} are.

Suppose now that actually the coefficients of P_i and P_{i-1} are polynomials in $\Gamma_1, \dots, \Gamma_{i-1}$, and let

$$\begin{aligned} \deg P_i &=: d_i, & \deg P_{i-1} &=: d_{i-1}, \\ P_i &= m_i x^{d_i} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots \right), \\ P_{i-1} &= m_{i-1} x^{d_{i-1}} \left(1 + \frac{b_1}{x} + \frac{b_2}{x^2} + \dots \right). \end{aligned}$$

Then

$$\left(\frac{P_i}{P_{i-1}} \right)^2 = \left(\frac{m_i}{m_{i-1}} \right)^2 x^{2(d_i - d_{i-1})} \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots \right),$$

where all $c_r \in K[\Gamma_1, \dots, \Gamma_{i-1}]$. A term $1/x$ does not occur since a function that is w.res. (at all $x_0 \in K$) is also w.res. at ∞ . If this expansion is integrated formally (no constant added yet) and multiplied with P_{i-1} the coefficients of \tilde{P}_{i+1} are seen to be in $K[\Gamma_1, \dots, \Gamma_{i-1}]$, as claimed. It is not visible from this construction that \tilde{P}_{i+1} is a polynomial with respect to x , but this information is already part of Theorem 4.

COROLLARY 5.1. *Suppose that w_0 is special and simple, $w_0 = v_0^2$, $v_0 = p_1/p_0$, where $p_0, p_1 \in K[x]$, $(p_0, p_1) = 1$ and $\deg p_0 \leq \deg p_1$. Then with every choice of the constants of integration we may indefinitely iterate in $K(x)$*

$$v_{n+1} = \frac{1}{v_n} \int v_n^2, \quad p_{n+1} = v_n p_n.$$

All v_n^2 are special rational functions, all p_n polynomials over K , and $B(p_{n-1}, p_n) = 0$ for all $n \in \mathbb{N}$.

Proof. Differentiation in $K[\Gamma_1, \dots, \Gamma_n][x]$ and $K[x]$ commutes with the homomorphism over K that takes Γ_i into $\gamma_i \in K$. We put

$$\begin{aligned} p_n(x) &:= P_n(\gamma_1, \dots, \gamma_{n-1}, x), \\ \tilde{p}_{n+1}(x) &:= \tilde{P}_{n+1}(\gamma_1, \dots, \gamma_{n-1}, x). \end{aligned}$$

Since (12) implies

$$(15) \quad \tilde{p}'_{n+1}p_{n-1} - \tilde{p}_{n+1}p'_{n-1} = p_n^2, \quad p'_{n+1}p_{n-1} - p_{n+1}p'_{n-1} = p_n^2,$$

\tilde{p}_{n+1} is one of the integrals $p_{n-1} \int (p_n/p_{n-1})^2$ and all are given as $p_{n+1} = \tilde{p}_{n+1} + \gamma_n p_{n-1} = P_{n+1}(\gamma_1, \dots, \gamma_n, x)$ with suitable $\gamma_n \in K$. As before (15) implies that all $v_n^2 = (p_{n+1}/p_n)^2$ are special.

Since $(P_{n-1}, P_n) = 1$ for all n , Theorem 4 adds to Theorem 5 the remark that $B(P_{n-1}, P_n) = 0$, and this relation specializes to

$$B(p_{n-1}, p_n) = 0.$$

3.3. Some remarks

3.3.1. Actually Corollary 5.1 is valid for $\deg p_0 > \deg p_1$ as well (see Theorem 9), but the proof of Theorem 5 fails at the point $m_i \in K$. We could obtain only $P_i \in K(\Gamma_1, \dots, \Gamma_{i-1})[x]$ and would not be able to specialize at will. A proof along similar lines can be given, but it is quite complicated. We formulate only its central lemma which may be of some interest itself.

LEMMA 3. *Let $\Gamma_1, \dots, \Gamma_j$ be independent indeterminates over K and let R, S be x -polynomials over $K[\Gamma_1, \dots, \Gamma_j]$. Assume that the differential equation*

$$(16) \quad y'R - yR' = S$$

has a polynomial solution $y_0(x)$ over $K(\Gamma_1, \dots, \Gamma_j)$. If the coefficients of R (i.e. of the x -powers) generate the 1-ideal, that is, $K[\Gamma_1, \dots, \Gamma_j]$, then (16) possesses a polynomial solution $y_1(x)$ over $K[\Gamma_1, \dots, \Gamma_j]$.

Lemma 3, incidentally, can also be used in the proof of Theorem 5. It replaces the part in which P_{i-1}, P_i are expanded at ∞ .

3.3.2. There is one peculiar feature about Corollary 5.1 that definitely prompts questioning: Though w_0 is required to be simple (and special, of course) it is claimed that the squid iteration continues indefinitely whether the further w_i are simple or not, and we know that they will in general not stay simple. The most drastic counterexample is the sequence

$$w_n = c_n^2 x^{2n} \quad \text{with } c_n = (1 \cdot 3 \cdot \dots \cdot (2n-1))^{-1}.$$

Could it be then, that we may drop the assumption that w_0 be simple? Or replace it by asking for a B -representation of v_0 ? Neither is correct as can be seen from the example (actually an instance of (4),(5))

$$(17) \quad w_0 = \left(\frac{x^3 - 1}{x} \right)^8, \quad \text{where } p_0 = x^{10}(x^3 - 1)^6, \quad q_0 = x^6(x^3 - 1)^{10}.$$

The computation is not as laborious as it may seem. First of all $B(p_0, q_0) = 0$ is a convenient application of Lemma 2 or, what can be derived from it easily,

LEMMA 4. If (a, b) is magic then

$$B(p^a q^b, p^b q^a) = (a + b)(pq)^{a+b-1} B(p, q).$$

Secondly, to check the residues of v_i^2 , it suffices to determine the Taylor expansions with modest accuracy. At zero we have

$$\begin{aligned} v_0^2 &= x^{-8}(1 - 8x^3 + 28x^6 + O(x^8)), \\ v_1^2 &= c_1 x^{-6}(1 - 20x^3 + O(x^6)), \\ v_2^2 &= c_2 x^{-4}(1 - 80x^3 + O(x^4)). \end{aligned}$$

As one sees, the squid iteration is terminated by the nonzero residue of v_2^2 (if not earlier by some residue of v_1^2 at another place) despite the specialty of w_0 and $B(p_0, q_0) = 0$.

3.3.3. Apart from the peculiar values connected with this particular example the above calculation reminds us that the residue of v_n^2 at some place, say zero, can be determined if sufficiently many derivatives of v_0 are known there together with the constants of integration, which in our example went into the remainders. If a local version of Corollary 5.1 could be proved, that is, a theorem in which the v_n are formal power series, then not only the global version should follow trivially, but also the distinction $\deg p_0 \leq$ or $>$ $\deg p_1$ would become immaterial.

Unfortunately, neither the proof of Theorem 5 given above nor the one just mentioned in 3.3.1 can be localized. Both of them depend strongly on properties of polynomials. A formal power series at $x = 0$ cannot be re-expanded at $x = \infty$ and Lemma 3 turns false if the words “polynomial solution” are replaced by “power series solution”. A counterexample here is $R = (x - \Gamma)^2$, $S = 1$, where all solutions are given by

$$y = \gamma(\Gamma^2 - 2\Gamma x + x^2) + \frac{1}{4\Gamma} \left(1 + \frac{x}{\Gamma} + \frac{x^2}{\Gamma^2} + \dots \right).$$

3.4. Local considerations

3.4.1. We consider formal power series

$$w = \sum_{n \geq N} a_n x^n, \quad N \in \mathbb{Z},$$

over our field K with $\text{char } K = 0$. These series form themselves a field $K[[x]]$ with a valuation $\text{ord } w = \min_{a_n \neq 0} n$. A series w is called *entire* if $\text{ord } w \geq 0$. Localizing our former notation we call w *w.res.* if $a_{-1} = 0$, *special* if w and w^{-1} are w.res., *simple* if $\text{ord } w \in \{-2, 0, +2\}$. Adjoining square roots to K as needed we may say that all simple series are squares, $w = v^2$. We use the notations $O(x^k), \sim cx^k$ in the obvious way, like thinking of $\lim_{x \rightarrow 0}$.

Differentiation

$$w' := \sum n a_n x^{n-1}$$

and integration

$$\int w := \sum \frac{1}{n+1} a_n x^{n+1} + \text{const}$$

(for w w.res.) have the well-known properties. In the case of entire w we write $\int_0^x w$ for the particular integral

$$\int_0^x w = \sum_{n \geq 0} \frac{1}{n+1} a_n x^{n+1}.$$

3.4.2. THEOREM 6. *If $v \in K[[x]]$, and v^2 is special and simple then any number of squids may be applied to v . More explicitly, with every choice of the constants of integration the recursion $v_0 := v$,*

$$(18) \quad v_{n+1} := \frac{1}{v_n} \int v_n^2$$

can be continued in $K[[x]]$ indefinitely producing special though not necessarily simple v_n^2 all the way.

Theorem 6 is contained in Theorem 7 below. We give an independent proof because it shows another aspect of our structure.

Proof. We have to show that always again v_{n+1}^2 is w.res. Specialty then follows easily since (18) implies

$$(19) \quad \begin{aligned} (v_n v_{n+1})' &= v_n^2, \\ \left(\frac{1}{v_n v_{n+1}} \right)' &= -\frac{v_n^2}{v_n^2 v_{n+1}^2} = -\frac{1}{v_{n+1}^2}, \end{aligned}$$

hence the v_{n+1}^{-2} are w.res. as well. In fact, (19) shows how to invert (18):

$$\frac{1}{v_n} = -v_{n+1} \int \frac{1}{v_{n+1}^2},$$

as was mentioned in the introduction.

We distinguish three cases depending on $\text{ord } v = -1, 0$ or $+1$. We treat $\text{ord } v = 0$ first. For convenience take $v_0 = v \sim 1$. Integrability is trivial as long as

$$v_{n+1} = \frac{1}{v_n} \int_0^x v_n^2.$$

Let k be the first index (if there is any) for which a nonzero constant of integration is chosen:

$$v_{k+1} = \frac{1}{v_k} \left(\int_0^x v_k^2 + \alpha \right), \quad \alpha \neq 0.$$

Then

$$(20) \quad v_n \sim c_n x^n, \quad c_n = (1 \cdot 3 \cdot \dots \cdot (2n-1))^{-1} \quad \text{for } 0 \leq n \leq k,$$

$$v_{k+1} = \frac{\alpha}{v_k} + O(x^{k+1}).$$

The latter is the first instance ($i = 0$) of the relation

$$(21) \quad v_{k+i+1} = (-1)^i \frac{\alpha}{v_{k-i}} + O(x^{k-i+1}) \quad \text{for } 0 \leq i \leq k,$$

which we shall prove now by induction.

Let $0 \leq i \leq k-1$. Then (21) and (20) imply

$$v_{k+i+1}^2 = \frac{\alpha^2}{v_{k-i}^2} + O(x).$$

Here $O(x)$ is trivially integrable and v_{k-i}^{-2} so because of (19),

$$\int v_{k+i+1}^2 = \frac{-\alpha^2}{v_{k-i-1}v_{k-i}} + O(x^2) + \text{const} = \frac{-\alpha^2}{v_{k-i-1}v_{k-i}} + O(1).$$

Thus

$$(22) \quad v_{k+i+2} = \frac{-\alpha^2}{v_{k-i-1}v_{k-i}v_{k+i+1}} + O\left(\frac{1}{v_{k+i+1}}\right).$$

From (21) and again (20) we get

$$v_{k+i+1} \sim \frac{\pm\alpha}{c_{k-i}x^{k-i}}$$

and

$$v_{k-i}v_{k+i+1} = (-1)^i \alpha + O(x^{2k-2i+1}).$$

Inserting both into (22) yields

$$\begin{aligned} v_{k+i+2} &= \frac{-\alpha^2}{v_{k-i-1}((-1)^i \alpha + O(x^{2k-2i+1}))} + O(x^{k-i}) \\ &= (-1)^{i+1} \frac{\alpha}{v_{k-i-1}} + O\left(\frac{x^{2k-2i+1}}{v_{k-i-1}}\right) + O(x^{k-i}) \\ &= (-1)^{i+1} \frac{\alpha}{v_{k-i-1}} + O(x^{k-i}), \end{aligned}$$

which is the next instance of (21).

In particular,

$$v_{2k+1} = \pm \frac{\alpha}{v_0} + O(x), \quad \text{ord } v_{2k+1} = 0,$$

whence the iteration can start anew.

Next take $\text{ord } v = -1$. We prefer to write $v = v_{-1}$ here and may assume $v_{-1} \sim -1/x$. Then, since v_{-1}^2 is w.res.,

$$v_{-1}^2 = \frac{1}{x^2} + O(1),$$

$$v_0 = \frac{1}{v_{-1}} \int v_{-1}^2 = \left(-\frac{1}{x} + \dots \right)^{-1} \left(-\frac{1}{x} + \dots \right) \sim 1,$$

and we are back to the former case.

Last, let $\text{ord } v = 1$. We write $v = v_1$ and assume $v_1 \sim x$. Since v_1^{-2} is w.res. we can set

$$v_0 := -\left(v_1 \int v_1^{-2} \right)^{-1}.$$

Then $v_0 \sim 1$ and

$$v_1 = \frac{1}{v_0} \int v_0^2,$$

according to (10). So v_1 is embedded into the sequence starting with v_0 . This ends the proof.

3.4.3. Rephrasing Theorem 6 we may say that any sequence of squids can be applied to any

$$v_{-1} = \frac{c}{x} + O(x) \quad \text{or} \quad v_0 \sim c \quad \text{or} \quad v_1 = cx + O(x^3)$$

(where always $c \neq 0$), for these representations are obviously equivalent to asking the residues of v_{-1}^2 and v_1^{-2} to vanish. But what are the possible v_2, v_3 etc. in the sequence (18)? The following definition and theorem will give the answer.

DEFINITION 6. Let us write Ev for any polynomial f of x^2 such that $f(0) = 0$; thus

$$a_2x^2 + a_4x^4 + \dots + a_{2n}x^{2n} = Ev.$$

For any $k \in \mathbb{Z}$ let \mathcal{H}_k be the set of $v \in K[[x]]$ of the form

$$v = x^k(c + Ev + O(x^{2|k|+1})), \quad c \in K, c \neq 0,$$

and write

$$\mathcal{H} := \bigcup_{k \in \mathbb{Z}} \mathcal{H}_k, \quad \mathcal{E} := \mathcal{H}_{-1} \cup \mathcal{H}_0 \cup \mathcal{H}_1.$$

Note that $\mathcal{H}_0 = \{v : \text{ord } v = 0\}$ and that the simple special series w are exactly the $w = v^2, v \in \mathcal{E}$.

THEOREM 7. Any sequence of squid operations may be applied to a series v if and only if $v \in \mathcal{H}$.

PROOF. If $v \in \mathcal{H}$ then v^2 is w.res., so all squids may be applied to v . \mathcal{H} is closed under all squids. In detail:

$$(23) \quad v \in \mathcal{H}_k, k \in \mathbb{N}_0 \Rightarrow \frac{1}{v} \int_0^x v^2 \in \mathcal{H}_{k+1},$$

$$(24) \quad v \in \mathcal{H}_k, k \in \mathbb{N}_0 \Rightarrow \frac{1}{v} \left(\int_0^x v^2 + \gamma \right) \in \mathcal{H}_k \quad \text{if } \gamma \neq 0,$$

$$(25) \quad v \in \mathcal{H}_{-k}, k \in \mathbb{N} \Rightarrow \frac{1}{v} \int v^2 \in \mathcal{H}_{-k+1}.$$

The interesting case is (23). Here (without loss of generality $c = 1$)

$$\begin{aligned} v &= x^k(1 + Ev + ax^{2k+1} + O(x^{2k+3})), \\ v^2 &= x^{2k}(1 + Ev + 2ax^{2k+1} + O(x^{2k+3})), \\ \int_0^x v^2 &= x^{2k+1} \left(\frac{1}{2k+1} + Ev + \frac{2a}{4k+2} x^{2k+1} + O(x^{2k+3}) \right), \\ \frac{1}{v} \int_0^x v^2 &= \frac{x^{k+1}}{2k+1} (1 + Ev + ax^{2k+1} + O(x^{2k+3})) \\ &\quad \times (1 + Ev + ax^{2k+1} + O(x^{2k+3}))^{-1} \\ &= \frac{x^{k+1}}{2k+1} (1 + Ev + O(x^{2k+3})) \in \mathcal{H}_{k+1}. \end{aligned}$$

With respect to (24) note that $\gamma v^{-1} \in \mathcal{H}_{-k}$ and

$$\frac{1}{v} \int_0^x v^2 = O(x^{k+1}) = x^{-k} O(x^{2k+1}).$$

While with any step (23) we gain two powers of x in the error term, at (25) the constant of integration brings it down again.

$$\begin{aligned} v^2 &= x^{-2k}(1 + Ev + O(x^{2k+1})), \\ \int v^2 &= x^{-2k+1} \left(\frac{-1}{2k-1} + Ev + O(x^{2k+1}) \right) + \text{const} \\ &= -\frac{1}{2k-1} x^{-2k+1} (1 + Ev + O(x^{2k-1})), \\ \frac{1}{v} \int v^2 &\in \mathcal{H}_{-k+1}. \end{aligned}$$

Now consider any $u \in K[[x]]$ such that any sequence of squids may be applied to it. If $\text{ord } u = 0$ then $u \in \mathcal{H}$. If $\text{ord } u = -k \in \mathbb{N}$, then applying k arbitrary squids yields

$$u_0 = u, u_1, \dots, u_k, \quad \text{ord } u_i = -k + i,$$

in particular $u_k \in \mathcal{H}_0$. Here $v_0 := u_k^{-1} \in \mathcal{H}$. Now we apply the proper inverse squids and find $1/u = \pm v_k \in \mathcal{H}$, hence also $u \in \mathcal{H}$. If $\text{ord } u = k \in \mathbb{N}$ then

$$u_1 := \frac{1}{u} \left(\int_0^x u^2 + 1 \right)$$

has $\text{ord } u_1 = -k$, so the same procedure works with $k + 1$ steps.

COROLLARY 7.1. \mathcal{H} is the closure of \mathcal{H}_0 , and a fortiori of \mathcal{E} , under all squid operations.

PROOF. If $u \in \mathcal{H}$ then also $1/u \in \mathcal{H}$. A suitable sequence of squids maps $1/u$ onto $v \in \mathcal{H}_0$, so the inverse squids will map $\pm 1/v$, which is in \mathcal{H}_0 , onto u .

Obviously, by Theorem 7 we are in a position to define the proper generalization of simple special series:

DEFINITION 7. A series $w \in K[[x]]$ will be called *semisimple special* (in short: sss) if $w = v^2$, $v \in \mathcal{H}$.

All simple special series are sss-series, and if $w = v^2$ is sss then $(\frac{1}{v} \int v^2)^2$ is sss again.

3.4.4. On the significance of the bilinear differential equation $B(p, q) = 0$. The local version of Corollary 2.1 states that in a B -representation of some $v \in K[[x]]$ the p and q are entire series.

THEOREM 8. The following are equivalent for $v \in K[[x]]$, $v \neq 0$:

- (i) v has a B -representation,
- (ii) v''/v is w.res.,
- (iii) if $v = cx^k(1 + ax + \dots)$, $c \neq 0$, then $ka = 0$.

PROOF. The residue of v''/v is easily calculated; it is $2ak$. Hence (ii) \Leftrightarrow (iii). Another simple calculation shows that if $v = q/p$, then

$$(26) \quad \begin{aligned} B(p, q) &= B(p, vp) = p^2v'' + 2(pp'' - p'^2)v, \\ \frac{B(p, q)}{pq} &= \frac{v''}{v} + 2\left(\frac{p'}{p}\right)' . \end{aligned}$$

Thus $v = q/p$ is a B -representation if and only if

$$(27) \quad \left(\frac{p'}{p}\right)' = -\frac{v''}{2v} .$$

As an immediate consequence v''/v is w.res. if v has a B -representation.

Let, on the other hand, v''/v be w.res.; then

$$\left(\frac{p'}{p}\right)' = -\frac{k(k-1)}{2} \cdot \frac{1}{x^2} + O(1)$$

can be solved for p :

$$\frac{p'}{p} = \binom{k}{2} \frac{1}{x} + c_0 + c_1x + \dots$$

Write

$$p = x^{\binom{k}{2}} r, \quad r = 1 + r_1x + r_2x^2 + \dots$$

Then the r_i can be determined recursively from

$$r' = r(c_0 + c_1x + \dots)$$

or explicitly from

$$r = \exp\left(c_0x + \frac{c_1}{2}x^2 + \dots\right).$$

Note that in accordance with Theorem 2

$$(\text{ord } p, \text{ord } q) = (\text{ord } p, \text{ord}(pv)) = \left(\binom{k}{2}, \binom{k+1}{2}\right)$$

is magic and p, q are entire.

The functions p, q are not uniquely determined. Due to the two integrations they contain an arbitrary common factor ae^{bx} .

COROLLARY 8.1. *If $w = v^2$ is sss in $K[[x]]$ then w has a B -representation there.*

Remark. We saw in 3.3.2 that the converse does not hold. Now it is easily seen why: Apart from the case $k = 0$ where neither condition really means a restriction, sss consists in the vanishing of exactly k coefficients, and B -representability of only one. In this connection it should be noted that

COROLLARY 8.2. *If v has a B -representation then so has v^m for any $m \in \mathbb{Z}$.*

Nothing alike holds for sss-series. The corollary follows from Theorem 8(iii) and can be made quite explicit by Lemma 4: If $v = p/q$, $B(p, q) = 0$ put $a = \binom{m}{2}$, $b = \binom{m+1}{2}$. Then

$$v^m = \frac{p^a q^b}{p^b q^a} \quad \text{and} \quad B(p^a q^b, p^b q^a) = 0.$$

4. Global again

4.1. The sss-functions in $K(x)$. Let \bar{K} be the algebraic closure of K . Then all definitions and statements made with respect to $K[[x]]$ can be applied in $\bar{K}(x)$ to any finite place. Thus our old notion “w.res.” means “w.res. everywhere”. In particular, we say that $w = v^2$, $v \in \bar{K}(x)$, is sss at $\omega \in \bar{K}$, $v \in \mathcal{H}(\omega)$ if the Taylor expansion of $v(\omega+x)$ is an sss-series according

to Definition 7. We define w to be (globally) sss if it is sss everywhere, i.e. we require

$$v \in \mathcal{H}^G := \bigcap_{\omega \in \bar{K}} \mathcal{H}(\omega).$$

COROLLARY 7.2. *Any sequence of squid operations may be applied to a function $v \in K(x)$ if and only if $v \in \mathcal{H}^G$.*

This follows trivially from Theorem 7. All we have to add is that a function $v \in K(x)$ has an integral in $K(x)$.

We define B -representations as before, replacing simply $K[[x]]$ by $K(x)$. Remember that by Theorem 2(ii) the p, q are necessarily polynomials.

COROLLARY 8.3. *If $w = v^2$ is sss in $K(x)$ then w has a B -representation there.*

Proof. We have v''/v w.res. (everywhere) since w is sss (everywhere). Write $w = v^2$,

$$v = \prod_i (x - a_i)^{\nu_i}, \quad \nu_i \in \mathbb{Z}, \quad a_i \neq a_j.$$

Putting

$$p := \prod_i (x - a_i)^{\binom{\nu_i}{2}}, \quad q = \prod_i (x - a_i)^{\binom{\nu_i+1}{2}}$$

we have $v = q/p$ to begin with. As before, (26) implies that $B(p, q)/pq$ is w.res. With Lemma 2 we compute

$$\begin{aligned} \frac{B(p, q)}{pq} &= \left(\sum \frac{\binom{\nu_i+1}{2}}{x - a_i} - \sum \frac{\binom{\nu_i}{2}}{x - a_i} \right)^2 + \left(\dots + \dots \right)' \\ &= \left(\sum \frac{\nu_i}{x - a_i} \right)^2 + \left(\sum \frac{\nu_i^2}{x - a_i} \right)' \\ &= \sum_{i \neq j} \frac{\nu_i \nu_j}{(x - a_i)(x - a_j)}. \end{aligned}$$

The relevant observation is now that $B(p, q)/pq$, if written in lowest terms, has a squarefree denominator. The residues can vanish only if $pq \mid B(p, q)$. Since $\deg(B(p, q)) < \deg(pq)$ this implies $B(p, q) = 0$.

THEOREM 9. *The functions $v \in K(x)$ for which v^2 is sss form the closure of the function 1 under all squids and inversion.*

Proof. Since 1 is obviously sss we may apply any sequence of squids to it. This was already proved in Corollary 5.1. If v is sss then so is v^{-1} .

Now suppose v^2 is sss, $v \in \mathcal{H}^G$ in other words. Then $v^{-1} \in \mathcal{H}^G$ as well. We choose $v_0 := v^{\pm 1}$ such that $\text{ord}_\infty v_0 \leq 0$. If $\text{ord}_\infty v_0 < 0$ then there is

one integral, it may be denoted by \int_{∞}^x , such that

$$\text{ord}_{\infty} \left(\int_{\infty}^x v_0^2 \right) = 2 \text{ord}_{\infty} v_0 + 1.$$

For

$$v_1 := \frac{1}{v_0} \int_{\infty}^x v_0^2$$

we have $\text{ord}_{\infty} v_1 = \text{ord}_{\infty} v_0 + 1$. We repeat this process until we have v_n with $\text{ord}_{\infty} v_n = 0$. By Corollary 8.3 there are $p, q \in K[x]$ such that $v_n = q/p$, $B(p, q) = 0$. Since $(\deg p, \deg q)$ is magic and on the other hand $\deg q - \deg p = \text{ord}_{\infty} v_n = 0$ it follows that p, q and therefore v_n are constants. Reversing our squid steps and adapting the constant factor in the general definition of a squid properly we obtain v or v^{-1} from 1.

We are now in a position to prove a satisfactory generalization of Theorem 5 and Corollary 5.1.

THEOREM 10. *There is a sequence of polynomials $P_i = P_i(\Gamma_1, \dots, \Gamma_{i-1}, x)$, $P_0 = P_1 = 1$, over \mathbb{Q} such that the functions*

$$W_i = \left(\frac{P_{i+1}}{P_i} \right)^2$$

on every specialization $\Gamma_i \rightarrow \gamma_i \in K$ produce sss-functions in $K(x)$, and any sss-function w in $K(x)$ is obtained by a suitable such specialization as

$$w(x) = c \left(\frac{P_{i+1}}{P_i}(\gamma_1, \dots, \gamma_i, x) \right)^{\pm 2}, \quad c \in K.$$

PROOF. The P_i are actually those of Theorem 5 in the special case $p_0 = p_1 = 1$. Since $p_0, p_1 \in \mathbb{Q}[x]$ the construction (13), (14) can be carried out over \mathbb{Q} ($\subset K$). Every specialization leaves (12) valid and produces a squid iteration inside $K(x)$,

$$(28) \quad p_{i+1} = p_{i-1} \int \left(\frac{p_i}{p_{i-1}} \right)^2,$$

or, differently written,

$$(29) \quad v_i = \frac{1}{v_{i-1}} \int v_{i-1}^2$$

with $v_i = p_{i+1}/p_i$. By Theorem 9 all these v_i are in \mathcal{H}^G . Conversely, any $v \in \mathcal{H}^G$ can be obtained as $v = cv_n^{\pm 1}$, v_n an element of a chain (29) with $v_0 = 1$. Inserting $p_0 = 1$, $p_{i+1} = p_i v_i$ takes us back to (28) and, as was shown in the proof of Corollary 5.1, any such sequence p_0, p_1, \dots is a specialization of P_0, P_1, \dots

4.2. Meromorphic functions. The basic statement to all of this paper, that a function may be integrated if and only if it is w.res., is also valid in the field $\text{Mer}(\mathcal{D})$ of all meromorphic functions on a region $\mathcal{D} \subset \mathbb{C}$ if \mathcal{D} is simply connected. Therefore Theorems 7 and 8 have obvious counterparts here. Again v^2 will be called sss at ω if the Taylor expansion of $v(\omega + x)$ is an sss-series, and globally sss if it is sss everywhere in \mathcal{D} , i.e. if

$$v \in \mathcal{H}^G := \bigcap_{\omega \in \mathcal{D}} \mathcal{H}(\omega).$$

Here again v is in \mathcal{H}^G exactly if any sequence of squids can be applied to it. Such a function also has a B -representation: $v = q/p$, $B(p, q) = 0$ where p and q are entire, i.e. holomorphic on \mathcal{D} . To see this, note that by Theorem 8 for $v \in \mathcal{H}^G$ we have v''/v w.res. at any $\omega \in \mathcal{D}$ and remember that all solutions p of the differential equation (27) are holomorphic in some neighbourhood of ω . So any solution of (27) can be analytically continued to all of \mathcal{D} . As in the local case, p and $q = vp$ are determined up to an arbitrary common factor ae^{bz} only.

If p_1/p_0 is a B -representation of v_0 then B -representations of the v_n , derived by (29), can be obtained in the form p_{n+1}/p_n with the p_n of (28). To see this we only need our old observation that $B(p_{n-1}, p_n) = 0$ and (28) imply $B(p_n, p_{n+1}) = 0$.

An interesting example is $v_0 = \tan$ on $\mathcal{D} = \mathbb{C}$. Not only is v^2 superspecial but in addition all powers of v^2 are sss-functions! At $z = 0$ this is clear because for all n and $k \in \mathbb{Z}$ we have $v^n \in \mathcal{H}_k$, and the other zeros and poles are obtained from these by simple translations. We conclude that starting from

$$v_0 = \tan^n, \quad p_0 = \cos^{\binom{n+1}{2}} \sin^{\binom{n}{2}}, \quad p_1 = \cos^{\binom{n}{2}} \sin^{\binom{n+1}{2}}$$

all v_i produced by (29) are meromorphic and all p_i from (28) are entire on all of \mathbb{C} .

Exactly the same arguments apply to $w = \wp - e_1$, where \wp is any Weierstraß elliptic function with a primitive pair of periods ω, ω' and $e_1 = \wp(\omega/2)$, say. Here again all w^n are sss. Starting from (27) a B -representation of $v = w^{1/2}$ can be given in terms of the corresponding σ -function:

$$p(z) = e^{-e_1 z^2/4} \sigma(z), \quad q(z) = e^{\eta z/2 - e_1 z^2/4} \sigma\left(\frac{\omega}{2}\right)^{-1} \sigma\left(z - \frac{\omega}{2}\right),$$

where $\eta = 2\frac{\sigma'}{\sigma}\left(\frac{\omega}{2}\right)$.

4.3. Squids and the Padé approximation of e^{2ix} . It is well-known that there are uniquely determined $\alpha_n \in \mathbb{C}[x]$ of degree n such that the Taylor expansion of

$$\Delta_n(x) := \alpha_n e^{ix} + \bar{\alpha}_n e^{-ix}$$

begins with

$$\frac{x^{2n+1}}{(2n+1)!} + \dots$$

In fact,

$$(30) \quad \Delta_n(x) = \frac{1}{2^{2n+1}n!^2} \int_{-x}^{+x} (x^2 - y^2)^n e^{iy} dy.$$

Also well-known and easily derived from (30) are some linear recursions:

$$(31) \quad 2n\Delta'_n = x\Delta_{n-1},$$

$$(32) \quad 2n\Delta_n = (2n-1)\Delta_{n-1} - x\Delta'_{n-1},$$

$$(33) \quad 2(n+1)\Delta_{n+1} - (2n+1)\Delta_n + \frac{x^2}{2n}\Delta_{n-1} = 0,$$

the last being an immediate consequence of (31) and (32). The following squid type recursion, however, may be new. Put

$$p_n(x) = d_n x^{\binom{n-1}{2}} \Delta_{n-1}(x) \quad \text{for } n \in \mathbb{N}, \quad p_0 = \cos,$$

with constants defined by

$$d_1 = d_2 = 1, \quad d_{n+2} = \frac{(n+1)d_{n+1}^2}{n(2n+1)d_n}.$$

Then

$$(34) \quad p_{n+2} = p_n \int_0^x \left(\frac{p_{n+1}}{p_n} \right)^2 \quad \text{for all } n \in \mathbb{N}_0.$$

This, since $p_1 = \sin$, identifies the v_n from

$$v_0 = \tan, \quad v_{n+1} = \frac{1}{v_n} \int_0^x v_n^2$$

as

$$v_n(x) = \frac{d_{n+1}}{d_n} x^{n-1} \frac{\Delta_{n+1}(x)}{\Delta_n(x)}.$$

Proof of (34). We claim that for all $n \in \mathbb{N}_0$

$$(35) \quad p'_{n+2} p_n - p_{n+2} p'_n = p_{n+1}^2.$$

The case $n = 0$ is easily checked separately. Now let $n \in \mathbb{N}$ and put

$$M := (d_{n+2}d_n)^{-1} x^{1-\binom{n+1}{2}-\binom{n-1}{2}} (p'_{n+2}p_n - p_{n+2}p'_n - p_{n+1}^2).$$

Then

$$\begin{aligned} M &= \left(\binom{n+1}{2} \Delta_{n+1} + x \Delta'_{n+1} \right) \Delta_{n-1} \\ &\quad - \Delta_{n+1} \left(\binom{n-1}{2} \Delta_{n-1} + x \Delta'_{n-1} \right) - \frac{n(2n+1)}{n+1} \Delta_n^2 \\ &= (2n-1) \Delta_{n+1} \Delta_{n-1} + x(\Delta'_{n+1} \Delta_{n-1} - \Delta_{n+1} \Delta'_{n-1}) - \frac{n(2n+1)}{n+1} \Delta_n^2. \end{aligned}$$

Eliminating Δ'_{n+1} by (31) and Δ'_{n-1} by (32) gives

$$\begin{aligned} M &= 2n \Delta_n \Delta_{n+1} + \frac{x^2}{2(n+1)} \Delta_n \Delta_{n-1} - \frac{n(2n+1)}{n+1} \Delta_n^2 \\ &= \frac{n}{n+1} \Delta_n \left(2(n+1) \Delta_{n+1} - (2n+1) \Delta_n + \frac{x^2}{2n} \Delta_{n-1} \right) = 0, \end{aligned}$$

by (33).

Finally, (34) follows from (35) because of $\text{ord}_0 p_n = \binom{n+1}{2}$; cf. (9).

5. Superspecial rational functions. Quite in contrast to the situation in $\text{Mer}(\mathbb{C})$ we have

THEOREM 11. *The only superspecial functions in $K(x)$ are the functions given in (2).*

The proof uses an old formula on the inversion of power series. Let

$$f(x) = x + a_2 x^2 + a_3 x^3 + \dots, \quad \varphi(y) = y + b_2 y^2 + b_3 y^3 + \dots$$

be inverses of each other. Then for all $n \in \mathbb{N}$

$$(36) \quad b_n = \frac{1}{n} \text{Res}(f^{-n}, 0).$$

This is proved most easily for series with a positive radius of convergence over the field $K = \mathbb{C}$. Substituting $x = \varphi(y)$ in Cauchy's formula gives

$$\text{Res}(f^{-n}, 0) = \frac{1}{2\pi i} \oint \frac{dx}{f^n(x)} = \frac{1}{2\pi i} \oint \frac{\varphi(y)' dy}{y^n} = n b_n.$$

Both $\text{Res}(f^{-n})$ and b_n can be obtained as polynomials in a_2, a_3, \dots, a_n with coefficients in \mathbb{Z} by purely formal operations. As we see, these polynomials are equal, whence (36) follows for all fields and irrespective of convergence.

Assume now that $w = p/q$, where $p, q \in K[x]$ are co-prime, is superspecial and that there is a pole of order k at 0. In $K[[x]] \supset K(x)$ we have an expansion

$$1/w = x^k (c_k + c_{k+1} x + \dots), \quad c_k \neq 0.$$

Without loss of generality we take $c_k = 1$. Then there is an $f(x) \in K[[x]]$ such that

$$1/w = f^k, \quad f = x + a_2x^2 + \dots$$

and f has an inverse

$$\varphi(y) = y + b_2y^2 + \dots$$

By (36) the assumption implies that $b_n = 0$ for all $n \equiv 0 \pmod k$. If ζ denotes a primitive k th root of unity we can express this by

$$\sum_{\kappa=1}^k \varphi(\zeta^\kappa y) = 0.$$

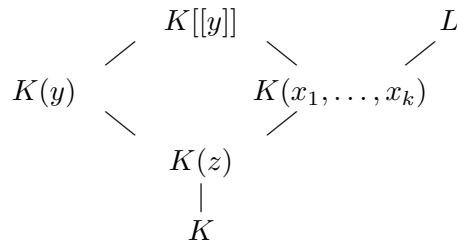
Writing $x_\kappa := \varphi(\zeta^\kappa y)$ we obtain $\sum_{\kappa=1}^k x_\kappa = 0$ together with $w(x_\kappa) = y^{-k}$. If the pole in question is at an arbitrary place ω , then $x_\kappa - \omega$ replace the x_κ , giving

$$(37) \quad \sum_{\kappa=1}^k x_\kappa = k\omega.$$

The x_κ are zeros, though not necessarily all zeros, of

$$g(x) := zp(x) + q(x), \quad z := y^k;$$

the rest will be called x_{k+1}, \dots, x_n .



By Gauß's Theorem and because p, q are coprime g is irreducible over $K(z)$. Let L denote the splitting field of g , N its degree over $K(z)$ and Tr the trace operator of L over $K(z)$. Since $\omega \in K$ and

$$\text{Tr } x_\kappa = \frac{N}{n} \sum_{\nu=1}^n x_\nu \quad \text{for all } \kappa = 1, \dots, n,$$

(37) implies

$$kN\omega = \text{Tr} \left(\sum_{\kappa=1}^k x_\kappa \right) = k \frac{N}{n} \sum_{\nu=1}^n x_\nu, \quad \sum_{\nu=1}^n x_\nu = n\omega.$$

If g is normalized into g^* , the (monic) minimal polynomial of the x_κ , then $\sum_{\nu=1}^n x_\nu$ is the second highest coefficient of $-g^*$; hence ω is uniquely determined by our function w . Moreover, changing from w to $1/w$ affects g^* only by substituting $1/z$ for z , which does not change the coefficient in question because it is in K , supposing that w had any pole at all. As w and $1/w$ together have at most one pole the theorem is proved.

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