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The number of squarefull numbers in an interval

by

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To the Spring Festival of 1989 when I was at Hefei

1. Introduction. A positive integer n is called *squarefull* if n having a divisor p implies that n also has a divisor p^2. Here p denotes a prime number. Let $Q(x)$ be the number of squarefull numbers not exceeding x. Let $h = x^{1/2+\theta}$, $0 < \theta < 1/2$. Asymptotic formulas (as $x \to \infty$) for the quantity $Q(x+h)-Q(x)$ were first investigated by means of the exponential sum method in P. Shiu [10] where it was proved that

$$Q(x+h) - Q(x) = \frac{1}{2}\frac{\zeta(3/2)}{\zeta(3)}x^{\theta}(1+o(1)) \tag{1}$$

for each number θ such that

$$1/6 > \theta > 0.1526\,.$$

(Note that for $1/2 > \theta \geq 1/6$, (1) follows at once from the asymptotic formula for $Q(x)$, cf. [10].) P. Shiu's result was improved by P. G. Schmidt [8], [9] to

$$1/6 > \theta > 0.1507 \quad \text{and} \quad 1/6 > \theta > 1/7 = 0.14285\ldots, \quad \text{resp.}$$

Independently, with the help of a corrected version of Theorem 1 of G. Kolesnik [4] and the exponent pair method, in [5] it was shown that (1) holds true whenever

$$1/6 > \theta > 0.14254\,.$$

As is well known, we have

$$Q(x+h) - Q(x) = \sum_{x<a^2b^3\leq x+h} |\mu(b)|\,, \tag{2}$$

where $\mu(\cdot)$ is the Möbius function. All the above research based on representing $|\mu(b)|$ by a standard summation, namely,

$$|\mu(b)| = \sum_{m^2\,|\,b} \mu(m)\,. \tag{3}$$

Then, by substituting (3) in (2), after some standard arguments, the problem is reduced to estimating certain multiple (in fact, triple) exponential sums, whose estimates are always unsatisfactory.

In this paper, we show that it is actually redundant to use (3), and one can obtain a far better range if one keeps the original expression (2). Let

$$\psi(\xi) = \xi - [\xi] - \tfrac{1}{2}$$

where $[\xi]$ is the integral part of a real number ξ, and let

$$R(X,\beta) = \sum_{n \le X^\alpha} \psi(Xn^{-\beta}), \quad \beta > 0, \ \alpha = 1/(\beta+1).$$

A simple argument enables one to deduce the following theorem.

THEOREM 1. *If σ is a number such that for any $\varepsilon > 0$ and any $\xi > 1$,*

$$R(\xi^{1/2}, 3/2) \ll \xi^{\sigma+\varepsilon}, \quad R(\xi^{1/3}, 2/3) \ll \xi^{\sigma+\varepsilon}, \tag{4}$$

then, for any number θ with $1/6 > \theta > \sigma + 2\varepsilon$, one has

$$Q(x + x^{1/2+\theta}) - Q(x) = \frac{1}{2}\frac{\zeta(3/2)}{\zeta(3)} x^\theta (1 + O(x^{-\varepsilon/2})).$$

Hence, the key to our problem is to find an optimal upper bound for $R(\xi^{1/2}, 3/2)$ and $R(\xi^{1/3}, 2/3)$. The sum $R(X,\beta)$ was first introduced in H. E. Richert [7], where it was estimated via the van der Corput–Phillips exponent pair method solely. In [10], P. Shiu showed that $\sigma \ge 0.1318162$ is admissible in (4). P. G. Schmidt [8] refined that to $\sigma \ge 27/205 = 0.13170\ldots$, and pointed out that even $\sigma \ge 0.13169\ldots$ is accessible for (4) by using van der Corput's method alone.

Note that in treating the error term occurring in the Dirichlet divisor problem, H. Iwaniec and C. J. Mozzochi [3] indeed found an estimate for $R(X,1)$:

PROPOSITION 1.

$$R(X,1) \ll X^{7/22+\varepsilon}. \tag{5}$$

The estimate (5) is substantially new as compared with the former developments. In view of the importance of $R(X,\beta)$ in various problems, especially in our current problem, in this paper I shall generalize the estimate (5) to every sum $R(X,\beta)$, $\beta > 0$. The following proposition will be proved.

PROPOSITION 2. *For any $\varepsilon > 0$,*

$$R(X,\beta) \ll x^{\tau(\beta)+\varepsilon}.$$

Here

$$\tau(\beta) = \begin{cases} \dfrac{7}{11(\beta+1)} & \textit{if } 0 < \beta \le 1, \\ \max(\tau_1(\beta), \tau_2(\beta)) & \textit{if } \beta > 1, \end{cases}$$

with

$$\tau_1(\beta) = \inf_{(k,\lambda)\in E}\left(\frac{7(\lambda-k)}{22\lambda-(15\beta+7)k+7(\beta-1)}\right),$$
$$\tau_2(\beta) = \inf_{(k,\lambda)\in E}\left(\frac{3\lambda+k}{4\lambda+(1-\beta)k+3\beta+1}\right),$$

where

$$E = E(\beta) = \{(k,\lambda) \mid (k,\lambda) \text{ is an exponent pair such that } \lambda \geq \beta k\},$$

and the infima are taken over all exponent pairs belonging to E.

Proposition 2 reveals that the estimate for $R(\xi^{1/2}, 3/2)$ is rather worse than that for $R(\xi^{1/3}, 2/3)$. Nevertheless, in conjunction with a neat exponent pair $(2/7, 4/7)$, it will be clear that Proposition 2 implies (4) with $\sigma = 14/107$; thus I obtain the following theorem:

THEOREM 2. *For any $\varepsilon > 0$, and any θ in the range*

$$\theta \geq 14/107 + \varepsilon = 0.13084\ldots + \varepsilon$$

we have

$$Q(x + x^{1/2+\theta}) - Q(x) = \frac{1}{2}\frac{\zeta(3/2)}{\zeta(3)}x^{\theta}(1 + O(x^{-\varepsilon/4}))\,.$$

I remark here that the number 14/107 is of course not best possible, and one can slightly reduce it by taking some more cumbersome exponent pairs.

2. The proof of Theorem 1. Put $B = x^{\theta-\varepsilon}$. We have

$$\begin{aligned}(6)\quad Q(x + x^{1/2+\theta}) - Q(x) &= \sum_{\substack{x<a^2b^3\leq x+x^{\theta+1/2}\\ b\leq B}} |\mu(b)| + \sum_{\substack{x<a^2b^3\leq x+x^{\theta+1/2}\\ b>B}} |\mu(b)| \\ &= \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say}\,.\end{aligned}$$

Clearly, one has

$$\begin{aligned}\sum\nolimits_1 &= \sum_{b\leq B} |\mu(b)| \sum_{(xb^{-3})^{1/2}<a\leq((x+h)b^{-3})^{1/2}} 1 \\ &= \sum_{b\leq B} |\mu(b)| \left(\frac{(x+h)^{1/2} - x^{1/2}}{b^{3/2}} + O(1)\right).\end{aligned}$$

As

$$(x+h)^{1/2} - x^{1/2} = \tfrac{1}{2}x^{\theta}(1 + O(x^{\theta-1/2}))$$

and

$$\sum_{b\le B}\frac{|\mu(b)|}{b^{3/2}}=\sum_{b=1}^{\infty}\frac{|\mu(b)|}{b^{3/2}}+O(B^{-1/2})\,,\qquad \sum_{b=1}^{\infty}\frac{|\mu(b)|}{b^{3/2}}=\frac{\zeta(3/2)}{\zeta(3)}\,,$$

we have

$$\sum\nolimits_1=\frac{1}{2}x^{\theta}\frac{\zeta(3/2)}{\zeta(3)}(1+O(x^{-\varepsilon/2}))\,. \tag{7}$$

The advantage comes from "abandoning" $|\mu(b)|$ in $\sum_2$. One has

$$\sum\nolimits_2\le\sum_{\substack{x<a^2b^3\le x+h\\ b>B}}1=\sum_{B<b\le(x+h)^{1/5}}\ \sum_{(xb^{-3})^{1/2}<a\le((x+h)b^{-3})^{1/2}}1$$
$$+\sum_{a\le(x+h)^{1/5}}\ \sum_{(xa^{-2})^{1/3}<b\le((x+h)a^{-2})^{1/3}}1+O(1)\,.$$

As

$$\sum_{(xb^{-3})^{1/2}<a\le((x+h)b^{-3})^{1/2}}1=\frac{(x+h)^{1/2}-x^{1/2}}{b^{3/2}}+\psi\Big(\frac{x^{1/2}}{b^{3/2}}\Big)-\psi\Big(\frac{(x+h)^{1/2}}{b^{3/2}}\Big)\,,$$

$$\sum_{(xa^{-2})^{1/3}<b\le((x+h)a^{-2})^{1/3}}1=\frac{(x+h)^{1/3}-x^{1/3}}{a^{2/3}}+\psi((xa^{-2})^{1/3})-\psi(((x+h)a^{-2})^{1/3})\,,$$

one gets

$$\sum\nolimits_2\le R(x^{1/2},3/2)-R((x+h)^{1/2},3/2)+R(x^{1/3},2/3)-R((x+h)^{1/3},2/3)+O(x^{\theta-\varepsilon})\,. \tag{8}$$

From (6)–(8) and the assumption (4), one concludes that

$$Q(x+x^{1/2+\theta})-Q(x)=\frac{1}{2}\frac{\zeta(3/2)}{\zeta(3)}x^{\theta}(1+O(x^{-\varepsilon/2}))\,.$$

Theorem 1 is proved.

3. The proof of Proposition 2

3.0. *Introduction.* In analytic number theory, a variety of problems are reduced to exponential sums which can be effectively estimated by van der Corput's method. The exponent pair method was introduced by van der Corput in order that a better result might be gained for a concrete problem after a suitable iterative procedure, and it was simplified in E. Phillips [6].

To enhance the power of the method, a number of refinements have been developed. For example, the original Weyl inequality has been generalized so that one can shift several variables simultaneously. The work of E. Bombieri and H. Iwaniec [1] is somewhat pioneer in the sense that it preludes the possibility of an alternative approach to problems which formerly could only be treated via van der Corput's method or some refinements of it. However, the method of [1] is not altogether new in the field of trigonometric sums. In fact, starting with a Weyl shift without using the Cauchy inequality, and then approximating the Taylor coefficients by fractions, and finally appealing to some mean value theorems, all these features in [1] are not dissimilar from those which appear in I. M. Vinogradov's estimate for $\zeta(1+it)$ in [11] (which has never been improved since its establishment). While the method in [1] seems to work only for exponential sums of one variable, H. Iwaniec and C. J. Mozzochi [3] succeeded in a quite analogous manner with the very special multiple sum $R(X,1)$, and they got the estimate (5). In February 1989, I generalized their result to all sums $R(X,\beta)$, $\beta > 0$. As my generalization is useful for the problem of this paper, I present my proof of Proposition 2 here.

My proof of Proposition 2 mimics closely that of Proposition 1 given in [3]. A notable difference lies in treating the sum

$$\sum_{m\sim M} \min(1, \|xm^{-\beta}\|^{-1}Y^{-1}) .$$

In the case $\beta = 1$, this sum was estimated by an elementary argument in [3]. However, for $\beta \neq 1$, one has to appeal to its Fourier expansion, and employ the special expressions of its Fourier coefficients. (For more details, see next subsection.) In fact, the estimate of this sum will constitute just the bulk of Section 3.

Notations. For a real number ξ, put

$$\|\xi\| = \min_{n\in\mathbb{Z}} |n-\xi| ,$$

where $\mathbb{Z}$ is the set of all integers, and $e(\xi) = \exp(2\pi i\xi)$. C_i $(i \ge 1)$ denote absolute constants. The constants implied by the "O" or "$\ll$" symbols are absolute. $m \sim M$ means $M < m \le 2M$ and $m \asymp M$ means that $U \le m/M \le V$ for some absolute constants U and V. As above, ε is a given small positive number.

3.1. *The formulation of the method*. We have

$$R(X,\beta) = \sum_{M}\sum_{m\sim M} \psi(Xm^{-\beta}) + O(1) ,$$

where M takes the form $X^{\alpha}2^{-j}$, $j = 1, 2, \ldots$ By means of the familiar inequality

$$\psi(\xi) = \sum_{1\le|h|\le Y} \frac{e(h\xi)}{2\pi i h} + O\Big(\min\Big(1, \frac{1}{Y\|\xi\|}\Big)\Big)$$

and the Fourier expansion

$$\min\Big(1, \frac{1}{Y\|\xi\|}\Big) = \sum_{h=-\infty}^{+\infty} a(h)e(h\xi)\,,$$

where Y is an arbitrary positive number, and

$$a(h) = \frac{1}{\pi Y h}\int_{Y^{-1}}^{1/2} \frac{\sin(2\pi h\theta)}{\theta^2}\,d\theta \ll \min\Big(\frac{\ln(2+Y)}{Y}, \frac{1}{|h|}, \frac{Y}{h^2}\Big),$$

we get

$$\begin{aligned}(9)\quad R(X,M,\beta) &:= \sum_{m\sim M} \psi(Xm^{-\beta})\\ &= O\Big(\sum_{1\le h\le Y}\sum_{m\sim M}\frac{e(Xhm^{-\beta})}{h}\Big)\\ &\quad + O\Big(\sum_{1\le h\le Y^2} f(h)\sum_{m\sim M} e(hXm^{-\beta})\Big) + O(MY^{-1}\ln(2+Y))\end{aligned}$$

where, for $\xi \ne 0$,

$$f(\xi) = \frac{1}{\pi\xi Y}\int_{Y^{-1}}^{1/2}\frac{\sin(2\pi\xi\theta)}{\theta^2}\,d\theta + \frac{2\cos(\pi\xi)}{(\pi\xi)^2 Y} \tag{10}$$

$$= \frac{Y\cos(2\pi\xi Y^{-1})}{2(\pi\xi)^2} - \frac{1}{(\pi\xi)^2 Y}\int_{Y^{-1}}^{1/2}\frac{\cos(2\pi\xi\theta)}{\theta^3}\,d\theta\,. \tag{11}$$

It is easy to verify that, for $\xi > 1$, $Y > 1$,

$$f(\xi) \ll \min(1/\xi, Y/\xi^2)\,, \qquad f'(\xi) \ll 1/\xi^2\,, \tag{12}$$

$$f''(\xi) \ll 1/(Y\xi^2) + Y/\xi^4\,, \qquad f'''(\xi) \ll 1/(Y\xi)^2 + Y/\xi^5\,. \tag{13}$$

Now it is clear that Proposition 2 is a consequence of the following two lemmas, which are valid whenever $M \ll X^\alpha$.

Lemma 1. *We have*

$$x^{-\varepsilon}R(X,M,\beta) \ll (XM^{1-\beta})^{7/22} + (X^3M^{-1-3\beta})^{1/4}\,.$$

Lemma 2. *For an exponent pair* (k,λ),

$$x^{-\varepsilon}R(X,M,\beta) \ll (X^kM^{\lambda-\beta k})^{1/(1+k)}\,.$$

The proof of Lemma 2 is routine. In fact, from (9) we get

$$R(X,M,\beta) \ll MY^{-1}\ln(2+Y) + \sum_{1\le h\le Y^2} \min\left(\frac{1}{h}, \frac{Y}{h^2}\right)\Big|\sum_{m\sim M} e(hXm^{-\beta})\Big| .$$

If (k,λ) is an exponent pair in the sense of [6], then

$$\sum_{m\sim M} e(hXm^{-\beta}) \ll (hXM^{-\beta-1})^k M^\lambda ,$$

and Lemma 2 follows by taking $Y = (X^{-k}M^{1+k-\lambda+\beta k})^{1/(1+k)}$.

Thus we only need to prove Lemma 1. Let $Y = M(XM^{1-\beta})^{-7/22}$. Obviously we can assume that $Y \ge 100$. Let

$$R_1(X,M,\beta) = \sum_{1\le h\le Y}\sum_{m\sim M} \frac{e(Xhm^{-\beta})}{h} ,$$

$$R_2(X,M,\beta) = \sum_{1\le h\le Y^2} f(h)\sum_{m\sim M} e(hXm^{-\beta}) .$$

We shall only estimate $R_2(X,M,\beta)$, because $R_1(X,M,\beta)$ can be dealt with similarly and more easily. Let $\chi(\cdot)$ be a C^∞ function such that

$$\chi(x) = 0 \quad \text{if } x \ge 4 , \qquad 0 < \chi(x) \le 1 \quad \text{if } 2 \le x < 4 ,$$
$$\chi(x) = 1 - \chi(2x) \quad \text{if } 1 < x \le 2 , \qquad \chi(x) = 0 \quad \text{if } x \le 1 ,$$

then

$$\sum_H \chi\left(\frac{x}{H}\right) = 1 \qquad \text{for all } x > 0 ,$$

where H runs through the sequence $\{2^j : j \in \mathbb{Z}\}$. Hence one sees that

$$R_2(X,M,\beta) \ll \ln x|S(H,M,X)| + (XM^{1-\beta})^{7/22}$$

for some $H = 2^j \in [1, Y^2]$, where

$$S(H,M,X) = \sum_h f(h)\chi\left(\frac{h}{H}\right)\sum_{m\sim M} e(hXm^{-\beta}) .$$

Let

$$Q(m) = \sum_h f(h)\chi\left(\frac{h}{H}\right)e(hXm^{-\beta}) .$$

Then

$$S(H,M,X) = \sum_{m\sim M} Q(m) , \qquad Q(m) \ll \min(1, YH^{-1}) .$$

For this H, we set the choice

$$N = \max(H, (MH^{-1})^{1/2}, M^{1+2\beta/5}(XH)^{-2/5}) ,$$
$$D = \min(H, Y, H^{-1}X^{-1}M^{\beta+2})$$

(our choice implies that $N = O(MX^{-\varepsilon})$). Adopting the arguments in Sections 5 and 6 of [3], we obtain

$$X^{-\varepsilon}|S(H,M,X)| \ll \sum_{1\le c\le C_1 G} \sum_{a\asymp cXM^{-\beta-1}} B(m_0) + \max_C \Big(\frac{G}{C} \sum_{\substack{c\sim C \\ (c,a)=1}} \sum_{a\asymp A} F(m_0)\Big) + N\,.$$

Here the maximum is taken over numbers C of the form 2^j, $j\in\mathbb{Z}$, such that $C_2 G \le C \le D$, and G, m_0, A are defined as follows:

$$G = \frac{M^{\beta+2}}{XND}\,, \quad m_0 = m_0\Big(\frac{a}{c}\Big) = \Big[\Big(\frac{Xc\beta}{a}\Big)^{\alpha}\Big]\,, \quad A = CXM^{-\beta-1}\,,$$

$B(m_0)$ is a number such that for any integers L_1 and L_2 with $|L_1|, |L_2| \ll M^{\beta+2}/(XcD)$, we have

$$\Big|\sum_{L_1\le r\le L_2} Q(m_0+r)\Big| \ll B\Big(m_0\Big(\frac{a}{c}\Big)\Big)\,;$$

and $F(m_0)$ is as follows:

$$F(m_0) = \sum_n Q(m_0+n)g(n)\,, \quad g(n) = \sigma\Big(\frac{n}{N}\Big)\,,$$

where $\sigma(\cdot)$ is also some C^∞ function, whose support is contained in an interval $[C_3, C_4]$.

3.2. *The estimate for the sum involving $B(\cdot)$.* In this subsection, we prove

Lemma 3.

$$\sum_{1\le c\le C_1 G} \sum_{a\asymp cXM^{-1-\beta}} B(m_0) \ll MY^{-1} + Y(X^{19}H^{19}M^{-30-19\beta})^{1/10} + (X^{-1}H^{-11}M^{10+\beta})^{1/10}\,.$$

Proof. From (12) we see that, for any L_1, L_2,

$$\Big|\sum_{L_1\le r\le L_2} Q(m_0+r)\Big| \ll \min\Big(\frac{1}{H}, \frac{Y}{H^2}\Big) \sum_{h\asymp H} \Big|\sum_{L_1\le r\le L_2} e\Big(\frac{hX}{(m_0+r)^\beta}\Big)\Big|\,.$$

Writing

$$m_0 = \Big(\frac{\beta cX}{a}\Big)^{\alpha} - v\,, \quad 0\le v<1\,, \tag{14}$$

it is easy to verify that

$$\frac{hX}{(m_0+r)^\beta} = \frac{hX}{m_0^\beta} - \frac{a}{c}hr + R(r)\,,$$

where

$$R(r)=\beta hrX\biggl(\frac{1}{(m_0+v)^{\beta+1}}-\frac{1}{m_0^{\beta+1}}\biggr)+\frac{hX}{m_0^\beta}\biggl(\biggl(1+\frac{r}{m_0}\biggr)^{-\beta}+\frac{\beta r}{m_0}-1\biggr).$$

For $|r|\ll M^{\beta+2}/(XDc)$, we have

$$R'(r)\ll 1/c\,,\qquad R''(r)\ll HXM^{-\beta-2}\,.$$

Let

$$\omega(r)=\max(0,1+\min(r-L_1,L_2-r,0))\,.$$

(We can assume that L_1 and L_2 are integers.) By using the Poisson summation formula and the familiar estimates for trigonometric integrals, we can obtain, as in Section 7 of [3],

$$\begin{aligned}\sum_{L_1\le r\le L_2} e\biggl(\frac{hX}{(m_0+r)^\beta}\biggr)&=e\biggl(\frac{hX}{m_0^\beta}\biggr)\sum_{k\equiv -ah\ (\mathrm{mod}\ c)}\int\omega(r)e\biggl(R(r)+\frac{kr}{c}\biggr)\,dr\\&\ll\sum_{k\equiv -ah\ (\mathrm{mod}\ c)} I(k)\,,\end{aligned}$$

where

$$I(k)=\begin{cases}\min(c|k|^{-1},c^2k^{-2}) & \text{if } |k|>C_5HD^{-1},\\ (HXM^{-\beta-2})^{-1/2} & \text{if } |k|\le C_5HD^{-1}.\end{cases}$$

Hence we can deduce that

$$\begin{aligned}\Bigl|\sum_{L_1\le r\le L_2}Q(m_0+r)\Bigr|&\ll \min\biggl(\frac{1}{H},\frac{Y}{H^2}\biggr)\sum_{h\asymp H}\ \sum_{k\equiv -ah\ (\mathrm{mod}\ c)} I(k)\\&\ll c^{-1}\min(1,YH^{-1})\sum_k I(k)\\&\ll c^{-1}D^{-1}H^{1/2}X^{-1/2}M^{1+\beta/2}\min(1,YH^{-1})\,.\end{aligned}$$

Note that the bound given above is independent of L_1, L_2. Thus

$$\begin{aligned}&\sum_{1\le c\le C_1G}\ \sum_{a\asymp cXM^{-\beta-1}} B(m_0)\\&\ll \min(1,YH^{-1})N^{-1}H^{1/2}M^{2+\beta/2}X^{-1/2}D^{-2}\\&\ll \min(1,YH^{-1})N^{-1}H^{1/2}M^{2+\beta/2}X^{-1/2}(H^{-2}+Y^{-2}+H^2X^2M^{-2\beta-4})\\&\ll \min(1,YH^{-1})((X^{-1}H^{-11}M^{10+\beta})^{1/10}+(X^{-1}H^9M^{10+\beta}Y^{-20})^{1/10}\\&\qquad+(X^{19}M^{-30-19\beta}H^{29})^{1/10})\,,\end{aligned}$$

and Lemma 3 follows.

Note that in the above argument we have assumed that $Y^2\ll X^{-1}M^{\beta+2}$, which ensures that $D\gg 1$. This assumption is permissible, for otherwise one has $M^{4\beta+7} \ll X^4$, and, by choosing $(k,\lambda) = (2/7, 4/7)$ in Lemma 2, one finds that

$$X^{-\varepsilon}R(X,M,\beta) \ll (XM^{1-\beta})^{2/9}M^{2/9} \ll (XM^{1-\beta})^{10/33} \ll (XM^{1-\beta})^{7/22},$$

hence Lemma 1 trivially holds.

3.3. *The contribution from the sum involving $F(\cdot)$.* In this section, we shall prove the following estimate.

LEMMA 4. *Let C be any number such that $C_2G \le C \le D$. Then*

$$\begin{aligned} X^{-\varepsilon}\Big(\frac{G}{C}\sum_{\substack{c\sim C\\(c,a)=1}}\sum_{a\asymp A}F(m_0)\Big) &\ll (XM^{1-\beta})^{7/22} \\ &\quad + (X^3M^{-1-3\beta})^{1/4} + (X^2M^{5-2\beta}H^{-3})^{1/10} \\ &\quad + Y((H^2X^3M^{-3\beta-3})^{1/4} + (H^{-3}X^2M^{-2\beta})^{1/5} \\ &\quad + (H^3XM^{-2-\beta})^{1/2}) + (H^9Y^{-5}X^4M^{-4\beta})^{1/10}. \end{aligned}$$

It will be clear that Lemma 4 is a consequence of the next two lemmas.

LEMMA 5. *Suppose $C_2G \le C \le D$, $c \sim C$, $a \asymp A$, $(c,a) = 1$. Then*

$$\begin{aligned} F(m_0) &= \frac{1}{2(\eta c)^{1/2}}\sum_{r\asymp L}\sum_{k\asymp K}k^{-1/2}e(x_1kr + x_2r + x_3rk^{1/2} + x_4rk^{-1/2}) \\ &\quad \times \chi\Big(\frac{rk^{-1/2}}{2H(\eta c)^{1/2}}\Big)\sigma\Big(\frac{k^{1/2}}{N(\eta c)^{1/2}}\Big)f\Big(\frac{rk^{-1/2}}{2(\eta c)^{1/2}}\Big) \\ &\quad + O(CH^{1/2}Y\max(Y^{-5/2}, H^{-5/2})) + O(\min(1, YH^{-1})\mathcal{R}), \end{aligned}$$

where

$$K = N^2CXM^{-\beta-2}, \qquad L = HCNXM^{-\beta-2},$$

$$\kappa = \frac{cx}{m_0^\beta} - \Big[\frac{cx}{m_0^\beta}\Big], \qquad \eta = \frac{1}{2}\beta(\beta+1)X^{-\alpha}\Big(\frac{a}{c\beta}\Big)^{1+\alpha},$$

$\overline{a}$ is the unique solution of the congruence $a\overline{a} \equiv 1 \pmod{c}$ with $1 \le \overline{a} < c$, $b = [cX/m_0^\beta]$, v is as in (14), *and*

$$x_1 = \frac{\overline{a}}{c}, \quad x_2 = \frac{\overline{a}b + v}{c}, \quad x_3 = -\frac{1}{(\eta c^3)^{1/2}}, \quad x_4 = \frac{\kappa}{2(\eta c^3)^{1/2}},$$

$$\begin{aligned} \mathcal{R} = CH^{-3/2}N^{-1}X^{-1/2}M^{1+\beta/2} &+ N^{-2}(H^{-1}X^{-1}M^{2+\beta})^{3/2} \\ &+ N(HXM^{-\beta-2})^{3/2} + (HXM^{-4-\beta})^{1/2}N^3. \end{aligned}$$

Proof. The arguments in what follows are clear in view of Sections 8 to 12 of [3]. For $n \asymp N \ll MX^{-\varepsilon}$, we have the expansion

$$\frac{hX}{(m_0+n)^\beta} = \frac{hX}{m_0^\beta} + \gamma n + \delta n^2 + t(n)\,,$$

where

$$\gamma = -h\left(\frac{a}{c} + v\beta(\beta+1)\left(\frac{a}{c\beta}\right)^{1+\alpha} X^{-\alpha}\right) = -h\left(\frac{a}{c} + 2v\eta\right), \tag{15}$$

$$\delta = h\frac{\beta(\beta+1)}{2}\left(\frac{a}{c\beta}\right)^{1+\alpha} X^{-\alpha} = h\eta \asymp HXM^{-\beta-2}\,, \tag{16}$$

$$\begin{aligned} t(n) = {} & hn\beta(\beta+1)v\left(\frac{a}{c\beta}\right)^{1+\alpha} X^{-\alpha}\left(1 - \left(1 - v\left(\frac{a}{c\beta X}\right)^{\alpha}\right)^{-\beta-2}\right) \\ & + hn\beta X\left((\beta+1)vm_0^{-\beta-2} + \frac{1}{(m_0+v)^{\beta+1}} - \frac{1}{m_0^{\beta+1}}\right) \\ & + hXm_0^{-\beta}\left(\left(1+\frac{n}{m_0}\right)^{-\beta} - 1 + \frac{\beta n}{m_0} - \frac{\beta(\beta+1)}{2}\left(\frac{n}{m_0}\right)^2\right) \\ & + hn^2\frac{1}{2}\beta(\beta+1) \\ & \times \left(\frac{a}{c\beta}\right)^{1+\alpha} X^{-\alpha}\left(-1 + \left(1 - v\left(\frac{a}{\beta cX}\right)^{\alpha}\right)^{-\beta-2}\right). \end{aligned}$$

From the expression of $t(n)$, we can obtain the estimates

$$t(n) \ll HN^3XM^{-\beta-3}\,, \qquad t'(n) \ll HN^2XM^{-\beta-3}\,.$$

Hence, by partial summation, one gets

$$\begin{aligned} F(m_0) &= \sum_h f(h)\chi\left(\frac{h}{H}\right)e\left(\frac{hX}{m_0^\beta}\right)\sum_n \sigma\left(\frac{n}{N}\right)e(\gamma n + \delta n^2 + t(n)) \\ &= \sum_h f(h)\chi\left(\frac{h}{H}\right)e\left(\frac{hX}{m_0^\beta}\right)\left(\sum_n \sigma\left(\frac{n}{N}\right)e(\gamma n + \delta n^2) \right. \\ &\qquad \left. + O((HXN^6M^{-\beta-4})^{1/2})\right) \\ &= \sum_h f(h)\chi\left(\frac{h}{H}\right)e\left(\frac{hX}{m_0^\beta}\right)\sum_n \sigma\left(\frac{n}{N}\right)e(\gamma n + \delta n^2) \\ &\qquad + O(\min(1, YH^{-1})(HXN^6M^{-\beta-4})^{1/2})\,. \end{aligned} \tag{17}$$

From (15), (16), we find that

$$\gamma = -(ha+\varrho)/c\,, \quad \varrho = -2cv\delta \ll HDXM^{-\beta-2} \ll 1\,,$$

$$\delta cN \gg GHXNM^{-\beta-2} \gg \frac{M^{\beta+2}}{XND}HXNM^{-\beta-2} \gg \frac{H}{D} \gg 1\,,$$

thus, as in Section 9 of [3], we deduce that

$$(18)\quad \sum_n \sigma\Big(\frac{n}{N}\Big) e(\gamma n + \delta n^2) = \Big(\frac{i}{2\delta}\Big)^{1/2} \sum_{\substack{r\equiv ah \ (\mathrm{mod}\ c)\\ |r|\ll \delta CN}} \bigg(\sigma\Big(\frac{r+\varrho}{2\delta cN}\Big) e\Big(-\frac{(r+\varrho)^2}{4\delta c^2}\Big) + O(N^{-2}H^{-1}X^{-1}M^{\beta+2})\bigg).$$

On account of

$$\sigma\Big(\frac{r+\varrho}{2\delta cN}\Big) = \sigma\Big(\frac{r}{2\delta cN}\Big) + O(N^{-1}),$$
$$e\Big(-\frac{(r+\varrho)^2}{4\delta c^2}\Big) = e\Big(-\frac{r^2+2r\varrho}{4\delta c^2}\Big) + O(HXM^{-\beta-2}),$$

we get

$$(19)\quad \sum_n \sigma\Big(\frac{n}{N}\Big) e(\gamma n + \delta n^2) = \Big(\frac{i}{2h\eta}\Big)^{1/2} \sum_{r\equiv ah \ (\mathrm{mod}\ c)} \sigma\Big(\frac{r}{2\delta cN}\Big) e\Big(-\frac{r^2+2r\varrho}{4\delta c^2}\Big) + O((H\eta)^{-1/2}(N^{-2}H^{-1}X^{-1}M^{\beta+2} + NH^2X^2M^{-2\beta-4})).$$

We get, by the Poisson summation formula,

$$\sum_h h^{-1/2} f(h)\chi\Big(\frac{h}{H}\Big) e\Big(\frac{hX}{m_0^\beta}\Big) \sum_{r\equiv ah \ (\mathrm{mod}\ c)} \sigma\Big(\frac{r}{2\delta cN}\Big) e\Big(-\frac{r^2+2r\varrho}{4\delta c^2}\Big)$$
$$= \sum_{r\asymp L} e\Big(\frac{r(\bar a b + v)}{c}\Big) \sum_{h\equiv \bar a r \ (\mathrm{mod}\ c)} \sigma\Big(\frac{r}{2hcN\eta}\Big) h^{-1/2} f(h) \times \chi\Big(\frac{h}{H}\Big) e\Big(\frac{h\kappa}{c} - \frac{r^2}{4h\eta c^2}\Big)$$
$$= \frac{1}{c}\sum_{r\asymp L} e\Big(\frac{r(\bar a b + v)}{c}\Big) \sum_k e\Big(\frac{rk\bar a}{c}\Big) J(k-\kappa, r),$$

where the integral $J(\cdot,\cdot)$ is given by

$$J(k-\kappa, r) = \int_0^\infty \xi^{-1/2} f(\xi)\chi\Big(\frac{\xi}{H}\Big)\sigma\Big(\frac{r}{2cN\eta\xi}\Big) e\Big(-\frac{k-\kappa}{c}\xi - \frac{r^2}{4\eta c^2}\xi^{-1}\Big)\, d\xi.$$

If $k > C_6 K$ or $k < C_7 K$, then for $r \asymp L$, $\xi \asymp H$, one has

$$\left|\frac{r^2}{4\eta c^2\xi^2} - \frac{k-\kappa}{c}\right| \gg \frac{1}{c}(|k| + K)\,.$$

Integration by parts, gives, in view of (12), the estimate

$$J(k-\kappa, r) \ll \frac{c^2 H^{-5/2}}{(|k|+K)^2}\,,$$

thus

$$\Big(\sum_{k>C_6K} + \sum_{k<C_7K}\Big) J(k-\kappa, r) \ll C^2 H^{-5/2} K^{-1}\,,$$

and, consequently, one obtains

$$\begin{aligned}(20)\qquad &\sum_h h^{-1/2} f(h)\chi\Big(\frac{h}{H}\Big) e\Big(\frac{hX}{m_0^\beta}\Big) \\ &\qquad\qquad \times \sum_{r\equiv ah \pmod c} \sigma\Big(\frac{r}{2\eta cNh}\Big) e\Big(-\frac{r^2+2r\varrho}{4hc^2\eta}\Big) \\ &= \frac{1}{c}\sum_{r\asymp L, k\asymp K} e\Big(\frac{r(\overline{a}b + v + \overline{a}k)}{c}\Big) J(k-\kappa, r) + O(CN^{-1}H^{-3/2})\,.\end{aligned}$$

Put

$$P(\xi) = \xi f(\xi)\chi\Big(\frac{\xi}{H}\Big)\sigma\Big(\frac{r}{2cN\eta\xi}\Big)\,.$$

By (12) and (13), we find that, for $\xi \asymp H$,

$$P'(\xi) \ll H^{-1}\,, \qquad P''(\xi) \ll \frac{1}{YH}(\max(1, YH^{-1}))^2\,,$$
$$P'''(\xi) \ll \frac{1}{HY^2}(\max(1, YH^{-1}))^3\,.$$

Thus, by taking

$$a = -\frac{r^2}{4\eta c^2}\,, \qquad b = -\frac{k-\kappa}{c}$$

in Lemma 11.1 of [3], we get

$$\begin{aligned}(21)\qquad &J(k-\kappa, r) \\ &= (2\eta i)^{1/2}\frac{c}{r} e\Big(-\frac{r}{c}\Big(\frac{k-\kappa}{\eta c}\Big)^{1/2}\Big) P\Big(\frac{r}{2c}\Big(\frac{c}{\eta(k-\kappa)}\Big)^{1/2}\Big) + R_P(a,b)\,,\end{aligned}$$

where

$$\begin{aligned}(22)\qquad R_P(a,b) &\ll (b^{-3/2} + a^{-1/2}b^{-2})(\|P''\|\,\|P'''\|)^{1/2} \\ &\ll H^{-1/2}(N^2XM^{-\beta-2}Y)^{-3/2}(\max(1, YH^{-1}))^{5/2}\,.\end{aligned}$$

Since

$$P\left(\frac{r}{2c}\left(\frac{c}{\eta(k-\kappa)}\right)^{1/2}\right)=P\left(\frac{r}{2c}\left(\frac{c}{\eta k}\right)^{1/2}\right)+O(K^{-1}),\tag{23}$$

the lemma follows from (17) to (23).

Now let

$$\mathcal{M}(a,c)=\sum_{r\asymp L}\sum_{k\asymp K}k^{-1/2}e(x_1kr+x_2r+x_3rk^{1/2}+x_4rk^{-1/2})$$
$$\times\chi\left(\frac{rk^{-1/2}}{2H(\eta c)^{1/2}}\right)\sigma\left(\frac{k^{1/2}}{N(\eta c)^{1/2}}\right)f\left(\frac{rk^{-1/2}}{2(\eta c)^{1/2}}\right).$$

Lemma 6. *For any C in the range $C_2G\le C\le D$, one has*

$$\frac{G}{\sqrt{XM^{-\beta-2}C^3}}\sum_{\substack{c\sim C\\(c,a)=1}}\sum_{a\asymp A}|\mathcal{M}(a,c)|$$
$$\ll X^{\varepsilon}((XM^{1-\beta})^{7/22}+(X^3M^{-1-3\beta})^{1/4}$$
$$+Y(H^2X^3M^{-3-3\beta})^{1/4}+(X^2M^{5-2\beta}H^{-3})^{1/10}).$$

Proof. First we assume that $Y\le H\le Y^2$. We have

$$\chi(\xi)=\int_{-\infty}^{+\infty}\widetilde{\chi}(it)\xi^{-it}\,dt,\qquad \sigma(\xi)=\int_{-\infty}^{+\infty}\widetilde{\sigma}(it)\xi^{-it}\,dt,$$

where $\widetilde{\chi}$, $\widetilde{\sigma}$ are the Millins transforms of χ and σ, such that

$$\int_{-\infty}^{+\infty}|\widetilde{\chi}(it)|\,dt\ll 1,\qquad \int_{-\infty}^{+\infty}|\widetilde{\sigma}(it)|\,dt\ll 1.$$

Hence, for some t_1 and t_2, we have

$$\sum_{\substack{c\sim C\\(c,a)=1}}\sum_{a\asymp A}|\mathcal{M}(a,c)|\ll\sum_{\substack{c\sim C\\(c,a)=1}}\sum_{a\asymp A}\Bigg|\sum_{k\asymp K}\sum_{r\asymp L}k^{-\frac12+\frac12 i(t_1-t_2)}\tag{24}$$
$$\times r^{-it_1}e(x_1kr+x_2r+x_3rk^{1/2}+x_4rk^{-1/2})f\left(\frac{r}{2(\eta ck)^{1/2}}\right)\Bigg|.$$

By means of the expression (11) for $f(\cdot)$, we get

$$\sum_{\substack{c\sim C\\(c,a)=1}}\sum_{a\asymp A}|\mathcal{M}(a,c)|\tag{25}$$
$$\ll CXM^{-\beta-2}\left(Y\mathcal{S}_1(C)+\frac{1}{Y}\int_{Y^{-1}}^{1/2}\mathcal{S}_2(C,\theta)\theta^{-3}\,d\theta\right),$$

where

$$\mathcal{S}_1(C) = \sum_{\substack{c\sim C\\(c,a)=1}} \sum_{a\asymp A} \Big| \sum_{r\asymp L} \sum_{k\asymp K} k^{\frac12+\frac{i}{2}(t_1-t_2)} r^{-2-it_1} \times e(x_1kr + x_2r + x_3rk^{1/2} + x_4'rk^{-1/2}) \Big| ,$$

$$\mathcal{S}_2(C,\theta) = \sum_{\substack{c\sim C\\(c,a)=1}} \sum_{a\asymp A} \Big| \sum_{r\asymp L} \sum_{k\asymp K} k^{\frac12+\frac{i}{2}(t_1-t_2)} r^{-2-it_1} \times e(x_1kr + x_2r + x_3rk^{1/2} + x_4''(\theta)rk^{-1/2}) \Big| ,$$

$$x_4' = x_4 \pm \frac{1}{2Y(\eta c)^{1/2}} , \qquad x_4''(\theta) = x_4 \pm \frac{\theta}{2(\eta c)^{1/2}} .$$

We proceed to estimate $\mathcal{S}_2(C,\theta)$ by means of Lemma 2.4 of [1]. We observe that the quantity $x_1kr + x_2r + x_3rk^{1/2} + x_4''(\theta)rk^{-1/2}$ is just the inner product of the two vectors $(x_1, x_2, x_3, x_4''(\theta))$ and $(kr, r, rk^{1/2}, rk^{-1/2})$; thus Lemma 2.4 of [1] gives

$$\mathcal{S}_2^4(C,\theta) \ll (CAKL^{-4})^2 \prod_{j=1}^{4} (1 + X_jY_j) B_1B_2 , \tag{26}$$

with

$$\begin{aligned} X_1 &= X_2 = 1 , \quad X_3 = (C^3XM^{-\beta-2})^{-1/2} , \\ X_4 &= X_4(\theta) = (C^3XM^{-\beta-2})^{-1/2}(1 + C\theta) , \end{aligned} \tag{27}$$

$$Y_1 = KL , \quad Y_2 = L , \quad Y_3 = LK^{1/2} , \quad Y_4 = LK^{-1/2} , \tag{28}$$

B_1 is the number of pairs (a,c), (a',c'), with a, $a' \asymp A$, c, $c' \sim C$, such that the following inequalities hold simultaneously:

$$\|x_1(a,c) - x_1(a',c')\| \ll (KL)^{-1} ,$$
$$|x_3(a,c) - x_3(a',c')| \ll (K^{1/2}L)^{-1} ,$$

or, equivalently,

$$\left\| \frac{\overline{a}}{c} - \frac{\overline{a}'}{c'} \right\| \ll \Delta_1 , \qquad \left| \frac{c}{c'} - \frac{g(c/a)}{g(c'/a')} \right| \ll \Delta_2 ,$$

where

$$g(\xi) = \xi^{(1+\alpha)/3} , \qquad \Delta_1 = (X^2C^2N^3H)^{-1}M^{2\beta+4} ,$$
$$\Delta_2 = (XHN^2)^{-1}M^{\beta+2} ;$$

thus, by Lemma 2.4 of [2], B_1 can be estimated as follows:

$$(29)\qquad \begin{aligned} B_1 &\ll CA + C^2A^2\Delta_1\Delta_2 + \Delta_1^2A^2C^2 + C^2 + \Delta_2A^2 \\ &\ll C^2XM^{-\beta-1}(1 + X^{-2}M^{2\beta+5}H^{-2}N^{-5} + MH^{-1}N^{-2}) \\ &\quad + H^{-2}N^{-6}X^{-2}M^{2\beta+6} \\ &\ll C^2XM^{-\beta-1} + H^{-2}X^{-2}N^{-6}M^{2\beta+6}\,; \end{aligned}$$

B_2 is the number of 8-tuples $(k_1, k_2, k_3, k_4, r_1, r_2, r_3, r_4)$ such that $k_1, k_2, k_3, k_4 \asymp K$, $r_1, r_2, r_3, r_4 \asymp L$, and

$$\begin{gathered} r_1 + r_2 = r_3 + r_4\,, \qquad k_1r_1 + k_2r_2 = k_3r_3 + k_4r_4\,, \\ k_1^{1/2}r_1 + k_2^{1/2}r_2 = k_3^{1/2}r_3 + k_4^{1/2}r_4 + O(X_3^{-1})\,; \end{gathered}$$

Theorem 14.1 of [3] gives

$$(30)\qquad B_2 \ll (KL)^{2+\varepsilon}(1 + HN^{-1} + CH^{-1}) \ll (KL)^{2+\varepsilon}\,.$$

From (26) to (30), we obtain (be sure that $X_jY_j \gg 1$)

$$\begin{aligned} X^{-\varepsilon}\mathcal{S}_2^4(C,\theta) &\ll (CAL^{-1})^2K^5(XC^3M^{-\beta-2})^{-1}(1 + C\theta) \\ &\qquad \times (C^2XM^{-\beta-1} + (HXN^3M^{-\beta-3})^{-2}) \\ &\ll (CN^2XM^{-\beta-3/2})^4H^{-2}(1 + C\theta) \\ &\qquad \times (C^2XM^{-\beta-1} + (HXN^3M^{-\beta-3})^{-2})\,, \end{aligned}$$

$$(31)\qquad \begin{aligned} \mathcal{S}_2(C,\theta) &\ll CN^2X^{1+\varepsilon}M^{-\beta-3/2}H^{-1/2}(1 + C^{1/4}\theta^{1/4}) \\ &\qquad \times (C^2XM^{-\beta-1} + H^{-2}X^{-2}M^{2\beta+6}N^{-6})^{1/4}\,. \end{aligned}$$

As $C \le D \le Y$, by taking $\theta = 1/Y$ in (31), we get

$$(32)\qquad \begin{aligned} \mathcal{S}_1(C) &\ll CN^2X^{1+\varepsilon}M^{-\beta-3/2}H^{-1/2} \\ &\qquad \times (C^2XM^{-\beta-1} + H^{-2}X^{-2}M^{2\beta+6}N^{-6})^{1/4}\,. \end{aligned}$$

From (25), (31) and (32), we conclude that

$$\begin{aligned} &\frac{G}{\sqrt{XM^{-\beta-2}C^3}} \sum_{\substack{c\sim C \\ (c,a)=1}} \sum_{a\asymp A} |\mathcal{M}(a,c)| \\ &\quad \ll \frac{M^{\beta+2}}{XND}\frac{1}{\sqrt{XM^{-\beta-2}C^3}}C^2X^2M^{-2\beta-7/2}H^{-1/2}YN^2 \\ &\qquad \times (C^2XM^{-\beta-1} + H^{-2}X^{-2}M^{2\beta+6}N^{-6})^{1/4}X^\varepsilon \\ &\quad \ll X^\varepsilon(X^{3/4}M^{-(3\beta+3)/4}YH^{-1/2}N + M^{1-\beta}H^{-1}YN^{-1/2}D^{-1/2})\,, \end{aligned}$$

and the required estimate follows in view of the values for D, N, Y. For the case $1 \le H \le Y$, we use the expression (10) for the function $f(\cdot)$ in (24), and we get

$$\sum_{\substack{c\sim C}}\sum_{\substack{a\asymp A\\(c,a)=1}}|\mathcal{M}(a,c)| \ll Y^{-1}\Big((CXM^{-\beta-2})^{1/2}\int_{Y^{-1}}^{1/2}\mathcal{S}_3(C,\theta)\theta^{-2}\,d\theta + CXM^{-\beta-2}\mathcal{S}_4(C)\Big), \tag{33}$$

where

$$\mathcal{S}_3(C,\theta)=\sum_{\substack{c\sim C}}\sum_{\substack{a\asymp A\\(c,a)=1}}\Big|\sum_{r\asymp L}\sum_{k\asymp K}k^{\frac12 i(t_1-t_2)}r^{-1-it_1} \times e(x_1kr+x_2r+x_3rk^{1/2}+x_4'(\theta)rk^{-1/2})\Big|,$$

$$\mathcal{S}_4(C)=\sum_{\substack{c\sim C}}\sum_{\substack{a\asymp A\\(c,a)=1}}\Big|\sum_{r\asymp L}\sum_{k\asymp K}k^{\frac12+\frac12 i(t_1-t_2)}r^{-2-it_1} \times e(x_1kr+x_2r+x_3rk^{1/2}+x_4''rk^{-1/2})\Big|,$$

$$x_4'(\theta)=x_4\pm\frac{\theta}{2(\eta c)^{1/2}},$$

$$x_4''=x_4\pm\frac{1}{4(\eta c)^{1/2}}.$$

As before, we can deduce that

$$\mathcal{S}_3(C,\theta)\ll(C^3X^3M^{-3\beta-5}H)^{1/2}N^2(1+C\theta)^{1/4} \times(C^2XM^{-\beta-1}+(HXM^{-\beta-3}N^3)^{-2})^{1/4}X^{\varepsilon} \tag{34}$$

and

$$\mathcal{S}_4(C)\ll C^{5/4}N^2X^{1+\varepsilon}M^{-\beta-3/2}H^{-1/2} \times(C^2XM^{-\beta-1}+H^{-2}X^{-2}M^{2\beta+6}N^{-6})^{1/4}, \tag{35}$$

and the required estimate follows from (33)–(35). The proof of Lemma 6 is finished.

Remark. As is clear from the above proof, the idea here is to put (10) or (11) into (24), so that one can separate the variables c and $rk^{-1/2}$ inside $f(r/(2(\eta ck)^{1/2}))$ suitably.

Proof of Lemma 4. By Lemmas 5 and 6, we are left with estimating the contribution from the sum of the error terms given by Lemma 5. We have, for $C_2G \le C \le D$,

$$\begin{aligned}\frac{G}{C}\sum_{c\sim C}\sum_{a\asymp A}&(CYH^{-2}+CH^{1/2}Y^{-3/2}+\min(1,YH^{-1})\mathcal{R})\\ &\ll \frac{M}{N}(DYH^{-2}+DH^{1/2}Y^{-3/2}\\ &\quad+\min(1,YH^{-1})(DH^{-3/2}N^{-1}X^{-1/2}M^{\beta/2+1}\\ &\quad+N^{-2}(HXM^{-\beta-2})^{-3/2}\\ &\quad+N(HXM^{-\beta-2})^{3/2}+N^3(HXM^{-4-\beta})^{1/2}))\\ &\ll Y(H^{-3}X^2M^{-2\beta})^{1/5}+(H^9X^4M^{-4\beta}Y^{-5})^{1/10}+(XM^{-\beta}Y)^{3/10}\\ &\quad+M^{1+3\beta/10}(HX)^{-3/10}+Y(HX^3M^{-4-3\beta})^{1/2}\\ &\quad+Y(H^3XM^{-2-\beta})^{1/2}+(XM^{-\beta})^{1/2},\end{aligned}$$

and Lemma 4 follows by considering that $H \ll Y^2$, $M \ll X^\alpha$, and $M^{4\beta+7} \gg X^4$ (which is permissible, see the end of Section 3.2).

3.4. Proof of Lemma 1. By the arguments in Section 3.1, to prove Lemma 1, it suffices to establish the following estimate for $S(H,M,X)$.

LEMMA 7. *We have*

$$S(H,M,X) \ll X^\varepsilon((XM^{1-\beta})^{7/22}+(X^3M^{-1-3\beta})^{1/4}).$$

Proof. From (12) and the exponent pair $(1/2,1/2)$, we infer that

$$S(H,M,X) \ll \min(1,YH^{-1})(HXM^{-\beta})^{1/2}. \tag{36}$$

From the starting inequality for $S(H,M,X)$ in Section 3.1, and Lemmas 3 and 4, we get

$$\begin{aligned}&X^{-\varepsilon}S(H,M,X)\\ &\quad\ll (XM^{1-\beta})^{7/22}+(X^3M^{-1-3\beta})^{1/4}+\mathcal{R}^+(H)+\mathcal{R}^-(H),\end{aligned} \tag{37}$$

where

$$\begin{aligned}\mathcal{R}^+(H) = {}&Y(X^{19}H^{19}M^{-30-19\beta})^{1/10}+Y(H^2X^3M^{-3\beta-3})^{1/4}\\ &+Y(H^3XM^{-2-\beta})^{1/2}+(H^9Y^{-5}X^4M^{-4\beta})^{1/10}\end{aligned} \tag{38}$$

and

$$\begin{aligned}\mathcal{R}^-(H) = {}&(X^{-1}H^{-11}M^{10+\beta})^{1/10}+(X^2M^{5-2\beta}H^{-3})^{1/10}\\ &+Y(H^{-3}X^2M^{-2\beta})^{1/5}.\end{aligned} \tag{39}$$

From (36) and (37), we have

(40) $X^{-\varepsilon}S(H,M,X) \ll (XM^{1-\beta})^{7/22} + (X^3M^{-1-3\beta})^{1/4} + \mathcal{R}^+ + \mathcal{R}^-\,,$

where, by (38) and (39),

$$\mathcal{R}^+ = \min(Y(H^{-1}XM^{-\beta})^{1/2}\,,\, \mathcal{R}^+(H)) \ll \sum_{i=1}^{4}\mathcal{R}_i\,, \tag{41}$$

$$\mathcal{R}^- = \min((HXM^{-\beta})^{1/2}\,,\, \mathcal{R}^-(H)) \ll \sum_{i=5}^{7}\mathcal{R}_i\,, \tag{42}$$

and (provided that $(XM^{1-\beta})^{41} \le M^{110}$, see the end of Section 3.2)

$$\begin{aligned}\mathcal{R}_1 &= \min(Y(H^{-1}XM^{-\beta})^{1/2}, Y(X^{19}H^{19}M^{-30-19\beta})^{1/10})\\ &\le Y(H^{-1}XM^{-\beta})^{\varphi_1/2}(X^{19}H^{19}M^{-30-19\beta})^{\omega_1/10}\\ &= Y(XM^{-\beta})^{19/24}M^{-5/8} \le (XM^{1-\beta})^{7/22}\end{aligned} \tag{43}$$

with $(\varphi_1,\omega_1) = (19/24, 5/24)$,

$$\begin{aligned}\mathcal{R}_2 &= \min(Y(H^{-1}XM^{-\beta})^{1/2}, Y(H^2X^3M^{-3\beta-3})^{1/4})\\ &\le Y((H^{-1}XM^{-\beta})^{1/2})^{\varphi_2}((H^2X^3M^{-3\beta-3})^{1/4})^{\omega_2}\\ &= Y(X^5M^{-3-5\beta})^{1/8} \le (XM^{1-\beta})^{7/22}\end{aligned} \tag{44}$$

with $(\varphi_2,\omega_2) = (1/2, 1/2)$,

$$\begin{aligned}\mathcal{R}_3 &= \min(Y(H^{-1}XM^{-\beta})^{1/2}, Y(H^3XM^{-2-\beta})^{1/2})\\ &\le Y((H^{-1}XM^{-\beta})^{1/2})^{\varphi_3}((H^3XM^{-2-\beta})^{1/2})^{\omega_3}\\ &= Y(XM^{-\beta-1/2})^{1/2} \le (XM^{1-\beta})^{7/22}\end{aligned} \tag{45}$$

with $(\varphi_3,\omega_3) = (3/4, 1/4)$,

$$\begin{aligned}\mathcal{R}_4 &= \min(Y(H^{-1}XM^{-\beta})^{1/2}, (H^9Y^{-5}X^4M^{-4\beta})^{1/10})\\ &\le ((Y^2H^{-1}XM^{-\beta})^{1/2})^{\varphi_4}((H^9Y^{-5}X^4M^{-4\beta})^{1/10})^{\omega_4}\\ &= (XYM^{-\beta})^{13/28} \le (XM^{1-\beta})^{7/22}\end{aligned} \tag{46}$$

with $(\varphi_4,\omega_4) = (9/14, 5/14)$,

$$\begin{aligned}\mathcal{R}_5 &= \min((HXM^{-\beta})^{1/2}, (X^{-1}H^{-11}M^{10+\beta})^{1/10})\\ &\le ((HXM^{-\beta})^{1/2})^{\varphi_5}((X^{-1}H^{-11}M^{10+\beta})^{1/10})^{\omega_5}\\ &= (XM^{1-\beta})^{5/16}\end{aligned} \tag{47}$$

with $(\varphi_5,\omega_5) = (11/16, 5/16)$,

$$\begin{aligned}\mathcal{R}_6 &= \min((HXM^{-\beta})^{1/2}, (X^2M^{5-2\beta}H^{-3})^{1/10})\\ &\le ((HXM^{-\beta})^{1/2})^{\varphi_6}((X^2M^{5-2\beta}H^{-3})^{1/10})^{\omega_6}\\ &= (XM^{1-\beta})^{5/16}\end{aligned} \tag{48}$$

with $(\varphi_6, \omega_6) = (3/8, 5/8)$, and

$$\begin{aligned}(49)\qquad \mathcal{R}_7 &= \min((HXM^{-\beta})^{1/2}, (Y^5H^{-3}X^2M^{-2\beta})^{1/5}) \\ &\le ((HXM^{-\beta})^{1/2})^{\varphi_7}((Y^5H^{-3}X^2M^{-2\beta})^{1/5})^{\omega_7} \\ &= (YXM^{-\beta})^{5/11} \le (XM^{1-\beta})^{7/22}\,,\end{aligned}$$

with $(\varphi_7, \omega_7) = (6/11, 5/11)$.

Lemma 7 now follows from (40)–(49). The proof of Proposition 2 is therefore complete.

4. Proof of Theorem 2. By Proposition 2, we find that

$$(50)\qquad R(\xi^{1/3}, 2/3) \ll \xi^{7/55+\varepsilon}\,,$$

and, by choosing $(2/7, 4/7) \in E(3/2)$, we can verify that

$$\tau_1(3/2) \le 28/107\,, \qquad \tau_2(3/2) \le 28/107\,;$$

thus

$$(51)\qquad R(\xi^{1/2}, 3/2) \ll \xi^{14/107+\varepsilon}\,.$$

In view of (50), (51) and the fact

$$7/55 = 0.12727\ldots,\ \ 14/107 = 0.13084\ldots,$$

(4) holds with $\sigma = 14/107$, hence Theorem 2 follows from Theorem 1.

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