ACTA ARITHMETICA
LXIV.1 (1993)

A note on the diophantine equation $\frac{x^m-1}{x-1}=y^n$

by

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1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively, and let $\mathbb{P}$ be the set of primes and prime powers. The solutions (1) (x,y,m,n) of the equation

$$\frac{x^m-1}{x-1}=y^n\,,\qquad x>1\,,\ y>1\,,\ m>2\,,\ n>1\,, \tag{1}$$

which satisfy $x\in\mathbb{P}$ and $y\in\mathbb{P}$ are connected with many questions in number theory and group theory (see [2], [3], [4] and [7]). In [8], the authors proved that equation (1) has only finitely many solutions (x,y,m,n) for fixed $x\in\mathbb{P}$ or $y\in\mathbb{P}$. In this note we prove the following thorem.

Theorem. *If (x,y,m,n) is a solution of equation (1) satisfying $x\in\mathbb{P}$ and $y\equiv 1 \pmod{x}$, then $x^m<C$, where C is an effectively computable absolute constant.*

2. Preliminaries. For any real numbers α, β and γ, the hypergeometric function $F(\alpha,\beta,\gamma,z)$ is defined by the series

$$F(\alpha,\beta,\gamma,z)=1+\sum_{i=1}^{\infty}\biggl(\prod_{j=0}^{i-1}\frac{(\alpha+j)(\beta+j)}{\gamma+j}\biggr)\frac{z^i}{i!} \tag{2}$$

and satisfies the differential equation

$$z(z-1)F''+((\alpha+\beta+1)z-\gamma)F'+\alpha\beta F=0\,.$$

Let $n,t,t_1,t_2\in\mathbb{N}$ be such that $n>1$, $t>1$ and $t_1+t_2=t$. Further, let $G(z)=F(-t_2-1/n,-t_1,-t,z)$, $H(z)=F(-t_1+1/n,-t_2,-t,z)$ and

$$E(z)=\frac{F(t_2+1,t_1+(n-1)/n,t+2,z)}{F(t_2+1,t_1+(n-1)/n,t+2,1)}\,.$$

Supported by the National Natural Science Foundation of China.

(1) Throughout this paper, “solution” and “positive solution” are abbreviations for “integer solution” and “positive integer solution” respectively.

From (2), we have

$$\binom{t}{t_1} G(z) = \sum_{i=0}^{t_1} \binom{t_2 + 1/n}{i} \binom{t-i}{t_2} (-z)^i \,, \tag{3}$$

$$\binom{t}{t_1} H(z) = \sum_{i=0}^{t_2} \binom{t_1 - 1/n}{i} \binom{t-i}{t_1} (-z)^i \,. \tag{4}$$

This implies that $G(z)$ and $H(z)$ are polynomials of degree t_1 and t_2 respectively. The proofs of the following two lemmas may be found in [9].

LEMMA 1. $G(z) - (1-z)^{1/n} H(z) = z^{t+1} G(1) E(z)$. ■

LEMMA 2. *Let* $\overline{G}(z) = F(-t_2 + 1 - 1/n, -t_1 - 1, -t, z)$, $\overline{H}(z) = F(-t_1 - 1 + 1/n, -t_2 + 1, -t, z)$ *and*

$$\overline{E}(z) = \frac{F(t_2, t_1 + 1 + (n-1)/n, t+2, z)}{F(t_2, t_1 + 1 + (n-1)/n, t+2, 1)} \,.$$

Then $\overline{G}(z) H(z) - G(z) \overline{H}(z) = \lambda z^{t+1}$ *for some non-zero constant* λ. ■

LEMMA 3. *Let* $a, b, k, l_0 \in \mathbb{Z}$ *with* $k > 0$, *and let*

$$L = \prod_{l=l_0}^{l_0+k-1} (al + b) \,.$$

If p *is a prime with* $p \nmid a$ *and* $p^\alpha \parallel k!$, *then* $p^\alpha \mid L$.

Proof. Since $p \nmid a$, the congruence

$$ax + b \equiv 0 \pmod{p^r} \tag{5}$$

is solvable for any $r \in \mathbb{N}$. Let $N(r)$ denote the number of solutions of (5) which satisfy $l_0 \le x \le l_0 + k - 1$. Then

$$N(r) \ge \left[\frac{k}{p^r} \right] . \tag{6}$$

If $p^\beta \parallel L$, from (6) we get

$$\beta = \sum_{r=1}^{\infty} N(r) \ge \sum_{r=1}^{\infty} \left[\frac{k}{p^r} \right] = \alpha \,. \ ■$$

LEMMA 4. *If* n *is a prime*, *then*

$$n^{i+[i/(n-1)]} \binom{t_2 + 1/n}{i} \in \mathbb{Z} \,, \qquad n^{i+[i/(n-1)]} \binom{t_1 - 1/n}{i} \in \mathbb{Z}$$

for any $i \in \mathbb{N}$.

Proof. Let p be a prime, and let $p^{\alpha_i} \parallel i!$ for any $i \in \mathbb{N}$. By Lemma 3, if $p \neq n$, then

$$p^{\alpha_i} \Bigm| \prod_{j=0}^{i-1} (n(t_2 - j) + 1), \quad i \in \mathbb{N}.$$

If $p = n$, then

$$\alpha_i = \sum_{r=1}^{\infty} \left[\frac{i}{n^r}\right] < \sum_{r=1}^{\infty} \frac{i}{n^r} = \frac{i}{n-1}, \quad i \in \mathbb{N}.$$

Therefore,

$$n^{i+[i/(n-1)]} \binom{t_2 + 1/n}{i} \in \mathbb{Z}, \quad i \in \mathbb{N}.$$

Similarly, we can prove

$$n^{i+[i/(n-1)]} \binom{t_1 - 1/n}{i} \in \mathbb{Z}$$

for any $i \in \mathbb{N}$. ∎

LEMMA 5. *If* $|z| \geq 2$ *and* $[t/2] \geq t_1 \geq [t/2] - 1$, *then*

$$\left|\binom{t}{t_1} G(z)\right| < 2^{t-1}\left(\frac{t}{2} + 2\right)|z|^{t_1}, \quad \left|\binom{t}{t_1} H(z)\right| < 2^{t-1}|z|^{t_2}.$$

Proof. By (3), we get

$$\left|\binom{t}{t_1} G(z)\right| < \sum_{i=0}^{t_1} \binom{t_2+1}{i}\binom{t-i}{t_2}|z|^i = \sum_{i=0}^{t_1} \frac{t_2+1}{t_2-i+1}\binom{t-i}{t_1}\binom{t_1}{i}|z|^i.$$

Notice that $(t_2+1)/(t_2-t_1+1) \leq t/2+2$ and $\binom{t-i}{t_1} \leq 2^{t-i-1}$ $(i = 0, 1, \ldots, t_1)$. If $|z| \geq 2$, then we have

$$\left|\binom{t}{t_1} G(z)\right| < 2^{t-1}\left(\frac{t}{2} + 2\right)\left(1 + \frac{|z|}{2}\right)^{t_1} \leq 2^{t-1}\left(\frac{t}{2} + 2\right)|z|^{t_1}.$$

Similarly, from (4) we get

$$\left|\binom{t}{t_1} H(z)\right| < \sum_{i=0}^{t_2} \binom{t_2}{i}\binom{t-i}{t_1}|z|^i < 2^{t-1}\left(1 + \frac{|z|}{2}\right)^{t_2} \leq 2^{t-1}|z|^{t_2}. \ ∎$$

LEMMA 6. *Let* $D \in \mathbb{N}$ *be square free, and let* $k \in \mathbb{Z}$ *with* $\gcd(k, 2D) = 1$. *Let* $K = \mathbb{Q}(\sqrt{D})$, *and let* $h(D)$ *denote the class number of* K. *Further, let* $u_1 + v_1\sqrt{D}$ *be the fundamental solution of the equation*

$$u^2 - Dv^2 = 1. \tag{7}$$

If $|k| > 1$ *and* (X, Y, Z) *is a solution of the equation*

$$X^2 - DY^2 = k^Z, \quad \gcd(X, Y) = 1, \quad Z > 0, \tag{8}$$

then

$$Z = Z_1 t\,, \qquad X + Y\sqrt{D} = (X_1 \pm Y_1\sqrt{D})^t (u + v\sqrt{D})\,, \tag{9}$$

where $t \in \mathbb{N}$, (u,v) *is a solution of* (7), (X_1, Y_1, Z_1) *is a positive solution of* (8) *which satisfies* $Z_1 \mid 3h(D)$ *and*

$$1 < \left|\frac{X_1 + Y_1\sqrt{D}}{X_1 - Y_1\sqrt{D}}\right| < (u_1 + v_1\sqrt{D})^2\,. \tag{10}$$

Proof. Since $\gcd(X,Y) = \gcd(k, 2D) = 1$, $X + Y\sqrt{D}$ and $X - Y\sqrt{D}$ are relatively prime in $\mathbb{Z}[\omega]$, where

$$\omega = \begin{cases} (1+\sqrt{D})/2 & \text{if } D \equiv 1 \pmod 4,\\ \sqrt{D} & \text{otherwise.}\end{cases}$$

From $[X + Y\sqrt{D}][X - Y\sqrt{D}] = [k]^Z$ we get $Z = Z_1 t$, $Z_1, t \in \mathbb{N}$, $Z_1 \mid h(D)$ and $[X + Y\sqrt{D}] = [\alpha]^t$, where $\alpha \in \mathbb{Z}[\omega]$. This implies that

$$X + Y\sqrt{D} = \lambda\left(\frac{X_0 + Y_0\sqrt{D}}{2}\right)^t, \tag{11}$$

where λ is a unit in $\mathbb{Z}[\omega]$ with norm one and $X_0, Y_0 \in \mathbb{Z}$ satisfy

$$\begin{gathered} X_0^2 - DY_0^2 = 4k^{Z_1}\,, \qquad X_0 \equiv Y_0 \pmod 2\,,\\ \gcd(X_0, Y_0) = \begin{cases} 1 & \text{if } D \equiv 1 \pmod 4,\ 2 \nmid X_0,\\ 2 & \text{otherwise.}\end{cases}\end{gathered} \tag{12}$$

If $D \not\equiv 1 \pmod 4$, then from (11) and (12) we get

$$X + Y\sqrt{D} = (X_0' + Y_0'\sqrt{D})^t (u' + v'\sqrt{D})\,, \tag{13}$$

where (u', v') is a solution of (7) and $X_0', Y_0' \in \mathbb{Z}$ satisfy

$$X_0'^2 - DY_0'^2 = k^{Z_1}\,, \qquad \gcd(X_0', Y_0') = 1\,. \tag{14}$$

Since $|k| > 1$, there exists a unique solution (u'', v'') of (7) such that $X_1 \pm Y_1\sqrt{D} = (X_0' + Y_0'\sqrt{D})(u'' + v''\sqrt{D})$ satisfies $X_1, Y_1 \in \mathbb{N}$ and (10). We also get (9) from (13). By the same argument, we can prove the lemma in the case that $D \equiv 1 \pmod 4$ and $2 \mid X_0$.

Since $2 \nmid k$, we see from (12) that if $D \equiv 1 \pmod 4$ and $2 \nmid X_0$, then $D \not\equiv 1 \pmod 8$.

If $D \equiv 1 \pmod 4$, $2 \nmid X_0$ and $3 \mid t$, then from (12) we get

$$\left(\frac{X_0 + Y_0\sqrt{D}}{2}\right)^3 = X_1' + Y_1'\sqrt{D}\,,$$

where $X_1', Y_1' \in \mathbb{Z}$ satisfy

$$X_1'^2 - DY_1'^2 = k^{3Z_1}\,, \qquad \gcd(X_1', Y_1') = 1\,, \qquad 3Z_1 \mid 3h(D)\,.$$

Using the same method, we can prove the lemma in this case.

If $D \equiv 1 \pmod 4$, $2 \nmid X_0$ and $3 \nmid t$, then

$$\left(\frac{X_0 + Y_0\sqrt{D}}{2}\right)^t = \frac{X' + Y'\sqrt{D}}{2}, \tag{15}$$

where $X', Y' \in \mathbb{Z}$ with $2 \nmid X'Y'$. We see from (11) and (15) that $\lambda = (U + V\sqrt{D})/2$ where (U, V) is a solution of the equation

$$U^2 - DV^2 = 4, \quad \gcd(U, V) = 1. \tag{16}$$

For a suitable $\delta \in \{1, -1\}$, we have

$$\left(\frac{X_0 + Y_0\sqrt{D}}{2}\right)\left(\frac{U + \delta V\sqrt{D}}{2}\right) = X_0' + Y_0'\sqrt{D},$$

where $X_0', Y_0' \in \mathbb{Z}$ satisfy (14). Thus, the lemma also holds in this case. ∎

LEMMA 7. *$h(D) < \sqrt{D}(\log 4D + 2)/\log(u_1 + v_1\sqrt{D})$.*

Proof. This follows immediately from Theorem 12.10.1 and Theorem 12.13.3 of [5]. ∎

LEMMA 8 ([5, Theorem 12.13.4]). *$\log(u_1 + v_1\sqrt{D}) < \sqrt{D}(\log 4D + 2)$.* ∎

LEMMA 9 ([1, Theorem 2]). *Let $\alpha_1, \ldots, \alpha_r$ be algebraic numbers with heights $H_1, \ldots, H_r$ respectively, and let $A_i = \max(4, H_i)$ $(i = 1, \ldots, r)$. If $A_1 \le \ldots \le A_{r-1} \le A_r$ and $\Lambda = b_1 \log \alpha_1 + \ldots + b_r \log \alpha_r \ne 0$ for some $b_1, \ldots, b_r \in \mathbb{Z}$, then*

$$|\Lambda| > \exp\Big(-(16dr)^{200r}(\log B)\Big(\prod_{i=1}^{r} \log A_i\Big)\Big(\log \prod_{j=1}^{r-1} \log A_j\Big)\Big),$$

where d is the degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)$ and $B = \max(4, |b_1|, \ldots, |b_r|)$. ∎

LEMMA 10 ([10]). *Let $a \in \mathbb{Z}$ be non-zero, and let $f(y) \in \mathbb{Z}(y)$ have degree n and at least two simple zeros. If (x, y, m) is a solution of the equation*

$$ax^m = f(y), \quad x > 1,\ y > 0,\ m > 0,$$

then

$$m < \exp\left(\frac{C'n^5(\log 3H)^2}{\log(n \log 3H)}\right)(\log 3|a|)(\log\log 3|a|)^2,$$

where H is the height of $f(y)$ and C' is an effectively computable constant. ∎

3. The proofs. By [6], the only solutions of equation (1) with $2 \mid n$ are given by $(x, y, m, n) = (7, 20, 4, 2)$ and $(3, 11, 5, 2)$. By Theorem 5 of [8], we see that the theorem holds for $2 \mid m$. We now proceed to prove it for $2 \nmid mn$.

When $2 \nmid n$, n has an odd prime factor q. If (x, y, m, n) is a solution of (1), then $(x, y^{n/q}, m, q)$ is a solution with the same x^m. We can therefore assume that n is an odd prime.

Here and below, let C_i $(i = 1, 2, \ldots)$ denote some effectively computable absolute constants. We now prove the following conclusions.

Assertion 1. *Let* (x, y, m, n) *be a solution of equation* (1) *such that* $x \in \mathbb{P}$ *and* $y \equiv 1 \pmod{x}$. *If* $n > C_1$, *then*

$$x < n^{10/9}\,. \tag{17}$$

Proof. By the assumption,

$$x = p^r\,, \qquad p \text{ is a prime, } r \in \mathbb{N}\,, \tag{18}$$

where r satisfies

$$r \ge \begin{cases} 1 & \text{if } n \ne p, \\ 2 & \text{it } n = p, \end{cases} \tag{19}$$

since $y^p \equiv 1 \pmod{p^2}$ if $y \equiv 1 \pmod{p}$. From (1) we get

$$(1 - p^r)y^n = 1 - p^{rm}\,. \tag{20}$$

Let $\mathbb{Q}_p$, $\mathbb{Z}_p$ be the p-adic number field and the p-adic integer ring respectively. For any $\alpha \in \mathbb{Q}_p$, let $\upsilon(\alpha)$ denote the p-adic valuation of α, and let $\|\alpha\|_p = p^{-\upsilon(\alpha)}$. Since $y \equiv 1 \pmod{x}$, from (20) we get

$$(1 - p^r)^{1/n} y = \begin{cases} 1 + p^{rm}\theta & \text{if } n \ne p, \\ 1 + p^{rm-1}\theta & \text{if } n = p, \end{cases} \tag{21}$$

where $\theta \in \mathbb{Z}_p$. Let $t = m - 1$, and let $t_1, t_2 \in \mathbb{N}$ such that

$$t_1 + t_2 = m - 1\,, \qquad (m-1)/2 \ge t_1 \ge (m-3)/2\,. \tag{22}$$

Put $z = p^r$. By Lemma 1, from (21) we get

$$\begin{aligned} \left\| y \binom{t}{t_1} p^{rm} G(1) E(p^r) \right\|_p &= \left\| y \binom{t}{t_1} G(p^r) - (1-p^r)^{1/n} y \binom{t}{t_1} H(p^r) \right\|_p \\ &= \begin{cases} \left\| y \binom{t}{t_1} G(p^r) - \binom{t}{t_1} H(p^r) - p^{rm}\theta \binom{t}{t_1} H(p^r) \right\|_p & \text{if } n \ne p, \\ \left\| y \binom{t}{t_1} G(p^r) - \binom{t}{t_1} H(p^r) - p^{rm-1}\theta \binom{t}{t_1} H(p^r) \right\|_p & \text{if } n = p. \end{cases} \end{aligned} \tag{23}$$

By Lemma 4, we see from (3) and (4) that the power series expansions of $\binom{t}{t_1} G(n^2 z)$, $\binom{t}{t_1} H(n^2 z)$ and $(1 - n^2 z)^{1/n}$ in z have integer coefficients. Therefore, the power series of $\binom{t}{t_1} G(1) E(z)$ in z has rational coefficients with denominators being powers of n. Moreover, the denominator of the coefficient of z^i $(i \ge 0)$ does not exceed $n^{3i/2}$. This implies that if z satisfies (19) then the power series of $\binom{t}{t_1} G(1) E(z)$ converges in $\mathbb{Q}_p$ and

$$\left\| \binom{t}{t_1} G(1) E(z) \right\|_p \le 1\,.$$

On using (23), we get

$$\left\| y\binom{t}{t_1}G(p^r) - \binom{t}{t_1}H(p^r)\right\|_p \le p^{-rm+1}\,. \tag{24}$$

Let

$$N = n^{t_2+[t_2/(n-1)]}\left(y\binom{t}{t_1}G(p^r) - \binom{t}{t_1}H(p^r)\right).$$

Then $N \in \mathbb{Z}$ by Lemma 4. Further, by Lemma 2, there exists at least one pair (t_1,t_2) for which $N \neq 0$. It follows from (24) that

$$n^{t_2+[t_2/(n-1)]}\left(y\left|\binom{t}{t_1}G(x)\right| + \left|\binom{t}{t_1}H(x)\right|\right) \ge |N| \ge p^{rm-1} \ge x^{m-1}. \tag{25}$$

On applying Lemma 5 to (25), we get

$$2^{m-1}n^{\frac{m+3}{2}\cdot\frac{n}{n-1}}\left(\binom{\frac{m+3}{2}}{}x^{\frac{m}{n}+\frac{m-1}{2}} + x^{\frac{m+3}{2}}\right) \ge x^{m-1}\,. \tag{26}$$

If $n > C_1$, we deduce (17) from (26). ■

Assertion 2. *Let* (x,y,m,n) *be a solution of equation* (1) *which satisfies* (17). *If* $n > C_2$, *then* $\log y > n^{10}$.

P r o o f. From (1) we get

$$\begin{aligned} 0 < \Lambda &= m\log x - \log(x-1) - n\log y \\ &= \frac{2}{2x^m-1}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{1}{2x^m-1}\right)^{2i} < \frac{2}{x^m} < \frac{2}{y^n}\,. \end{aligned} \tag{27}$$

On the other hand, according to Lemma 9 we obtain

$$\begin{aligned} \Lambda > \exp(-48^{600}(\log x)(\log(x-1))(\log y)(\log\log x + \log\log(x-1)) \\ \times(\log\max(m,n)))\,. \end{aligned}$$

On combining this with (27) we get

$$1 + 2\cdot 48^{600}(\log x)(\log(x-1))(\log\log x)(\log\max(m,n)) > n\,. \tag{28}$$

Substituting (17) into (28) gives

$$1 + 2\cdot 48^{600}(\log n^{10/9})^2(\log\log n^{10/9})(\log\max(m,n)) > n\,. \tag{29}$$

If $m \le n$, then (29) is impossible for $n > C_2$. Hence $m > n$, and $\log m > n^{1/2}$ by (29). From (1), $y^n = x^{m-1} + \ldots + x + 1 > x^{m-1}$. Therefore we obtain $\log y > (m-1)\log x/n > n^{10}$. ■

Assertion 3. *If* (x,y,m,n) *is a solution of* (1) *with* x *being a square, then* (1) *has a solution* (x_1,y_1,m,n) *such that*

$$x = x_1^{2^r}\,,\quad r\in\mathbb{N}\,,\ x_1\in\mathbb{N}\,,\ x_1 \text{ is non-square.} \tag{30}$$

Proof. Since $x > 1$, there exists x_1 which satisfies (30). Since $2 \nmid m$, we have

$$\frac{x^m - 1}{x - 1} = \frac{x_1^m - 1}{x_1 - 1} \prod_{j=0}^{r-1} \frac{x_1^{2^j m} + 1}{x_1^{2^j} + 1},$$

where $(x_1^m - 1)/(x_1 - 1)$, $(x_1^{2^j m} + 1)/(x_1^{2^j} + 1)$ $(j = 0, \ldots, r-1)$ are coprime positive integers. The result follows at once. ∎

Assertion 4. *Let* (x, y, m, n) *be a solution of equation* (1) *which satisfies* (17). *If* x *is non-square*, *then* $n < C_3$.

Proof. Since x is non-square, we deduce from (18) and (1) that $2 \nmid r$ and

$$\left(\frac{x^{(m+1)/2} - 1}{x - 1}\right)^2 - p\left(p^{(r-1)/2}\frac{x^{(m-1)/2} - 1}{x - 1}\right)^2 = y^n.$$

This implies that $((x^{(m+1)/2} - 1)/(x - 1),\ p^{(r-1)/2}(x^{(m-1)/2} - 1)/(x - 1), n)$ is a solution of the equation

$$X^2 - pY^2 = y^Z, \quad \gcd(X, Y) = 1,\ Z > 0.$$

On applying Lemma 6, we have

(31) $$n = Z_1 t,$$

(32) $$\frac{x^{(m+1)/2} - 1}{x - 1} + \frac{x^{(m-1)/2} - 1}{x - 1}\sqrt{x} = (X_1 \pm Y_1\sqrt{p})^t(u + v\sqrt{p}),$$

where $t, X_1, Y_1, Z_1 \in \mathbb{N}$ satisfy

(33) $$X_1^2 - pY_1^2 = y^{Z_1}, \quad \gcd(X_1, Y_1) = 1,$$

(34) $$1 < \left|\frac{X_1 + Y_1\sqrt{p}}{X_1 - Y_1\sqrt{p}}\right| < (u_1 + v_1\sqrt{p})^2,$$

(35) $$3h(p) \equiv 0 \pmod{Z_1},$$

(u, v) is a solution of the equation

(36) $$u^2 - pv^2 = 1,$$

and $u_1 + v_1\sqrt{p}$ is the fundamental solution of (36). Recall that n is an odd prime. By Lemma 7, if x satisfies (17), then $n \nmid h(p)$. Hence $Z_1 = 1$ and $t = n$ by (31) and (35).

Let

(37) $$\varepsilon = X_1 + Y_1\sqrt{p}, \quad \overline{\varepsilon} = X_1 - Y_1\sqrt{p},$$

(38) $$\varrho = u_1 + v_1\sqrt{p}, \quad \overline{\varrho} = u_1 - v_1\sqrt{p},$$

$$A = \frac{x^{(m+1)/2} - 1}{x - 1}, \quad B = \frac{x^{(m-1)/2} - 1}{x - 1}.$$

Since

$$1 < \frac{A+B\sqrt{x}}{A-B\sqrt{x}} = \left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)\left(\frac{\sqrt{x^m}-1}{\sqrt{x^m}+1}\right) < \frac{\sqrt{x}+1}{\sqrt{x}-1}$$
$$\le \left\{\begin{matrix} u_1+v_1\sqrt{p} & \text{if } x \le 3 \\ 2.7 & \text{if } x > 3 \end{matrix}\right\} \le \varrho\,,$$

by (32) and (34), we get

$$A \pm B\sqrt{x} = \varepsilon^n \overline{\varrho}^s\,, \tag{39}$$

where $s \in \mathbb{Z}$ satisfies $0 \le s \le n$. Since $A = xB+1$, from (39) we get

$$0 < \Lambda = \left| n\log\frac{\varepsilon}{\overline{\varepsilon}} - 2s\log\varrho \mp \log\frac{\sqrt{x}+1}{\sqrt{x}-1}\right| \tag{40}$$
$$= \frac{2}{x^{m/2}}\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{1}{x^m}\right)^i < \frac{4}{x^{m/2}} < \frac{4}{y^{n/2}}\,.$$

Put $\alpha_1 = (\sqrt{x}+1)/(\sqrt{x}-1)$, $\alpha_2 = \varrho$, $\alpha_3 = \varepsilon/\overline{\varepsilon}$. Then by (33), (37) and (38), α_1,α_2 and α_3 satisfy $(x-1)\alpha_1^2 - 2(x+1)\alpha_1 + (x-1) = 0$, $\alpha_2^2 - 2u_1\alpha_2 + 1 = 0$ and $y\alpha_3^2 - 2(X_1^2+pY_1^2)\alpha_3 + y = 0$ respectively. This implies that $H_1 = 2(x+1)$, $H_2 = 2u_1$ and $H_3 = 2(X_1^2+pY_1^2) < 2(X_1+Y_1\sqrt{p})^2 < 2y\varrho^2$ by (34). Notice that the degree of $\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3) = \mathbb{Q}(\sqrt{p})$ is equal to 2. By Lemma 9,

$$\Lambda > \exp(-192^{600}(\log 2n)(\log 2(x+1))(\log 2u_1)(\log 2y\varrho^2)$$
$$\times(\log\log 2(x+1) + \log\log 2u_1))\,.$$

On combining this with (40) we get

$$1 + 192^{600}(\log 2n)(\log 2(x+1))(\log 2u_1)\left(1+\frac{\log 2\varrho^2}{\log y}\right) \tag{41}$$
$$\times(\log\log 2(x+1) + \log\log 2u_1) > \frac{n}{2}\,.$$

By Lemma 8, if x satisfies (17), then $\log 2u_1 < \log 2\varrho < \log 2 + n^{5/9}(\log 4n^{10/9} + 2)$. Hence, by Assertion 2, we have $\log 2\varrho^2 < \log y$ for $n > C_3$. Thus, by (41), we obtain

$$200^{600}(\log n)^4 > n^{4/9}\,.$$

This is impossible for $n > C_3$, which proves the assertion. ∎

Proof of Theorem. Let (x,y,m,n) be a solution of equation (1) such that $x \in \mathbb{P}$ and $y \equiv 1 \pmod{x}$. By Assertions 3 and 4, we obtain $n < C_3$. Further, by Assertion 1, $x < C_4$. Furthermore, by Lemma 10, $m < C_5$. To sum up, we get $x^m < C$. ∎

Acknowledgements. The author would like to thank the referee for his valuable suggestions.

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Received on 13.9.1991
and in revised form on 28.10.1992 (2177)