ACTA ARITHMETICA
LXIII.4 (1993)

Some remarks on the Erdős–Turán conjecture

by

Martin Helm (Mainz)

Notation. In additive number theory an increasing sequence of natural numbers is called an *asymptotic basis* of order h of $\mathbb{N}$ if every sufficiently large $n \in \mathbb{N}$ can be written as the sum of h elements of A.

Let $r_n(h, A)$ denote the number of representations of n as $n = a_1 + \ldots + a_h$ with $a_1, \ldots, a_h \in A$ and $a_1 \le \ldots \le a_h$.

If A satisfies $r_n(h, A) \le g\ \forall n \in \mathbb{N}$, where g is a natural constant, then A is called a $B_h[g]$-*sequence* (and in the special case $g = 1$ a B_h-*sequence*).

Furthermore, for any given sequence A of natural numbers and any $m \in \mathbb{N}$ we define

$$\delta_A(m) := |\{(a_i, a_j) : a_i, a_j \in A,\ m = a_j - a_i\}|,$$

and for a given $N \in \mathbb{N}$,

$$h_A(m) := |\{(a_i, a_j) : a_i, a_j \in A \cap [1, N^2],\ m = a_j - a_i\}|.$$

Introduction. A famous conjecture of Erdős and Turán [2] asserts that there exists no asymptotic basis of order 2 of $\mathbb{N}$ that is a $B_2[g]$-sequence at the same time. Erdős shows (see [4]) that if A is an arbitrary sequence of natural numbers satisfying $\liminf_{n\to\infty} A(n)/\sqrt{n} > 0$ and N is a given natural number then

$$H_A(N) := \sum_{m=1}^{N} h_A(m) \gg N \log N\,, \tag{1}$$

which proves the above hypothesis in the special case $g = 1$.

Almost all known results on $B_h[g]$-sequences are based on considerations concerning the representation of certain natural numbers as a difference of elements of a given sequence A.

Therefore for a further proof of the Erdős–Turán conjecture it is very interesting to decide whether Erdős' estimate (1) is sharp with respect to magnitude. Here we prove by means of an explicit construction that (1) is indeed sharp in the above sense.

Furthermore, (1) unfortunately does not render possible an estimation of $h_A(m)$ for any specific $m \in [1, N]$ but only provides some average information; in particular, it is not possible to decide whether any specific $m \in [1, N]$ for a given N satisfies e.g. $h_A(m) > c \log N$ or not, where c is a constant. Here we prove — again by means of an explicit construction — the existence of two increasing sequences of natural numbers B and M satisfying

$$\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} > 0, \qquad \liminf_{n\to\infty} \frac{M(n)}{\log n} > 0$$

and

$$\delta_B(m_j) \equiv 1 \qquad \forall j \ge j_0.$$

THEOREM. *There exists an infinite sequence of natural numbers A satisfying*

$$\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n}} > 0 \tag{2}$$

and

$$H_A(N) = \sum_{m=1}^{N} h_A(m) \ll N \log N\,. \tag{3}$$

Proof. We will prove the above theorem by constructing an infinite sequence A of natural numbers as an infinite countable union of finite B_2-sets (also called Sidon sets).

Let μ and ν be arbitrary natural numbers satisfying

$$\nu > 2 \quad \text{and} \quad \mu > 1.$$

Now a sequence $(n_j)_{j\in\mathbb{N}}$ is defined inductively as follows:

$$\begin{aligned} n_1 &:= 1, \\ n_{2j} &:= \mu n_{2j-1}\,, \qquad j \in \mathbb{N}\,, \\ n_{2j+1} &:= \nu n_{2j}\,, \qquad j \in \mathbb{N}\,. \end{aligned}$$

Therefore

$$\begin{aligned} n_{2j} &= \mu^j \nu^{j-1}\,, \qquad j \in \mathbb{N}\,, \\ n_{2j+1} &= \mu^j \nu^j\,, \qquad j \in \mathbb{N}_0\,. \end{aligned}$$

We define

$$I_k := \,]n_{k-1}, n_k[\qquad \forall k \ge 2\,.$$

A well-known result of Erdős and Chowla [1], [2] states that

$$\liminf_{n\to\infty} \frac{F_2(n)}{\sqrt{n}} \ge 1\,, \tag{4}$$

where $F_2(n)$ denotes the maximum number of elements that can be selected from the set $1, 2, \ldots, n$ to form a B_2-sequence. Since the "B_2-property" of a finite set is invariant under translations, for any $j \in \mathbb{N}$ (4) proves the existence of a Sidon set $S_{2j} \subseteq I_{2j} =]n_{2j-1}, n_{2j}[$ such that

$$|S_{2j}| \gg \sqrt{n_{2j} - n_{2j-1}} = \sqrt{\mu - 1}\sqrt{n_{2j-1}} \qquad (j \to \infty). \tag{5}$$

We define

$$A := \bigcup_{j=1}^{\infty} S_{2j}$$

and have to show that A satisfies the conditions (2) and (3).

Proof of (2). For any $m \in \mathbb{N}$ with $m > \mu$ there exists a $j_0 \in \mathbb{N}$ with

$$n_{2j_0} \le m < n_{2j_0+2}\,.$$

Therefore

$$\sqrt{m} < \sqrt{n_{2j_0+2}} \ll \sqrt{n_{2j_0-1}}\,,$$

and on the other hand,

$$A(m) \ge A(n_{2j_0}) - A(n_{2j_0-1}) = |S_{2j_0}| \gg \sqrt{n_{2j_0-1}}\,.$$

Thus $\liminf_{n\to\infty} A(n)/\sqrt{n} > 0$ and (2) holds.

Proof of (3). For a given $N \in \mathbb{N}$ with $N > \mu$ there exists a $j_1 \in \mathbb{N}$ such that

$$n_{2j_1-2} \le N < n_{2j_1}$$

and for any $m \in [1, N]$ we define

$$h_A^1(m) := |\{(a_i, a_k) : a_i, a_k \in [1, n_{2j_1}] \cap A \text{ and } m = a_k - a_i\}|,$$
$$h_A^2(m) := |\{(a_i, a_k) : a_i \in [1, N^2] \cap A, a_k \in]n_{2j_1}, N^2] \text{ and } m = a_k - a_i\}|,$$
$$H_A^1(N) := \sum_{m=1}^{N} h_A^1(m), \qquad H_A^2(N) := \sum_{m=1}^{N} h_A^2(m).$$

Consequently,

$$h_A(m) = h_A^1(m) + h_A^2(m), \qquad H_A(N) = H_A^1(N) + H_A^2(N).$$

Estimation of $H_A^1(N)$. Obviously

$$H_A^1(N) < (A(n_{2j_1}))^2 \ll n_{2j_1} \ll N\,.$$

Estimation of $H_A^2(N)$. Since $\nu > 2$, for any $j \ge j_1$ the length of the gap between two consecutive Sidon sets S_{2j+2} and S_{2j} is bigger than $n_{2j_1+1} - n_{2j_1} > n_{2j_1} > N$.

Therefore a number $m \in [1, N]$ can be represented as a difference of two elements a_i, a_k of A with $a_k > n_{2j_1}$ only if a_k and a_i are elements of the same Sidon subset of A.

Let Θ_{N^2} be the number of Sidon subsets S_{2j} of A satisfying

$$S_{2j} \cap [1, N^2] \neq \emptyset .$$

Then the B_2-property of all Sidon subsets of A leads to

$$h_A^2(m) \leq \Theta_{N^2} \qquad \forall m \in [1, N]$$

and consequently,

$$H_A^2(N) = \sum_{m=1}^{N} h_A^2(m) \leq N\Theta_{N^2} . \tag{6}$$

Estimation of Θ_{N^2}. For given $N \in \mathbb{N}$ with $N > \mu$, there exists a $j_2 \in \mathbb{N}$ so that

$$n_{2j_2-2} \leq N^2 < n_{2j_2} \;\Rightarrow\; \mu^{j_2-1}\nu^{j_2-2} \leq N^2 \;\Rightarrow\; j_2 \ll \log N$$

and as $\Theta_{N^2} \leq j_2$

$$\Rightarrow\; \Theta_{N^2} \ll \log N . \tag{7}$$

Thus (6) and (7) imply $H_A^2(N) \ll N \log N$ and

$$H_A(N) = H_A^1(N) + H_A^2(N) \ll N \log N ,$$

which completes the proof.

Corollary. *There exist two infinite increasing sequences B and M of natural numbers satisfying*

$$\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} > 0 , \tag{8}$$

$$\liminf_{n\to\infty} \frac{M(n)}{\log n} > 0 \tag{9}$$

and

$$\delta_B(m) \equiv 1 \qquad \forall m \in M . \tag{10}$$

Proof. Let A be the infinite sequence of natural numbers generated by the construction of the above theorem in the special case $\mu = 7/4$ and $\nu = 4$ (where the inductive definition of the sequence $(n_j)_{j\in\mathbb{N}}$ is supplemented by the definition $n_2 := 2$ which does not restrict at all the applicability of the proof).

Consequently, in this special case we define

$$\begin{aligned} n_1 &:= 1, \quad & n_{2j+1} &:= \nu n_{2j}, \quad & j &\in \mathbb{N}, \\ n_2 &:= 2, \quad & n_{2j} &:= \mu n_{2j-1}, \quad & j &\geq 2 . \end{aligned}$$

Thus

$$n_{2j} = 2 \cdot 7^{j-1}, \qquad n_{2j+1} = 8 \cdot 7^{j-1}, \qquad j \in \mathbb{N} .$$

For any $j \in \mathbb{N}$ we define

$$D_{2j} := \{m \in \mathbb{N} : \exists a_i, a_j \in S_{2j} \text{ and } m = a_j - a_i\}\,.$$

Since S_{2j} is a Sidon set and

$$\lim_{j\to\infty} \frac{|S_{2j}|}{\sqrt{n_{2j} - n_{2j-1}}} = 1$$

there exists a $j_0 \in \mathbb{N}$ so that

$$|D_{2j}| = \binom{|S_{2j}|}{2} > n_{2j-2} \qquad \forall j \geq j_0\,.$$

Since, on the other hand,

$$m \in D_{2j} \;\Rightarrow\; \quad m < n_{2j} - n_{2j-1} = 3n_{2j-2} \qquad \forall j \in \mathbb{N}\,,$$

for any $j \geq j_0$ there exists at least one $m_j \in D_{2j}$ satisfying

$$n_{2j-2} < m_j < 3n_{2j-2}\,.$$

Let M be defined as the sequence $m_{j_0}, m_{j_0+1}, m_{j_0+2}, \ldots$ Now we will construct a subsequence B of A by eliminating a negligible number of elements of A so that B will still satisfy

$$\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} > 0$$

and $\delta_B(m_j) = 1$ will hold for all $j \geq j_0$.

Since, according to the definition,

$$n_{2j-2} < m_j \qquad \forall j \geq j_0,$$

we have

$$m_j = a_k - a_i \;\Rightarrow\; a_k > n_{2j-2}\,.$$

Therefore, since $A \cap]n_{2j-2}, n_{2j-1}[= \emptyset$, a_k satisfies

$$a_k > n_{2j-1} \quad \text{and} \quad a_k \notin \bigcup_{h=1}^{j-1} S_{2h}\,. \tag{11}$$

On the other hand, for $h \geq j$ the length of the gap between two consecutive Sidon subsets S_{2h-2}, S_{2h} is bigger than $n_{2j-1} - n_{2j-2} = 3n_{2j-2}$. Therefore as according to the definition $m_j < 3n_{2j-2}$ for any $j \geq j_0$, m_j can occur as a difference $a_k - a_i$, $a_i, a_k \in A$, only if both a_k and a_i are elements of the same Sidon subset S_{2h}, $h \geq j$.

Therefore the B_2-property of the sets S_{2j} implies that

$$|D_{2j} \cap M| \leq j \qquad \forall j \in \mathbb{N}\,.$$

Thus $\forall j \geq j_0$ it is possible to construct a new Sidon set S'_{2j} from S_{2j} by eliminating less than j elements of S_{2j} so that

$$D'_{2j} \cap M = \{m_j\}\,,$$

where

$$D'_{2j} := \{m \in \mathbb{N} : \exists (a_i, a_k) \in S'_{2j},\ m = a_k - a_i\}\,.$$

We define

$$B := \bigcup_{j=1}^{\infty} S'_{2j}, \qquad \text{where } S'_{2j} := S_{2j}, \qquad 1 \le j < j_0\,.$$

Obviously $\delta_B(m_j) = 1\ \forall j \ge j_0$ and

$$|S'_{2j}| \gg |S_{2j}| \ \Rightarrow \liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} > 0\,.$$

Furthermore, $\Theta_n \gg \log n$ $(n \to \infty)$, where Θ_n is the number of Sidon subsets S'_{2j} of B satisfying $S'_{2j} \cap [1,n] \ne \emptyset$. Consequently

$$\liminf_{n\to\infty} \frac{M(n)}{\log n} > 0\,,$$

which completes the proof.

References

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FACHBEREICH MATHEMATIK
JOHANNES GUTENBERG-UNIVERSITÄT MAINZ
SAARSTR. 21
D-6500 MAINZ, GERMANY

Received on 4.8.1992
and in revised form on 24.9.1992 (2289)