ACTA ARITHMETICA
LXIII.4 (1993)

On B_{2k}-sequences

by

Martin Helm (Mainz)

Introduction. An old conjecture of P. Erdős repeated many times with a prize offer states that the counting function $A(n)$ of a B_r-sequence A satisfies

$$\liminf_{n\to\infty} \frac{A(n)}{n^{1/r}} = 0\,.$$

The conjecture was proved for $r = 2$ by P. Erdős himself (see [5]) and in the cases $r = 4$ and $r = 6$ by J. C. M. Nash in [4] and by Xing-De Jia in [2] respectively. A very interesting proof of the conjecture in the case of all even $r = 2k$ by Xing-De Jia is to appear in the Journal of Number Theory [3].

Here we present a different, very short proof of Erdős' hypothesis for all even $r = 2k$ which we developped independently of Jia's version.

Notation and terminology. We call a set of positive integers A a B_r-*sequence* if the equation $n = a_1 + \ldots + a_r$, $a_1 \le \ldots \le a_r$, $a_i \in A$, has at most one solution for all n.

We define

$$\begin{aligned}
&B = kA = \{a_1 + \ldots + a_k : a_i \in A\},\\
&S = \{(a_1, \ldots, a_k; a'_1, \ldots, a'_k) : a_i, a'_i \in A \cap [1, N^2]\,,\\
&\qquad\qquad 1 \le (a_1 + \ldots + a_k) - (a'_1 + \ldots + a'_k) \le N\}\,,\\
&S' = \{(b_i, b_j) : 1 \le b_j - b_i \le N,\ b_i, b_j \in B \cap [1, N^2]\}\,.
\end{aligned}$$

Theorem. *Let A be a B_{2k}-sequence such that*

$$A(n^2) \ll (A(n))^2\,.$$

Then

$$\frac{A(n)}{n^{1/(2k)}}(\log n)^{1/(2k)} < \infty\,. \tag{1}$$

P r o o f. Erdős showed (see [5]) that every B_2-sequence A satisfies

$$\frac{A(n)}{n^{1/2}}(\log n)^{1/2} < \infty\,. \tag{2}$$

Using an idea of Erdős on which the proof of (2) is based (see [1, pp. 89–90]) in this case we get

$$|S'| \gg \tau_B(N)^2 N$$

where

$$\tau_B(N) = \inf_{n>N} \frac{B(n)}{n^{1/2}}(\log n)^{1/2}\,.$$

Since

$$|S'| \le |S| \tag{3}$$

and as the B_{2k}-property of A implies

$$B(n) \gg (A(n))^k\,, \tag{4}$$

the proof of

$$|S| \ll N \tag{5}$$

will lead to $\tau_B(N) \ll 1$, which implies (1) immediately.

It remains to prove (5). Consider an arbitrary $2k$-tuple $(a_1,\ldots,a_k; a'_1,\ldots,a'_k)$ of S. It will be transformed into a new tuple according to the following procedure. Let u be the number of appearances of a_1 in $(a_1,\ldots,a_k)$ and let v be the number of appearances of a_1 in $(a'_1,\ldots,a'_k)$. Now a_1 will be eliminated $\min(u,v)$ times from $(a_1,\ldots,a_k)$ as well as from $(a'_1,\ldots,a'_k)$. In the next step the same procedure will be performed with the next component of $(a_1,\ldots,a_k)$ that is different from a_1, and so on till every component of $(a_1,\ldots,a_k)$ has been checked once. Eventually, the $2k$-tuple $(a_1,\ldots,a_k; a'_1,\ldots,a'_k)$ is transformed into a new $2j$-tuple $(a_{i1},\ldots,a_{ij}; a'_{h1},\ldots,a'_{hj})$ where j is the number of components of $(a_1,\ldots,a_k)$ and $(a'_1,\ldots,a'_k)$ that have not been dropped as above. Thus

$$\{a_{i1},\ldots,a_{ij}\} \cap \{a'_{h1},\ldots,a'_{hj}\} = \emptyset$$

for $1 \le j \le k$ as

$$(a_1+\ldots+a_k) - (a'_1+\ldots+a'_k) > 0 \qquad \forall (a_1,\ldots,a_k; a'_1,\ldots,a'_k) \in S\,.$$

Therefore it is possible to divide S into k disjoint classes $S_1,\ldots,S_k$, where S_j is the set of those $2k$-tuples of S whose corresponding tuple according to the above procedure of successive "truncation" consists of $2j$ components. Therefore

$$|S| = \sum_{j=1}^{k} |S_j|\,.$$

Since A is a B_{2k}-sequence,

$$|S_k| \ll N\,.$$

For if $(a_1,\dots,a_k;a'_1,\dots,a'_k)$ and $(b_1,\dots,b_k;b'_1,\dots,b'_k)$ belong to S_k and

$$(a_1+\dots+a_k)-(a'_1+\dots+a'_k)=(b_1+\dots+b_k)-(b'_1+\dots+b'_k)$$

then the B_{2k}-property of A in view of

$$\{a_1,\dots,a_k\}\cap\{a'_1,\dots,a'_k\}=\emptyset$$

and

$$\{b_1,\dots,b_k\}\cap\{b'_1,\dots,b'_k\}=\emptyset$$

implies that the numbers $(b_1,\dots,b_k)$ form a permutation of $(a_1,\dots,a_k)$ and also the numbers $(b'_1,\dots,b'_k)$ form a permutation of $(a'_1,\dots,a'_k)$.

For $j=1,\dots,k-1$ we define

$$\begin{aligned}\widehat{S}_j := \{(a_1,\dots,a_j;a'_1,\dots,a'_j) : a_i,a'_i\in A\cap[1,N^2]\,,\\ 1\le (a_1+\dots+a_j)-(a'_1+\dots+a'_j)\le N\,,\\ \{a_1,\dots,a_j\}\cap\{a'_1,\dots,a'_j\}=\emptyset\}\,.\end{aligned}$$

Since for every $(a_1,\dots,a_k;a'_1,\dots,a'_k)\in S_j$ the difference

$$(a_1+\dots+a_k)-(a'_1+\dots+a'_k)$$

may be written in the form

$$(a_{i1}-a'_{h1})+\dots+(a_{ij}-a'_{hj})+(a_{i,j+1}-a_{i,j+1})+\dots+(a_{ik}-a_{ik})$$

with

$$\{a_{i1},\dots,a_{ij}\}\cap\{a_{h1},\dots,a_{hj}\}=\emptyset\,,$$

we have

$$|S_j|\ll|\widehat{S}_j|(A(N^2))^{k-j}. \tag{6}$$

For every $(a_1,\dots,a_j;a'_1,\dots,a'_j)\in\widehat{S}_j$ let t be the number of different subsets of $\{A\cap[1,N]\}\setminus\{\{a_1,\dots,a_j\}\cup\{a'_1,\dots,a'_j\}\}$ consisting of $2(k-j)$ different elements. An easy combinatorial argument shows that

$$t\gg(A(N))^{2(k-j)}\,.$$

Thus there are $t\gg(A(N))^{2(k-j)}$ ways of transforming an element of $\widehat{S}_j$ into a tuple of S'_k where

$$\begin{aligned}S'_k := \{(a_1,\dots,a_k;a'_1,\dots,a'_k) : a_i,a'_i\in A\cap[1,N^2]\,,\\ 1\le (a_1+\dots+a_k)-(a'_1+\dots+a'_k)\le kN,\\ \{a_1,\dots,a_k\}\cap\{a'_1,\dots,a'_k\}=\emptyset\}\,.\end{aligned}$$

Obviously, since A is a B_{2k}-sequence,

$$|S'_k|\ll N\,.$$

In the course of this procedure for every $(a_1,\dots,a_k;a'_1,\dots,a'_k)\in S_j$ every $(a_1,\dots,a_k;a'_1,\dots,a'_k)\in S'_k$ can appear at most $\binom{k}{j}\binom{k}{j}$ times. Therefore

$$|\widehat{S}_j|(A(N))^{2(k-j)}\ll N\,.$$

Thus (6) and the assumption $(A(N))^2\gg A(N^2)$ imply

$$|\widehat{S}_j|(A(N^2))^{k-j}\ll N,\qquad j=1,\dots,k-1\,,$$

and therefore

$$|S_j|\ll N,\qquad j=1,\dots,k\,.$$

This implies (5) and thus the proof is complete.

COROLLARY. *Every B_{2k}-sequence A satisfies*

$$\liminf_{n\to\infty}\frac{A(n)}{n^{1/(2k)}}=0. \tag{7}$$

Proof. It is easy to see that every B_{2k}-sequence A satisfies $A(n)\ll n^{1/(2k)}$. Therefore assuming that there exists a B_{2k}-sequence A satisfying

$$\liminf_{n\to\infty}\frac{A(n)}{n^{1/(2k)}}>0 \tag{8}$$

A also satisfies $A(n^2)\ll(A(n))^2$. But then, as a consequence of the above theorem, (1) holds, which contradicts (8).

Remark. In the special case $r=4$ the more precise estimate for $\widehat{S}_1$,

$$|\widehat{S}_1|\sum_{l=1}^{N}A_l^2\ll N$$

with

$$A_l=|A\cap[(l-1)N,lN]|$$

shows that here the assumption $A(N^2)\ll(A(N))^2$ is not necessary. This result was already achieved by Nash.

The above theorem also holds for B_{2k}-sequences satisfying only the weaker condition $A(n^2)\le\Lambda(A(n))^2$ for infinitely many n where Λ is any positive constant.

Acknowledgments. I am very thankful to Prof. Gerd Hofmeister for steady encouragement and a lot of helpful comments.

References

[1] H. Halberstam and K. F. Roth, *Sequences*, Springer, New York 1983.
[2] X.-D. Jia, *On B_6-sequences*, J. Qufu Norm. Univ. Nat. Sci. 15 (3) (1989), 7–11.
[3] —, *On B_{2k}-sequences*, J. Number Theory, to appear.
[4] J. C. M. Nash, *On B_4-sequences*, Canad. Math. Bull. 32 (1989), 446–449.

[5] A. Stöhr, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe. II*, J. Reine Angew. Math. 194 (1955), 111–140.

FACHBEREICH MATHEMATIK
JOHANNES GUTENBERG-UNIVERSITÄT MAINZ
SAARSTR. 21
D-6500 MAINZ, GERMANY

Received on 10.7.1992
and in revised form on 24.9.1992 (2278)