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On the zeros of $\zeta(s) - a$

by

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1. Introduction. The main object of this paper is to prove the following theorem. (We write $s = \sigma + it$ as usual.)

THEOREM 1. *Let a be any non-zero complex constant. Let δ and μ be any two constants satisfying $0 < \delta \le 1/10$ and $0 < \mu \le 1/10$. Then for $T \ge T_0(\delta, \mu, a)$ (depending only on the constants indicated) there are at least $\ge CT^\mu$ distinct zeros of $\zeta(s) - a$ in the rectangle $(\sigma \ge 1-\delta,\ T \le t \le T + T^\mu)$ where $C\ (>0)$ is independent of T.*

Remark 1. As a complement to this theorem we can prove (by using some ideas of J. E. Littlewood) that the number of zeros (counted with multiplicity) of $\zeta(s) - a$ in $(\sigma \ge 1-\delta,\ T \le t \le T + T^\mu)$ is $O(T^\mu)$ for a certain constant $\delta = \delta(a, \mu) > 0$. (Thus there are $\gg T^\mu$ zeros of a fixed bounded (bound independent of T) order which depends on μ and a. The order, however, may depend on the rectangle.)

In fact, under fairly general conditions on a generalised Dirichlet series one of which being $\int_T^{T+T^\mu} |F(1-\delta_0+it)|^2 dt = O(T^\mu)$ (where $\delta_0 > 0$ is a suitable constant) there are at most $O(T^\mu)$ zeros (counted with multiplicity) of $F(s)$ in $(\sigma \ge 1-\delta,\ T \le t \le T + T^\mu)$ for every constant δ $(0 < \delta < \delta_0)$.

Remark 2. As can easily be seen, the theorem is equivalent to the one with $\mu = \delta$. But we have stated it in this way since we feel that it is possible to prove a uniform result in a certain range for δ with $\mu = \delta^{3/2-\varepsilon}$ for any fixed $\varepsilon > 0$. Note that Theorem 1 deals with *any non-zero complex constant* a, while in [3] we dealt with zeros of $\zeta'(s) - a$ *for any complex constant* a.

Remark 3. H. Bohr and B. Jessen have proved the remarkable result that the number of zeros (counted with multiplicity) of $\log \zeta(s) - a$ (with any complex constant a) in $(1/2 < \alpha < \sigma < \beta < 1,\ 0 \le t \le T)$ is $\sim K(a, \alpha, \beta)T$ as $T \to \infty$ for any two fixed constants α, β (see pp. 306–308 of [5]; a correction on p. 308: Jensen should read Jessen). But our Theorem 1 gives a new information which may be of some interest.

Remark 4. Our proof is sufficiently general and goes through for ζ and L-functions and ζ-function of any ray class in any algebraic number field. Actually in the last section we formulate a theorem which we can further generalise to some extent. However, if we are dealing with functions $f(s)$ like the zeta-function of a ray class where we do not have an Euler product we can only prove that $f(s)(f(s)-a)$ has $\gg T^\mu$ distinct zeros in the rectangle $(\sigma \geq 1-\delta,\ T \leq t \leq T+T^\mu)$. (The notation $\gg T^\mu$ means $\geq CT^\mu$ where C (>0) is independent of T.) In fact, if $f(s)$ has an Euler product we first prove that $f(s)(f(s)-a)$ has $\gg T^\mu$ distinct zeros and we recover that $f(s)-a$ has $\gg T^\mu$ zeros since by density results $f(s)$ has a smaller number of zeros for a suitable δ.

2. Some preparations. Throughout this paper we consider the function $F(s)=\sum_{n=1}^\infty a_n n^{-s}$ with the following two conditions.

(i) Let $a_1, a_2, \ldots$ be a sequence of complex numbers with n_0 the least integer for which $a_{n_0} \neq 0$ and n_1 the next least integer for which $a_{n_1} \neq 0$. Let $\sum_{n\leq x} |a_n|^2 \ll x^{1+\varepsilon}$ for every $\varepsilon>0$ and all $x \geq 1$.

(ii) Suppose $F(s)$ can be continued analytically in $(\sigma \geq 1-\eta,\ T-1 \leq t \leq T+T^\mu+1)$ for some fixed η $(0<\eta<1/(10A))$ and there $\max |F(s)| < T^A$ where A (≥ 1) is any positive constant.

We begin our preparations with

Lemma 1. *For some constant η' (with $0<\eta'<\eta/2$) we have, for all $\sigma \geq 1-\eta'$,*

$$\int_{T_1}^{T_2} |F(\sigma+it)|^2 dt = O(T^\mu),$$

where $T_1 = T+(\log T)^2$ and $T_2 = T+T^\mu-(\log T)^2$.

Remark. This lemma as well as the lemmas of this section go through for all functions of the form $F(s)=\sum_{n=1}^\infty a_n \lambda_n^{-s}$ where $1=\lambda_1<\lambda_2<\ldots$ is any sequence of real numbers with $C_1^{-1} \leq \lambda_{n+1}-\lambda_n \leq C_1$ where $C_1 \geq 1$ is any constant. Of course we have to assume (i) and (ii).

Proof. The proof follows from standard arguments. For example let t be in the range of integration. We start with

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s+w)X^w \Gamma(w)\,dw = \sum_{n=1}^\infty a_n n^{-s}\exp(-n/X) \qquad (X = T^{A\eta^{-2}})$$

and deform the line of integration to the w-contour obtained by joining the points $2-i\infty, 2-i(\log\ T)^2, 1-\eta-\sigma-i(\log T)^2, 1-\eta-\sigma+i(\log T)^2, 2+i(\log T)^2, 2+i\infty$ (by straight line segments) in this order. The pole at $w=0$

contributes $F(s)$. Rough estimations show that

$$F(s)=\sum_{n=1}^{\infty}a_n n^{-s}\exp(-n/X)+O(T^{-1})$$
$$=\sum_{n\le X^2}a_n n^{-s}\exp(-n/X)+O(T^{-1})\,.$$

To estimate the mean square of the last finite sum we use (for $m\neq n$)

$$\left|\frac{a_m\overline{a}_n\exp(-(m+n)/X)}{(mn)^\sigma\log(m/n)}\right|\le\frac{2|a_m\overline{a}_n\exp(-(m+n)/X)|}{(mn)^\sigma}$$

if $|\log(m/n)|\ge 1/2$. Otherwise we use $|\log(m/n)|\ge|(m-n)/(m+n)|$ and obtain the lemma with slight work.

LEMMA 2. *Consider the rectangle* $(\sigma\ge 1-\eta'/2,\ T+2(\log T)^2\le t\le T+T^\mu-2(\log T)^2)$. *Divide the t-range into abutting t-intervals I each of length H* (≥ 10) (*ignoring a bit at one end*). *Put* $M(I)=$ *maximum of* $|F(s)|$ *in* $(\sigma\ge 1-\eta'/2,\ t\in I)$. *Then*

$$\sum_I M(I)=O(T^\mu)\,.$$

Proof. Let $r=\eta'/2$ and $0<r_1<r$ and $z=x+iy$ a complex variable with $|z|\le r$. Then by Cauchy's theorem we have

$$(F(s))^2=\frac{1}{2\pi i}\int_{|z|=r_1}(F(s+z))^2\frac{dz}{z}$$

and so

$$|F(s)|^2\le\frac{1}{\pi r^2}\int\int_{|z|\le r}|F(s+z)|^2\,dx\,dy\,.$$

Note that $|F(s)|$ is bounded in $\sigma\ge 2$. Let now s run through points of $(1-\eta'/2\le\sigma\le 2,\ t\in I)$ where $\max|F(s)|$ is attained. Then we have

$$\sum_s M(I)\le\frac{2}{\pi r^2}\int\int|F(s)|^2\,dx\,dy\,,$$

the integral being taken over $(1-\eta'\le\sigma\le 2+\eta',\ T_1\le t\le T_2)$. By Lemma 1 this leads to Lemma 2.

LEMMA 3. *For at least* $\ge T^\mu H^{-1}(1+O(H^{-1}))$ *intervals I, we have* $M(I)\le H^2$.

Proof. By Lemma 2 the number of intervals I with $M(I)>H^2$ is $O(T^\mu H^{-2})$ and this proves the lemma.

LEMMA 4. *Let* $t_0\ge 100$, *let* $\delta,\ \delta',\ \delta''$ *be constants with* $\delta>\delta'>\delta''>0$ *and let* $D(s)$ *be any function analytic in* $(\sigma\ge 1-\delta,\ |t-t_0|\le C(\delta))$

where $C(\delta)$ is a large positive constant depending on δ, δ' and δ'' and D_0 to follow. In this region let the maximum of $|D(s)|$ be $\le M$ (≥ 30) and also $D(s) \ne 0$. Suppose further that for all σ exceeding a large positive constant D_0 we have $|\log D(s)| \le 1/2$. Then $\log D(s) = O(\log M)$ in $(\sigma \ge 1-\delta',\ |t-t_0| \le C(\delta)/2)$ and $\log D(s) = O(\log M)^\theta$ with a θ (<1) not depending on t_0 in $(\sigma \ge 1-\delta'',\ |t-t_0| \le C(\sigma)/3)$. Here the O-constants depend only on δ, δ', δ'' and D_0.

Remark. This lemma is the same as Lemma 1 of [3] with a slight change of notation.

Proof. This lemma is essentially due to J. E. Littlewood. See pages 336 and 337 of [5] for a proof which can be easily generalised to give this lemma.

Let a be any non-zero constant. Hereafter we put $F_1(s) = a_1^{-1}F(s)$ or $1 - a^{-1}F(s)$ according as $a_1 \ne 0$ or $a_1 = 0$. In any case $F_1(s)$ is a Dirichlet series of the type $\sum_{n=1}^{\infty} a'_n \lambda_n^{-s}$ with $a'_1 = \lambda_1 = 1$ (described in the remark below Lemma 1). We treat only the first case, i.e. $F_1(s)(F_1(s) - a_1^{-1}a)$ (hereafter we write a in place of $a_1^{-1}a$ in this case). In the second case we have to consider $F_1(s)(F_1(s)-1)$ and the treatment is exactly similar and we do not give details of proof in this case.

Lemma 5. *Consider the intervals I of Lemma 3. Then there exists a constant δ_1 (with $0 < \delta_1 < \delta$) with the following property. In order to prove that the number of distinct zeros of $F_1(s)(F_1(s)-a)$ in $(\sigma \ge 1-\delta,\ T \le t \le T + T^\mu)$ is $\gg T^\mu$, we can assume that there are at least $N \ge \frac{1}{4}T^\mu H^{-1}$ intervals I such that in $(\sigma \ge 1-\delta_1,\ t \in I)$ we have $F(s) = O(H^2)$ and also $F_1(s)(F_1(s)-a) \ne 0$. (We denote these intervals by J.)*

Proof. If at least $\ge \frac{1}{2}T^\mu H^{-1}$ of the intervals I of Lemma 3 have the property that $(\sigma \ge 1-\delta,\ t \in I)$ contains a zero of $F_1(s)(F_1(a)-a)$ then we are through by fixing H to be a large constant. Hence we may assume that the number of intervals I of Lemma 3 with the property that the rectangle $(\sigma \ge 1-\delta,\ t \in I)$ contains at least one zero is $\le \frac{1}{2}T^\mu H^{-1}$. The remaining intervals of Lemma 3 are $\ge \frac{1}{4}T^\mu H^{-1}$ in number and this proves the lemma.

Lemma 6. *Let J^* denote the interval J with t-intervals of length $1000(\log H)^2$ deleted from both ends. Then in $(\sigma \ge 1-\delta_1/1000,\ t \in J^*)$ we have $F(s) = O(\exp\exp((\log\log H)^\theta))$, where θ $(0 < \theta < 1)$ is independent of H and T.*

Proof. Let J_k $(k = 1, 2, \ldots, 5)$ denote the interval J with t-intervals of length $2k(\log H)^2$ deleted from both ends. We apply Lemma 4 to $F_1(s)$ and the rectangle $(\sigma \ge 1-\delta_1/2,\ t \in J_2)$. We see that in this rectangle $\log F_1(s) = O(\log H)$. Let $P(s) = (\log F_1(s) - \log a)(-\log a)^{-1}$ if $a \ne 1$

and otherwise $P(s) = g_1^s h_1 \log F_1(s)$ where g_1 and h_1 are suitable constants ($g_1 > 1$ and h_1 a non-zero complex constant) which secure the property that $P(s) \to 1$ as $\sigma \to \infty$. Now since $F_1(s) \neq a$ we have $P(s) \neq 0$ in $(\sigma \geq 1 - \delta_1/3,\ t \in J_3)$. So we can apply Lemma 4 and conclude that in $(\sigma \geq 1 - \delta_1/4,\ t \in J_4)$ we have $\log P(s) = O(\log\log H)$ and that in $(\sigma \geq 1 - \delta_1/5,\ t \in J_5)$ we have $|\log P(s)| \leq (\log\log H)^\theta$ for all large H. This leads to the lemma.

3. Titchmarsh series. In this section we impose some conditions on $F(s)$ and prove that for every one of the intervals J^* the maximum $m(J^*)$ of $|F(s)|$ taken over $(\sigma \geq 1 - \delta_1/1000,\ t \in J^*)$ exceeds $\exp((\log H)^\alpha)$ where α (> 0) is a constant independent of T and H. Plainly it suffices to prove this result for $F(s) + 1$ and so if $a_1 = 0$ we can consider $F(s) + 1$ and otherwise $a_1^{-1} F(s)$. Hence for this new function $a_1 = 1$, and we can apply the results of [4]. Put $(F(s))^k = \sum_{n=1}^\infty b_n n^{-s}$ where k is an integer satisfying $1 \leq k \leq \log H$. We impose some extra conditions on $F(s)$ so as to secure that the quantity Q defined by

$$Q = \max_{1 \leq k \leq \log H} \max_{\sigma \geq 1 - \delta_1/1000} \left(\frac{1}{|J^*|} \int_{t \in J^*} |F(s)|^{2k}\, dt \right)^{1/(2k)}$$

exceeds $\exp((\log H)^\alpha)$ where α is as required by us. According to the main result of [4] we have the following theorem.

THEOREM 2. *A lower bound for Q is given by*

$$Q \geq \left(C_2 \sum_{n \leq C_3 H} |b_n|^2 n^{-2\beta} \left(1 - \frac{\log n}{\log H} + \frac{1}{\log\log H} \right) \right)^{1/(2k)} \tag{1}$$

where C_2 (> 0), C_3 (> 0) are certain constants and $\beta = 1 - \delta_1/1000$.

Remark 1. Since we are going to apply (1) with k a positive constant power of $\log H$, it suffices to prove the lower bound

$$Q_1 = \Big(\sum_{n \leq C_3 H} |b_n|^2 n^{-2\beta} \Big)^{1/(2k)} > \exp((\log H)^\alpha)\,.$$

Incidentally we remark that the conjecture that in (1),

$$1 - \frac{\log n}{\log H} + \frac{1}{\log\log H}$$

can be replaced by 1 (made in [4]) is solved in fact in a stronger form in [2] by a simpler method.

Let $F(s) = P_\chi(s) + E(s)$ where $P_\chi(s) = \sum^* \chi(n) a_n n^{-s}$ where the asterisk indicates that n runs over a semigroup (with identity) generated by a set S of primes and χ is a complex-valued (restricted) multiplicative function and further the a_n are all real and non-negative. We suppose that $E(s) = \sum^* b'_n n^{-s}$ where b'_n are arbitrary complex numbers and the asterisk indicates that n runs through integers which have at least one prime factor not in S. Then

$$(F(s))^k = (P_\chi(s) + E(s))^k = (P_\chi(s))^k + Q(s)\,,$$

where $Q(s)$ is a Dirichlet series "with integers n not in $(P_\chi(s))^k$". Let $Q_\chi(s) = \sum' \chi(m) a_m m^{-s}$ where the accent indicates that m's run over square-free power products (times a fixed integer ≥ 1) of primes in S. Then we impose only the conditions

(iii) $|a_m| \gg m^{-\varepsilon}$ for all m,
(iv) $|\chi(m)| \gg m^{-\varepsilon}$ for all large m,
(v) $\sum_{p\in S,\, Y\leq p\leq 2Y} 1 \gg Y^{1-\varepsilon}$ for all large Y,

all valid for all $\varepsilon > 0$ (in addition to (i) and (ii) imposed at the beginning of Section 2). Then the following theorem holds.

Theorem 3. *We have the lower bound*

$$m(J^*) > \exp((\log H)^\alpha)$$

where α (>0) *is independent of* T *and* H.

Proof. Put $W_0 = \sum_{n\leq C_3 H} |b_n|^2 n^{-2\beta}$. Then we have

$$W_0 \geq \sum_{n\leq C_3 H} \Big| \chi(n) n^{-\beta} \sideset{}{'}\sum_{m_1\ldots m_k = n} a_{m_1}\ldots a_{m_k}\Big|^2 \geq \sum_{n\leq C_3 H} n^{-2\beta-\varepsilon} (d_k^*(n))^2\,,$$

where $d_k^*(n) = \sum_{m_1\ldots m_k = n} 1$, i.e. $d_k^*(n)$ is defined by

$$\sum_{n=1}^{\infty} d_k^*(n) n^{-s} = \Big(\sideset{}{'}\sum m^{-s}\Big)^k = \prod_{p\in S} (1+p^{-s})^k\,.$$

This leads to

$$W_0 \geq \prod (k^2 p^{-2\sigma}) \qquad (\sigma = \beta + \varepsilon)\,,$$

where the product is extended over all primes in S with $Y \leq p \leq 2Y$ and $Y = k^{1/\sigma - \varepsilon}$. Thus

$$W_0 \geq 3^{2Y^{1-\varepsilon}}$$

and hence

$$m(J^*) \geq W_0^{1/(2k)} \geq \exp(k^{1/\sigma - 1 - 2\varepsilon})\,.$$

We have still to satisfy $\prod p \leq C_3 H$ where the product is over all primes between Y and $2Y$. This leads to the following (we allow in fact a stronger) restriction on k, which is otherwise arbitrary:

$$(2Y)^{2Y} \le C_3 H\,,$$

which gives $k \le (\log H)^{\sigma-5\varepsilon}$. We can take for k the greatest integer with this property. Thus we obtain

$$m(J^*) \ge \exp((\log H)^{1-\sigma-100\varepsilon})\,.$$

Here we note that $\sigma = \beta + \varepsilon$, $\beta = 1 - \delta_1/1000$ and we can choose ε small enough. This leads to Theorem 3.

Remark. The conditions imposed on $F(s)$ here are more general than those mentioned in Remark 3 on p. 342 of [1].

4. Completion of the proof. We have proved (compare Lemma 6 and Theorem 3) that $F(s)(F(s) - a)$ has $\gg T^\mu$ distinct zeros in the rectangle $(\sigma \ge 1 - \delta, T \le t \le T + T^\mu)$ for a suitable constant $\delta = \delta(a, \mu) > 0$. On the other hand, if $F(s)$ has an Euler product of the type

$$F(s) = \prod_p \left(1 + \frac{a_p \chi(p)}{p^s} + \frac{a_{p^2}\chi(p^2)}{p^{2s}} + \ldots\right)$$

where $p^{-\varepsilon} a_p \chi(p), p^{-2\varepsilon} a_{p^2} \chi(p^2), \ldots$ are all $O_\varepsilon(1)$ say for every $\varepsilon > 0$ then the number of zeros of $F(s)$ (counted with multiplicity) in the same rectangle is $\le T^{C_4\delta}$ where C_4 is independent of T and μ. Hence by choosing δ smaller we can show that the number of zeros of $F(s)$ is $O(T^\nu)$ where $\nu = \mu/2$. This completes the proof of Theorem 1.

5. Further generalisations. We can consider the zeros of $F(s)(F(s) - G(s))$ where $G(s)$ is a generalised Dirichlet series (of the type described in remark below Lemma 1) which does not vanish (for example) in $\sigma \ge 3/4$ and there $\log G(s) = O(1)$. Of course we should have the conditions (i) to (v). However, we do not carry out the details.

References

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