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Effective simultaneous approximation of complex numbers by conjugate algebraic integers

by

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We study effectively the simultaneous approximation of $n-1$ different complex numbers by conjugate algebraic integers of degree n over $\mathbb{Z}(\sqrt{-1})$. This is a refinement of a result of Motzkin [2] (see also [3], p. 50) who has no estimate for the remaining conjugate. If the $n-1$ different complex numbers lie symmetrically about the real axis, then $\mathbb{Z}(\sqrt{-1})$ can be replaced by $\mathbb{Z}$.

In Section 1 we prove an effective version of a Kronecker approximation theorem; we start with an idea of H. Bohr and E. Landau (see e.g. [4]); later we use an estimate of A. Baker for linear forms with logarithms. This and also Rouché's theorem are then applied in Section 2 to give the result; the required irreducibility is guaranteed by the Schönemann–Eisenstein criterion.

1. On the Kronecker approximation theorem. Let $k \in \mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$, $v \in \mathbb{N}$, $U \in \mathbb{R}$, $U \geq 1$, $i := \sqrt{-1}$, $e(x) := \exp(2\pi i x)$ $(x \in \mathbb{R})$; let $p_1 < p_2 < \ldots < p_k$ be primes and

$$u_\nu \in \mathbb{Z}, \quad 0 < |u_\nu| \leq U, \quad \beta_\nu \in \mathbb{R} \qquad (\nu = 1, \ldots, k),$$

$$f(t) := 1 + e(t) + \sum_{\nu=1}^{k} e\left(t \frac{u_\nu}{v} \log p_\nu - \beta_\nu\right) \qquad (t \in \mathbb{R}).$$

With $\gamma_{-1} := 0$, $\beta_{-1} := 0$, $\gamma_0 := 1$, $\beta_0 := 0$, $\gamma_\nu := (u_\nu/v)\log p_\nu$ $(\nu = 1, \ldots, k)$ we have

$$f(t) = \sum_{\nu=-1}^{k} e(t\gamma_\nu - \beta_\nu)\,. \tag{1}$$

For $P \in \mathbb{N}$, $b \in \mathbb{Z}$, $B \in \mathbb{R}$, $B > 0$ let

$$J := \int_b^{b+B} |f(t)|^{2P}\, dt\,.$$

The multinomial theorem gives

$$f(t)^P = \sum_{\substack{j_{-1}+\ldots+j_k=P \\ j_\nu\ge 0\ (\nu=-1,0,\ldots,k)}}\!\!\cdots\!\!\sum \frac{P!}{j_{-1}!\ldots j_k!} e\Big(\sum_{\nu=-1}^{k} j_\nu(t\gamma_\nu - \beta_\nu)\Big).$$

For $\alpha \in \mathbb{C}$ denote by $\overline{\alpha}$ the complex conjugate of α; we have $|\alpha|^2 = \alpha\overline{\alpha}$. For $x \in \mathbb{R}$ we have $\overline{e(x)} = e(-x)$. With

$$\mathbf{j} = (j_{-1},\ldots,j_k) \in \mathbb{Z}^{k+2}, \qquad \mathbf{j}' = (j'_{-1},\ldots,j'_k) \in \mathbb{Z}^{k+2},$$

$$S(\mathbf{j},\mathbf{j}') := \sum_{\nu=-1}^{k} (j_\nu - j'_\nu)\gamma_\nu, \qquad T(\mathbf{j},\mathbf{j}') := \sum_{\nu=-1}^{k} (j_\nu - j'_\nu)\beta_\nu$$

we get

$$J = \sum_{\substack{j_{-1}+\ldots+j_k=P \\ j_\nu\ge 0\ (\nu=-1,\ldots,k) \\ j'_{-1}+\ldots+j'_k=P \\ j'_\nu\ge 0\ (\nu=-1,\ldots,k)}}\!\!\cdots\!\!\sum \frac{P!}{j_{-1}!\ldots j_k!}\frac{P!}{j'_{-1}!\ldots j'_k!} \int_b^{b+B} e(S(\mathbf{j},\mathbf{j}')t - T(\mathbf{j},\mathbf{j}'))\,dt.$$

We subdivide the multiple sum according as $\mathbf{j} = \mathbf{j}'$ or $\mathbf{j} \ne \mathbf{j}'$. We have $S(\mathbf{j},\mathbf{j}) = 0$, $T(\mathbf{j},\mathbf{j}) = 0$; but

$$\begin{aligned} S(\mathbf{j},\mathbf{j}') = 0 \ &\Rightarrow \exp(S(\mathbf{j},\mathbf{j}')) = 1 \\ &\Rightarrow \exp((j_0 - j'_0)v)p_1^{(j_1-j'_1)u_1}\ldots p_k^{(j_k-j'_k)u_k} = 1 \\ &\Rightarrow j_\nu = j'_\nu \qquad (\nu = 0,\ldots,k) \\ &\qquad \text{(by } vu_1\ldots u_k \ne 0 \text{ and by Lindemann)} \\ &\Rightarrow \mathbf{j} = \mathbf{j}' \qquad \Big(\text{by } \sum_{\nu=-1}^{k} j_\nu = \sum_{\nu=-1}^{k} j'_\nu = P\Big); \end{aligned}$$

we found

$$\mathbf{j} = \mathbf{j}' \ \Leftrightarrow\ S(\mathbf{j},\mathbf{j}') = 0.$$

This gives

$$\int_b^{b+B} e(S(\mathbf{j},\mathbf{j}')t - T(\mathbf{j},\mathbf{j}'))\,dt = B \qquad (\mathbf{j} = \mathbf{j}'),$$

$$\Big|\int_b^{b+B} e(S(\mathbf{j},\mathbf{j}')t - T(\mathbf{j},\mathbf{j}'))\,dt\Big| \le \frac{1}{\pi|S(\mathbf{j},\mathbf{j}')|} \qquad (\mathbf{j} \ne \mathbf{j}').$$

For $\mathbf{j} \ne \mathbf{j}'$ there exists by A. Baker (see [1, p. 22]) an effectively computable number $C(k,p_k) > 0$ with

$$|S(\mathbf{j},\mathbf{j}')|^{-1} < A := (2PUv)^{C(k,p_k)}.$$

We obtain

$$J \geq B \sum_{\substack{j_{-1}+\ldots+j_k=P \\ j_\nu \geq 0\ (\nu=-1,\ldots,k)}} \cdots \sum \left(\frac{P!}{j_{-1}!\ldots j_k!}\right)^2 - \frac{A}{\pi} \sum_{\substack{j_{-1}+\ldots+j_k=P \\ j_\nu \geq 0\ (\nu=-1,\ldots,k) \\ j'_{-1}+\ldots+j'_k=P \\ j_\nu \geq 0\ (\nu=-1,\ldots,k)}} \cdots \sum \frac{P!}{j_{-1}!\ldots j_k!} \frac{P!}{j'_{-1}!\ldots j'_k!}$$

where in the second multiple sum we have dropped the condition $\mathbf{j} \neq \mathbf{j}'$; to the first multiple sum we apply the Cauchy inequality and observe

$$\sum_{\substack{j_{-1}+\ldots+j_k=P \\ j_\nu \geq 0\ (\nu=-1,\ldots,k)}} \cdots \sum 1 \leq (P+1)^{k+2}.$$

This gives

$$J \geq \left(\frac{B}{(P+1)^{k+2}} - \frac{A}{\pi}\right)\left(\sum_{\substack{j_{-1}+\ldots+j_k=P \\ j_\nu \geq 0\ (\nu=-1,\ldots,k)}} \cdots \sum \frac{P!}{j_{-1}!\ldots j_k!}\right)^2;$$

since the last multiple sum equals $(1+\ldots+1)^P = (k+2)^P$, we have

$$J \geq \left(\frac{B}{(P+1)^{k+2}} - \frac{A}{\pi}\right)(k+2)^{2P}.$$

For some $\tau \in \mathbb{R}$, $b \leq \tau \leq b+B$, we have

$$|f(\tau)| = \sup_{b \leq t \leq b+B} |f(t)|;$$

this gives

$$J \leq B|f(\tau)|^{2P}.$$

We choose

$$B := A(P+1)^{k+2}.$$

This gives

$$\frac{B}{2(P+1)^{k+2}}(k+2)^{2P} \leq J \leq B|f(\tau)|^{2P},$$

$$|f(\tau)| \geq (k+2)\exp\left(-\frac{k+2}{2P}\log(2(P+1))\right)$$

$$> (k+2)\left(1 - \frac{k+2}{2P}\log(2(P+1))\right).$$

But $\log(2(P+1)) \le \frac{4}{3}\log P$ $(P \in \mathbb{R},\ P \ge 11)$. Setting

$$\mu := \frac{(k+2)^2 \log P}{3P} < 1$$

we obtain

$$|f(\tau)| > k + 2 - 2\mu\,.$$

(1) implies

$$f(t) = 1 + e(t\gamma_\nu - \beta_\nu) + \sum_{\substack{\mu=0 \\ \mu\ne\nu}}^{k} e(t\gamma_\mu - \beta_\mu)\,;$$

the triangle inequality and $|e(x)| = 1$ $(x \in \mathbb{R})$ give

$$|f(t)| \le k + |1 + e(t\gamma_\nu - \beta_\nu)| \qquad (\nu = 0, \ldots, k,\ t \in \mathbb{R})\,.$$

We obtain

$$|1 + e(\tau\gamma_\nu - \beta_\nu)| > 2 - 2\mu$$

and consequently

$$|\sin\pi(\tau\gamma_\nu - \beta_\nu)| < \sqrt{2\mu - \mu^2} < \sqrt{2\mu}\,;$$

denote by h_ν the nearest integer to $\tau\gamma_\nu - \beta_\nu$; we have

$$|\tau\gamma_\nu - \beta_\nu - h_\nu| \le 1/2 \qquad (\nu = 0, \ldots, k)\,.$$

Using

$$|\sin\pi x| \ge 2|x| \qquad (x \in \mathbb{R},\ |x| \le 1/2)\,,$$
$$|\sin\pi(x+h)| = |\sin\pi x| \qquad (x \in \mathbb{R},\ h \in \mathbb{Z})\,,$$

we obtain

$$2|\tau\gamma_\nu - \beta_\nu - h_\nu| \le |\sin\pi(\tau\gamma_\nu - \beta_\nu - h_\nu)| = |\sin\pi(\tau\gamma_\nu - \beta_\nu)| < \sqrt{2\mu}$$

$(\nu = 0, 1, \ldots, k)$; for $\nu = 0$ this implies

$$|\tau - h_0| < \sqrt{\mu}\,;$$

we replace τ by h_0 and with

$$\gamma^* := \sup_{\nu=1,\ldots,k} |\gamma_\nu|$$

we get by the triangle inequality

$$|h_0\gamma_\nu - \beta_\nu - h_\nu| < \mu^* := (1+\gamma^*)\sqrt{\mu} \qquad (\nu = 1, \ldots, k)\,.$$

Let $w \in \mathbb{R}$, $w \ge 1$; we are interested in the inequality

$$|h_0\gamma_\nu - \beta_\nu - h_\nu| < 1/w$$

with an effective estimate for h_0. We have

$$\gamma^* \le U \log p_k, \qquad \mu^* < 3U\sqrt{\mu}\log p_k\,,$$
$$\mu < \frac{(k+2)^2}{\sqrt{P}}, \qquad \mu^* < \widetilde{\mu} := \frac{3U(k+2)}{\sqrt[4]{P}}\log p_k\,.$$

The choice

$$P := [(3wU(k+2)\log p_k)^4] + 1$$

implies $P \ge 11$, $\mu < 1$, $\mu^* \le 1/w$. By $b \le \tau \le b+B$, $b \in \mathbb{Z}$, $h_0 \in \mathbb{Z}$, $|\tau - h_0| < 1$ we have $b \le h_0 < b+B+1$. By substitution, a bound for $B+1$ of the form $(2Uvw)^C$ can immediately be found. This proves

Theorem 1. *Let $k \in \mathbb{N}$, $v \in \mathbb{N}$, $U \in \mathbb{R}$, $U \ge 1$, $b \in \mathbb{Z}$, $w \in \mathbb{R}$, $w \ge 1$. Let $p_1 < \ldots < p_k$ be primes and*

$$u_\nu \in \mathbb{Z}, \quad 0 < |u_\nu| \le U, \quad \beta_\nu \in \mathbb{R} \qquad (\nu = 1, \ldots, k)\,.$$

Then there exist $h_\nu \in \mathbb{Z}$ $(\nu = 0, \ldots, k)$ and an effectively computable number $C'(k, p_k) > 0$, depending on k and p_k only, with

$$\left| h_0 \frac{u_\nu}{v} \log p_\nu - \beta_\nu - h_\nu \right| < \frac{1}{w} \qquad (\nu = 1, \ldots, k) \tag{2}$$

and

$$b \le h_0 \le b + (2Uvw)^{C'(k,p_k)}\,.$$

Theorem 1 is an effective Kronecker approximation theorem. If $p_1 < \ldots < p_k$ are the first k primes, then $C'(k, p_k)$ is an effectively computable $C''(k)$, depending on k only.

Let $m \in \mathbb{N}$ and $r_\nu \in \mathbb{Z}$, $0 \le r_\nu < m$ $(\nu = 0, \ldots, k)$. (2) is equivalent to

$$\left| (h_0 m + r_0)\frac{u_\nu}{v}\log p_\nu - \left(\beta_\nu m + r_0 \frac{u_\nu}{v}\log p_\nu - r_\nu\right) - (h_\nu m + r_\nu) \right| < \frac{m}{w}\,;$$

we write this as

$$\left| h_0' \frac{u_\nu}{v} \log p_\nu - \beta_\nu' - h_\nu' \right| < \frac{1}{w'} \qquad (\nu = 1, \ldots, k)\,. \tag{3}$$

Theorem 1 implies

Corollary 1. *Let $k \in \mathbb{N}$, $v \in \mathbb{N}$, $U \in \mathbb{R}$, $U \ge 1$, $b \in \mathbb{Z}$, $w' \in \mathbb{R}$, $w' \ge 1$; let $p_1 < \ldots < p_k$ be primes,*

$$u_\nu \in \mathbb{Z}, \quad 0 < |u_\nu| \le U, \quad \beta_\nu' \in \mathbb{R} \qquad (\nu = 1, \ldots, k)\,;$$

furthermore, let $m \in \mathbb{N}$ and $r_\nu \in \mathbb{Z}$, $0 \le r_\nu < m$ $(\nu = 0, \ldots, k)$. Then (3) *holds with $h_\nu' \equiv r_\nu \bmod m$ $(\nu = 0, \ldots, k)$ and*

$$b \le h_0'/m \le 1 + b + (2Uvmw')^{C'(k,p_k)}\,.$$

2. On a theorem of Motzkin. Let $n \in \mathbb{Z}$, $n > 1$,

$$f(z) := \prod_{j=1}^{n-1}(z - z_j) = z^{n-1} + a_1 z^{n-2} + \ldots + a_{n-1} \in \mathbb{C}[z]\,,$$
$$d(f) := \inf_{j \neq k}\{1, |z_j - z_k|\} > 0\,,$$
$$D(f) := \sup |z_j|, \qquad K(f) := \sup |a_j|\,.$$

We have $d(f) \le 2D(f)$ and

$$|a_j| \le \binom{n-1}{j}(1 + D(f))^j \le (2 + D(f))^{n-1} \qquad (j = 1, \ldots, n-1)\,,$$
$$K(f) \le (2 + D(f))^{n-1}\,.$$

Lemma 1. *Let* $g \in \mathbb{N}$,

$$F(z) = \prod_{j=1}^{g}(z - \alpha_j) \in \mathbb{C}[z], \qquad d(F) > 0\,,$$
$$F^*(z) \in \mathbb{C}[z] \quad \textit{with leading term } z^g\,,$$
$$\varrho \in \mathbb{R}, \qquad 0 < \varrho \le \tfrac{1}{4}d(F)\ (< 1)\,.$$

For $j \in \{1, \ldots, g\}$ *and*

$$B_j(F) := \left(\frac{d(F)}{2(|\alpha_j| + 2)}\right)^{g-1}, \qquad K(F^* - F) \le \varrho B_j(F)\,,$$

there exist $\alpha_j^* \in \mathbb{C}$ *with*

$$F^*(\alpha_j^*) = 0 \quad \textit{and} \quad |\alpha_j^* - \alpha_j| < \varrho\,;$$

furthermore,

$$F^*(z) = \prod_{j=1}^{g}(z - \alpha_j^*), \qquad d(F^*) > d(F)/2\,.$$

In short: *a small change in the coefficients of a polynomial implies a small change in its* (*simple*) *roots.*

Proof. Let $j \in \{1, \ldots, g\}$, $\widetilde{z} \in \mathbb{C}$, $|\widetilde{z} - \alpha_j| = \varrho$; then $|\widetilde{z}| < |\alpha_j| + 1$ and $|\widetilde{z} - \alpha_k| \ge d(F) - \varrho > d(F)/2$ $(k \ne j)$. We get

$$|(F^* - F)(\widetilde{z})| \le K(F^* - F)(1 + |\widetilde{z}| + \ldots + |\widetilde{z}|^{g-1})$$
$$< K(F^* - F)(|\alpha_j| + 2)^{g-1}\,,$$
$$|F(\widetilde{z})| = |\widetilde{z} - \alpha_j| \prod_{k \ne j} |\widetilde{z} - \alpha_k| \ge \varrho\left(\frac{d(F)}{2}\right)^{g-1},$$

and therefore

$$|(F^* - F)(\widetilde{z})| < |F(\widetilde{z})|\,.$$

By Rouché's theorem, there exists exactly one $\alpha_j^* \in \mathbb{C}$ with $|\alpha_j^* - \alpha_j| < \varrho$ and $F^*(\alpha_j^*) = 0$.

Let

$$B(f) := \left(\frac{d(f)}{2(D(f)+2)}\right)^{n-1} = \inf_{j=1,\ldots,n-1} B_j(f)\,.$$

Define $c_j \in \mathbb{R}$ by $a_j = c_{2j-1} + ic_{2j}$ $(j = 1, \ldots, n-1)$. Denote by p_j the jth prime; we have $p_j < (2j)^2$ $(j = 1, 2, \ldots)$. Let

$$\varrho \in \mathbb{R}, \quad 0 < \varrho \le d(f)/4, \quad \varepsilon := B(f)\varrho, \quad v := \left[\frac{4\log(4n)^2}{\varepsilon}\right] + 1\,.$$

Then

$$\frac{2}{v}\log p_j < \frac{\varepsilon}{2}$$

and there exist $u_j \in \mathbb{Z}$ such that for

$$c_j^* := \frac{2u_j + 1}{v}\log p_j \ne 0$$

we have

$$|c_j^* - c_j| < \varepsilon/2 \qquad (j = 1, \ldots, 2n-2)\,.$$

Let

$$a_j^* := c_{2j-1}^* + ic_{2j}^* \qquad (j = 1, \ldots, n-1)\,,$$
$$f^*(z) := z^{n-1} + a_1^* z^{n-2} + \ldots + a_{n-1}^*\,.$$

Then $K(f^* - f) < \varepsilon$. By Lemma 1, there exist $z_j^* \in \mathbb{C}$ with

$$f^*(z) = \prod_{j=1}^{n-1}(z - z_j^*), \qquad |z_j^* - z_j| < \varrho \qquad (j = 1, \ldots, n-1)\,,$$

hence

$$|z_j^* - z_k^*| > d(f) - 2\varrho \qquad (0 < j < k < n),$$
$$d(f)/2 \le d(f) - 2\varrho < d(f^*) < d(f) + 2\varrho \le 3d(f)/2\,.$$

Let $h_0 \in \mathbb{Z}$, $z_n^* := a_1^* - h_0$, $c_{2n-1}^* := 0$, $c_{2n}^* := 0$, $a_n^* := c_{2n-1}^* + ic_{2n}^*$,

$$g(z) := f^*(z)(z - z_n^*)\,;$$

with

$$b_j := a_j^* + a_{j-1}^*(h_0 - a_1^*) \qquad (j = 2, \ldots, n)$$

we have

$$g(z) - z^n - h_0 z^{n-1} = b_2 z^{n-2} + \ldots + b_n\,;$$

with

$$\beta_{2j-3} := -c^*_{2j-1} + c^*_{2j-3}c^*_1 - c^*_{2j-2}c^*_2\,,$$
$$\beta_{2j-2} := -c^*_{2j} + c^*_{2j-2}c^*_1 + c^*_{2j-3}c^*_2$$

we have

$$b_j = (h_0 c^*_{2j-3} - \beta_{2j-3}) + i(h_0 c^*_{2j-2} - \beta_{2j-2}) \qquad (j = 2, \ldots, n)\,.$$

Let $w \in \mathbb{R}$, $w \geq 1$; we apply Theorem 1 with $k = 2n-2$ and obtain $h_j \in \mathbb{Z}$ $(j = 0, \ldots, 2n-2)$ such that for

$$g^*(z) := z^n + h_0 z^{n-1} + (h_1 + ih_2)z^{n-2} + \ldots + (h_{2n-3} + ih_{2n-2}) \in (\mathbb{Z}[i])[z]$$

we have

$$K(g^* - g) < 2/w\,.$$

By Corollary 1 with $m = 9$ we can guarantee

$$h_0 \equiv h_1 \equiv \ldots \equiv h_{2n-3} \equiv 0 \bmod 9, \qquad h_{2n-2} \equiv 3 \bmod 9\,.$$

By the Schönemann–Eisenstein criterion for $3 \in \mathbb{Z}[i]$, g^* is irreducible over $\mathbb{Z}[i]$. Now

$$\begin{aligned} h_0 \geq b := [2n(D(f)+1)] > 0 \Rightarrow h_0 &> 2\sum_{j=1}^{n-1}(|z_j|+1) + 1 \geq 2\sum_{j=1}^{n-1}|z^*_j| + 1 \\ &\geq \Big|\sum_{j=1}^{n-1} z^*_j\Big| + |z^*_k| + 1 = |a^*_1| + |z^*_k| + 1 \\ &\Rightarrow |z^*_n - z_k| > 1 \qquad (k = 1, \ldots, n-1)\,; \end{aligned}$$

hence

$$d(g) = d(f^*)\,.$$

Let $\sigma \in \mathbb{R}$, $0 < \sigma \leq d(g)/4$; we have

$$\begin{aligned} B_j(g) &= \left(\frac{d(g)}{2(|z^*_j|+2)}\right)^{n-1} \qquad (j = 1, \ldots, n-1) \\ &> \widetilde{B}(f) := \left(\frac{d(f)}{4(D(f)+3)}\right)^{n-1}\,; \end{aligned}$$

let

$$w := \frac{2}{\sigma \widetilde{B}(f)}\,.$$

By Lemma 1, there exists $\zeta_j \in \mathbb{C}$ with

$$g^*(\zeta_j) = 0, \qquad |\zeta_j - z^*_j| < \sigma \qquad (j = 1, \ldots, n-1)\,,$$

hence

$$|\zeta_j - \zeta_k| > d(f^*) - 2\sigma \qquad (0 < j < k < n)\,.$$

Let $\eta \in \mathbb{R}$, $0 < \eta \le d(f)/4$, $\varrho := \eta/2$; then

$$|\zeta_j - z_j| < \eta \quad (j = 1, \ldots, n-1),$$
$$|\zeta_j - \zeta_k| > d(f) - 2\varrho - 2\sigma > d(f)/2 \quad (0 < j < k < n)$$

and obviously $\varrho \le d(f)/8 < d(f)/4$,

$$\sigma := \frac{\eta}{2} \le \frac{d(f)}{8} < \frac{d(f^*)}{4} = \frac{d(g)}{4}.$$

In c_j^* we certainly have

$$0 < |2u_j + 1| \le 2v(K(f) + 1) \le U := 2v(3 + D(f))^{n-1}.$$

In Corollary 1 we have

$$0 < b \le h_0/9 \le b + (2Uvw)^{5C''(2n-2)};$$

substitution gives

$$|h_0| < 2(2 \cdot 2v^2(3 + D(f))^{n-1}w)^{5C''};$$

but

$$0 < v < \frac{\log(4n)^2}{B(f)\eta} \cdot 16;$$

so the estimate for $|h_0|$ takes the form

$$|h_0| < (L(n, d(f), D(f))\eta^{-3})^{5C''}$$

where $L > 0$ is increasing in n, $1/d(f)$ and $D(f)$. For

$$S := \sup |a_j|, \quad S' := \sup |a_j^*|, \quad S'' := \sup |b_j|$$

we have

$$S' < S + 1 \quad (\text{since } K(f^* - f) < 1),$$
$$S'' < S' + S'(|h_0| + S') \quad (\text{by definition of } b_j),$$
$$|h_{2j-1} + h_{2j}i| < S'' + 1 \quad (j = 1, \ldots, n-1) \ (\text{since } K(g^* - g) < 1)$$

and g^* is effectively computable. This completes the proof of

THEOREM 2. *Let $n \in \mathbb{Z}$, $n > 1$, $z_j \in \mathbb{C}$ $(j = 1, \ldots, n-1)$,*

$$d := \inf_{j \ne k}\{1, |z_j - z_k|\} > 0, \quad D := \sup |z_j|, \eta \in \mathbb{R}, 0 < \eta \le d/4.$$

Then there exists an effectively computable polynomial $g^*(z) = z^n + e_1 z^{n-1} + \ldots + e_n$ with $e_j \in \mathbb{Z}[i]$ and with the properties:

(i) *g^* is irreducible over $\mathbb{Z}[i]$,*
(ii) *its suitably numbered roots $\zeta_1, \ldots, \zeta_n$ satisfy*

$$|\zeta_j - z_j| < \eta \quad (j = 1, \ldots, n-1).$$

This is a refinement of a result of Motzkin [2] who has no upper bound for $|\zeta_n|$.

THEOREM 3. *If in Theorem* 2 *the set* $\{z_1, \ldots, z_{n-1}\}$ *is symmetric about* $\mathbb{R} \subset \mathbb{C}$, *we have* $e_j \in \mathbb{Z}$ $(j = 1, \ldots, n)$ (*and* $\zeta_1, \ldots, \zeta_n$ *is a complete set of conjugate algebraic integers*).

Proof. In the proof of Theorem 2 we have

$$f(z) \in \mathbb{R}[z], \quad a_j^* = c_{2j-1}^* \quad (j = 1, \ldots, n-1),$$
$$f^*(z) \in \mathbb{R}[z], \quad \{z_1^*, \ldots, z_{n-1}^*\} \text{ symmetric about } \mathbb{R},$$
$$z_n^* := a_1^* - h_0 \in \mathbb{R}, \quad g(z) \in \mathbb{R}[z], \quad g^*(z) \in \mathbb{Z}[z].$$

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