ACTA ARITHMETICA
LXIII.3 (1993)

Integral spinor norms in dyadic local fields II

by

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In the previous paper [X] we have generalized the results of [BD] to a dyadic local field with $e = \operatorname{ord} 2 = 2$. In the present paper we generalize these results to an arbitrary dyadic local field, and we also point out that the bound for $\operatorname{ord}(dL)$ is the best possible. The results obtained are applied to improve the sufficient condition for the class number of an indefinite quadratic form over the ring of integers of a number field to be a divisor of the class number of the field, which is analogous to Satz 5 of [K].

Here we adopt the notations from [O] and [X]. In particular, F denotes a dyadic local field, ϑ the ring of integers in F, p the maximal ideal of ϑ, U the group of units in ϑ, $e = \operatorname{ord} 2$ the ramification index of 2 in F, π a fixed prime element in F, $D(\ ,\ )$ the quadratic defect function, Δ a fixed unit of quadratic defect 4ϑ, V a regular quadratic space over F with associated symmetric bilinear form $B(x,y)$, L a lattice on V, dL the determinant of L, $O^+(V)$ the group of rotations on V, $O^+(L)$ the corresponding subgroup of units of L, and $\theta(\ ,\ )$ the spinor norm function. We use the symbol $\langle a,b,c,\ldots\rangle$ for lattices, and $[a,b,c,\ldots]$ for spaces.

Lemma 1. *For any $i \geq 1$, $1+p^i$ is generated by $1+\lambda\pi^i$ with $\lambda \in U$.*

Proof. This follows from the identity

$$(1+\sigma\pi^{k+1}) = (1+\pi^k)(1+(1+\pi^k)^{-1}(\sigma\pi-1)\pi^k)\,.$$

Lemma 2. *Suppose $sL \subseteq \vartheta$ and $\operatorname{rank} L \geq 3$ and $e \geq 3$. If $\operatorname{ord}(dL) \leq 3$ then $\theta(O^+(L)) \supseteq U\dot{F}^2$.*

Proof. Let $L = L_1 \perp \ldots \perp L_t$ be a Jordan splitting of L. We assume $t \geq 2$ and $\operatorname{rank} L_i \leq 2$, $i = 1, \ldots, t$. Since $\operatorname{ord}(dL) \leq 3$, $t \leq 3$. We consider several cases.

(1) $L = L_1 \perp L_2$ where L_1 is unimodular with $\operatorname{rank} L_1 = 2$ and $L_2 = \vartheta x_2$ with $Q(x_2) = \varepsilon_2\pi$ and $\varepsilon_2 \in U$. Put $L_1 \cong A(a_1, -a_1^{-1}\delta_1)$ with the base $\{x_1, y_1\}$ and $0 \leq \operatorname{ord} a_1 \leq e$ and $D(1+\delta_1) = \delta_1\vartheta$.

If $\operatorname{ord}(-a_1^{-1}\delta_1) < e$, then $\operatorname{ord} a_1 \equiv \operatorname{ord}(-a_1^{-1}\delta_1) + 1 \bmod 2$.

When $\operatorname{ord} a_1$ is even, take $K = \vartheta x_1 \perp \vartheta x_2$. Note that any maximal vector of K gives rise to a symmetry of L. So $\theta(O^+(L)) \supseteq Q([1, \dot{a}_1\varepsilon_2\pi])$ which does not contain Δ, but Δ is in $\theta(O^+(L_1))$ by [H]. Therefore $\theta(O^+(L)) = \dot{F}$.

When $\operatorname{ord} a_1$ is odd, then $\operatorname{ord}(-a_1^{-1}\delta_1)$ is even. Take $K = \vartheta y_1 \perp \vartheta x_2$, and $\theta(O^+(L)) = \dot{F}$ by the same arguments as above.

If $\operatorname{ord}(-a_1^{-1}\delta_1) \geq e$, write $a_1 = \varepsilon_1 \pi^{r_1}$ and $-\varepsilon_1\varepsilon_2^{-1} = \eta^2 + \sigma\pi^d$ where d is an odd integer or $d \geq 2e$.

When r_1 is odd, consider a unimodular lattice $\overline{L}_1 = \vartheta(x_1 + \eta\pi^{(r_1-1)/2}x_2) + \vartheta y_1$ which splits L. Write $L = \overline{L}_1 \perp \vartheta\overline{x}_2$ with $Q(\overline{x}_2) = \overline{\varepsilon}_2\pi$. Note

$$\operatorname{ord}(Q(x_1 + \eta\pi^{(r_1-1)/2}x_2)) = r_1 + d\,.$$

If $r_1 + d \geq e$, then $\overline{L}_1 \cong A(0,0)$ or $A(2, 2\varrho)$ by [O, 93:11]. Therefore $\theta(O^+(L)) \supseteq \theta(O^+(\overline{L}_1)) = U\dot{F}^2$ by [H, Lemma 1]. Otherwise, $r_1 + d < e$ and $r_1 + d$ is even. Take $K = \vartheta(x_1 + \eta\pi^{(r_1-1)/2}x_2) \perp \vartheta\overline{x}_2$. Therefore $\theta(O^+(L)) = \dot{F}$.

When r_1 is even, take $K = \vartheta x_1 \perp \vartheta x_2$. So $\theta(O^+(L)) = \dot{F}$.

(2) $L = L_1 \perp L_2$ where L_1 is unimodular with $\operatorname{rank} L_1 = 2$ and $L_2 = \vartheta x_2$ with $Q(x_2) = \varepsilon_2\pi^2$ and $\varepsilon_2 \in U$. By the arguments similar to Case (1), we only need to consider $L_1 \cong A(\varepsilon_1, -\varepsilon_1^{-1}\delta_1)$ with the base $\{x_1, y_1\}$ where ε_1 is in U, $D(1 + \delta_1) = \delta_1\vartheta$ and $\operatorname{ord}(-\varepsilon_1^{-1}\delta_1) > e$. Put $D(\varepsilon_1\varepsilon_2) = p^t$ with $1 \leq t \leq 2e$ or $t = \infty$.

If $t \leq e - 1$, then $D(-\varepsilon_1\varepsilon_2) = D(\varepsilon_1\varepsilon_2) = p^t$. Take $K = \vartheta x_1 \perp \vartheta x_2$. Note that any maximal vector of K gives rise to a symmetry of L, so $\theta(O^+(L)) \supseteq Q([1, \dot{\varepsilon}_1\varepsilon_2])$. By [H, Lemma 3], there exists η in U such that $(\eta, -\varepsilon_1\varepsilon_2) = -1$ with $D(\eta) = p^{2e-t}$. Since $2e - t \geq e + 1$, η is in $\theta(O^+(L_1))$ by [H, Lemma 2]. Therefore $\theta(O^+(L)) = \dot{F}$.

If $t > e - 1$, write $\varepsilon_2^{-1}\varepsilon_1 = \xi^2 + \sigma\pi^t$ where ξ and σ are in U.

When e is odd, there exists u in ϑ such that

$$\lambda + 2\pi^{-e}\xi(\pi^{(e-1)/2} - \xi) - \sigma\pi^{t-e} - 2\pi^{-e}\varepsilon_2^{-1}u + \varepsilon_1^{-1}\varepsilon_2^{-1}\delta_1\pi^{-e}u^2 = 0$$

for any $\lambda \in \vartheta$ by Hensel's Lemma. Put $z = \pi x_1 + \pi u y_1 + (\pi^{(e-1)/2} - \xi)x_2 \in L$, and

$$\begin{aligned}
Q(z) &= \pi^2\varepsilon_1 + \pi^2u^2(-\varepsilon_1^{-1}\delta_1) + 2\pi^2u + (\pi^{(e-1)/2} - \xi)^2\varepsilon_2\pi^2 \\
&= \varepsilon_2\pi^2(\xi^2 + \sigma\pi^t + 2\varepsilon_2^{-1}u + (\pi^{(e-1)/2} - \xi)^2 - (\varepsilon_1\varepsilon_2)^{-1}\delta_1u^2) \\
&= \varepsilon_2\pi^2(\pi^{e-1} - 2\xi(\pi^{(e-1)/2} - \xi) + \sigma\pi^t + 2\varepsilon_2^{-1}u - (\varepsilon_1\varepsilon_2)^{-1}\delta_1u^2) \\
&= \varepsilon_2\pi^{e+1}(1 + \pi(-2\pi^{-e}\xi(\pi^{(e-1)/2} - \xi) \\
&\qquad + \sigma\pi^{t-e} + 2\pi^{-e}\varepsilon_2^{-1}u - (\varepsilon_1\varepsilon_2)^{-1}\delta_1\pi^{-e}u^2)) \\
&= \varepsilon_2\pi^{e+1}(1 + \lambda\pi)\,.
\end{aligned}$$

So τ_z is in $O(L)$ and $\theta(O^+(L)) \supseteq U\dot{F}^2$.

When e is even, there exists u in ϑ such that

$$\lambda + 2\pi^{1-e}\xi(\pi^{(e-2)/2} - \xi) - \sigma\pi^{t-e+1} - 2\pi^{-e}\varepsilon_2^{-1}u + (\varepsilon_1\varepsilon_2)^{-1}\delta_1\pi^{-e-1}u^2 = 0$$

for any $\lambda \in \vartheta$ by Hensel's Lemma if $\operatorname{ord}(\delta_1) \geq e+2$. Put

$$z = \pi x_1 + uy_1 + (\pi^{(e-2)/2} - \xi)x_2 \in L\,.$$

Since $Q(z) = \varepsilon_2\pi^e(1+\lambda\pi)$ by a direct computation, τ_z is in $O(L)$ and $\theta(O^+(L)) \supseteq U\dot{F}^2$.

Now we treat the case of $\operatorname{ord}(-\varepsilon_1^{-1}\delta_1) = e+1$. For any $\lambda \in U$, write

$$(-(\varepsilon_1\varepsilon_2)^{-1}\delta_1\pi^{-e-1})^{-1} = \alpha^2 + \beta\pi^d$$

where α and β are in U, and $d \geq 1$. By Hensel's Lemma, there exists u in ϑ such that

$$(\varepsilon_1\varepsilon_2)^{-1}(\delta_1\pi^{-e-1})\beta\pi^{d-1} + (2\pi^{-e}\alpha\varepsilon_2^{-1} - 2\xi\pi^{-e/2})u + (\sigma\pi^{t+2-e} + 2\xi^2\pi^{2-e})u^2 = 0\,.$$

Put $z = \pi^2ux_1 + \alpha y_1 + (\pi^{(e-2)/2} - \xi\pi u)x_2 \in L$. So $Q(z) = \varepsilon_2\pi^e(1+\lambda\pi)$ and τ_z is in $O(L)$. Therefore we obtain $\theta(O^+(L)) \supseteq U\dot{F}^2$ by Lemma 1.

(3) $L = L_1 \perp L_2$ where L_1 is unimodular with $\operatorname{rank} L_1 = 2$ and $L_2 = \vartheta x_2$ with $Q(x_2) = \varepsilon_2\pi^3$ and $\varepsilon_2 \in U$. By the arguments similar to Case (1), we only need to consider $L_1 \cong A(\varepsilon_1\pi_1, -\varepsilon_1^{-1}\pi^{-1}\delta_1)$ with the base $\{x_1, y_1\}$ where ε_1 is in U, $D(1+\delta_1) = \delta_1\vartheta$ and $\operatorname{ord}(-\varepsilon_1^{-1}\pi^{-1}\delta_1) > e$. Put $D(\varepsilon_1\varepsilon_2) = p^t$ with $1 \leq t \leq 2e$ or $t = \infty$.

If $t \leq e-2$, take $K = \vartheta x_1 \perp \vartheta x_2$. By the same arguments as in Case (2), we have $\theta(O^+(L)) = \dot{F}$.

If $t > e-2$, write $\varepsilon_2^{-1}\varepsilon_1 = \xi^2 + \sigma\pi^t$ where ξ and σ are in U.

When e is even, there exists u in ϑ such that

$$\lambda + 2\pi^{1-e}\xi(\pi^{(e-2)/2} - \xi) - \sigma\pi^{t-e+1} - 2\pi^{-e}\varepsilon_2^{-1}u + (\varepsilon_1\varepsilon_2)^{-1}\delta_1\pi^{-e-1}u^2 = 0$$

for any $\lambda \in \vartheta$ by Hensel's Lemma. Put $z = \pi x_1 + \pi u y_1 + (\pi^{(e-2)/2} - \xi)x_2 \in L$ and $Q(z) = \varepsilon_2\pi^{e+1}(1+\lambda\pi)$; so τ_z is in $O(L)$ and $\theta(O^+(L)) \supseteq U\dot{F}^2$.

When e is odd, there exists u in ϑ such that

$$\lambda + 2\pi^{2-e}\xi(\pi^{(e-3)/2} - \xi) - \sigma\pi^{t-e+2} - 2\pi^{-e}\varepsilon_2^{-1}u + (\varepsilon_1\varepsilon_2)^{-1}\pi^{-e-2}\delta_1u^2 = 0$$

for any $\lambda \in \vartheta$ by Hensel's Lemma if $\operatorname{ord}(-\varepsilon_1^{-1}\pi^{-1}\delta_1) > e+1$. Put

$$z = \pi x_1 + uy_1 + (\pi^{(e-3)/2} - \xi)x_2 \in L\,.$$

Since $Q(z) = \varepsilon_2\pi^e(1+\pi\lambda)$, τ_z is in $O(L)$ and $\theta(O^+(L)) \supseteq U\dot{F}^2$.

Now we treat the case of $\operatorname{ord}(-\varepsilon_1^{-1}\pi^{-1}\delta_1) = e+1$. For any $\lambda \in U$, write

$$\lambda(-(\varepsilon_1\varepsilon_2)^{-1}\pi^{-2-e}\delta_1)^{-1} = \alpha^2 + \beta\pi^d$$

where α and β are in U and $d \geq 1$. By Hensel's Lemma, there exists u in ϑ such that

$$\beta\pi^d(\varepsilon_1\varepsilon_2)^{-1}\pi^{-2-e}\delta_1 + (2\varepsilon_2^{-1}\alpha\pi^{-e} - 2\xi\pi^{(1-e)/2})u + (\sigma\pi^{t-e+2} + 2\pi^{2-e}\xi^2)u^2 = 0\,.$$

Put $z = \pi u x_1 + \alpha y_1 + (\pi^{(e-3)/2} - \xi u)x_2 \in L$ and $Q(z) = \varepsilon_2\pi^e(1+\lambda\pi)$. So τ_z is in $O(L)$ and $\theta(O^+(L)) \supseteq U\dot{F}^2$ by Lemma 1.

(4) $L = L_1 \perp L_2$ where L_1 is unimodular with $\operatorname{rank} L_1 = 2$ and L_2 is p-modular with $\operatorname{rank} L_2 = 2$. Write $L_1 \cong A(\varepsilon_1\pi^{r_1}, -\varepsilon_1^{-1}\pi^{-r_1}\delta_1)$ with the base $\{x_1, y_1\}$ and $0 \leq r_1 \leq e$ and $D(1+\delta_1) = \delta_1\vartheta$. $L_2 \cong \pi A(\varepsilon_2\pi^{r_2}, -\varepsilon_2^{-1}\pi^{-r_2}\delta_2)$ with the base $\{x_2, y_2\}$ and $0 \leq r_2 \leq e$ and $D(1+\delta_2) = \delta_2\vartheta$.

If $r_1 \equiv r_2 \bmod 2$, take $K = \vartheta x_1 \perp \vartheta x_2$. By the same arguments as in Case (1), we obtain $\theta(O^+(L)) = \dot{F}$.

If $r_1 \equiv r_2 + 1 \bmod 2$, write $-\varepsilon_1\varepsilon_2^{-1} = \xi^2 + \sigma\pi^d$ with $\xi, \sigma \in U$ and $d \geq 1$.

When $\operatorname{ord}(-\varepsilon_1^{-1}\pi^{-r_1}\delta_1) < e$, take $K = \vartheta y_1 \perp \vartheta x_2$; thus $\theta(O^+(L)) = \dot{F}$.

When $\operatorname{ord}(-\varepsilon_2^{-1}\pi^{-r_2}\delta_2) < e$, take $K = \vartheta x_1 \perp \vartheta y_2$; thus $\theta(O^+(L)) = \dot{F}$.

Otherwise, we take $\overline{L}_1 = \vartheta(x_1 + \xi\pi^{(r_1-r_2-1)/2}x_2) + \vartheta y_1$ splitting L if $r_1 \geq r_2+1$, or $\overline{L}_2 = \vartheta(x_2 + \xi^{-1}\pi^{(r_2-r_1-1)/2}x_1) + \vartheta y_2$ splitting L if $r_1 < r_2+1$ by [O, 82:15].

When $\operatorname{ord}(Q(x_1 + \xi\pi^{(r_1-r_2-1)/2}x_2)) = r_1 + d \geq e$ or $\operatorname{ord}(Q(x_2 + \xi^{-1}\pi^{(r_2-r_1+1)/2}x_1)) = r_2 + d + 1 \geq e$, then $\overline{L}_1$ or $\overline{L}_2 \cong A(0,0)$ or $A(2, 2\varrho)$, and $\theta(O^+(L)) \supseteq \theta(O^+(\overline{L}_1)) = U\dot{F}^2$ or $\theta(O^+(\overline{L}_2)) = U\dot{F}^2$.

If $\operatorname{ord}(Q(x_1 + \xi\pi^{(r_1-r_2-1)/2}x_2)) = r_1 + d < e$ or $\operatorname{ord}(Q(x_2 + \xi^{-1}\pi^{(r_2-r_1+1)/2}x_1)) = r_2 + d + 1 < e$, then $L \cong \overline{L}_1 \perp L_2'$ or $L_1' \perp \overline{L}_2$ respectively and we repeat the above arguments until we obtain the results as desired.

(5) $L = L_1 \perp L_2 \perp L_3$ where L_1 is unimodular with $\operatorname{rank} L_1 = 2$, and $L_i = \vartheta x_i$ with $Q(x_i)\vartheta = p^{i-1}$, $i = 2, 3$. Then $\theta(O^+(L)) \supseteq \theta(O^+(L_1 \perp L_2)) \supseteq U\dot{F}^2$ by Case (1).

(6) $L = L_1 \perp L_2$ with $\operatorname{rank} L_1 = 1$ and $\operatorname{rank} L_2 = 2$. We scale the dual lattice of L by π and reduce to Case (1).

(7) $L = L_1 \perp L_2 \perp L_3$ with $\operatorname{rank} L_i = 1$, $i = 1, 2, 3$. So $L_i = \vartheta x_i$ with $Q(x_i) = p^{i-1}$, $i = 1, 2, 3$; and $\theta(O^+(L)) = \dot{F}$ by [X, Theorem 3.1].

We point out that the bound $\operatorname{ord}(dL) \leq 3$ given in the above lemma cannot be unconditionally improved for any $e \geq 3$ in view of the following example.

Example. Suppose $L \cong A(1, \pi^{2e-1}) \perp \langle \pi^4 \rangle$ with the base $\{x, y, z\}$ and $e \geq 3$. Then $\theta(O^+(L)) \subseteq (1+p^2)\dot{F}^2$.

Proof. First we prove that $O(L)$ is generated by the symmetries of L.

Take σ in $O(L)$. Write $\sigma x = ax+by+cz$. So $1-a^2 = 2ab+b^2\pi^{2e-1}+c^2\pi^4 \in p^3$ and $(1-a) \in p^2$. We can assume $\operatorname{ord} b \le 1$, otherwise, instead of σ we consider $\tau_{\pi^{[e/2]}x+y}\sigma$ if necessary and $\tau_{\pi^{[e/2]}x+y} \in O(L)$. Since $Q(\sigma x - x) = 2((1-a)-b)$, $\tau_{\sigma x - x} \in O(L)$. Therefore we assume $\sigma x = x$, $\sigma y = \alpha x+\beta y+\gamma z$. So

$$\alpha + \beta = 1\,, \qquad \pi^{2e-1} = \alpha^2 + 2\alpha\beta + \beta^2\pi^{2e-1} + \gamma^2 Q(z)$$

and

$$Q(\sigma y - y) = 2\alpha(-1+\pi^{2e-1})\,.$$

When $\operatorname{ord}\alpha \le 4$, then $\tau_{\sigma y - y} \in O(L)$. So $\sigma = \tau_{\sigma y-y}$ or $\tau_{\sigma y - y}\tau_z$.

When $\operatorname{ord}\alpha > 4$, put $\xi = 1+\pi^{[(e-2)/2]}$ and $u = \pi^2 x - \pi^2 y + \xi z$. Then

$$Q(u) = \pi^4(1+\xi^2) + \pi^{2e+3} - 2\pi^4 = \pi^{4+2[(e-2)/2]} + 2\pi^{4+[(e-2)/2]} + \pi^{2e+3}\,.$$

So $\tau_u \in O(L)$ and $\tau_u(x) = x$. Write $\tau_u\sigma(y) = \alpha' x + \beta' y + \gamma' z$. We can check $\operatorname{ord}\alpha' \le 3$. Therefore we obtain the result as desired by the above arguments.

It is not difficult to check $Q(v) \in (1+p^2)\dot{F}^2$ for any maximal vector v of L which gives rise to a symmetry of L. So we obtain $\theta(O^+(L)) \subseteq (1+p^2)\dot{F}^2$.

LEMMA 3. *Suppose* $sL \subseteq \vartheta$ *and* $\operatorname{rank} L \ge 4$ *and* $e \ge 3$. *If* $\operatorname{ord}(dL) \le 7$ *then* $\theta(O^+(L)) \supseteq U\dot{F}^2$.

Proof. Using the above Lemma 2, considering components and dual lattices whenever necessary, there remain two cases to be treated.

(1) $L = L_1 \perp L_2$ where L_1 is a binary unimodular lattice and L_2 is a binary p^2-modular lattice. Write $L_1 \cong A(\varepsilon_1\pi^{r_1}, -\varepsilon_1^{-1}\pi^{-r_1}\delta_1)$ with base $\{x_1, y_1\}$ and $0 \le r_1 \le e$ and $D(1+\delta_1) = \delta_1\vartheta$. $L_2 \cong \pi^2 A(\varepsilon_2\pi^{r_2}, -\varepsilon_2^{-1}\pi^{-r_2}\delta_2)$ with base $\{x_2, y_2\}$ and $0 \le r_2 \le e$ and $D(1+\delta_2) = \delta_2\vartheta$.

By the same arguments as in Lemma 2, Case (4), and in [H, Lemma 1, Prop. C], and considering the dual lattice of L if necessary, we only need to consider the case $0 \le r_1 = r_2 \le e-2$ and $\operatorname{ord}(-\varepsilon_1^{-1}\pi^{-r_1}\delta_1) > e$ and $\operatorname{ord}(-\varepsilon_2^{-1}\pi^{-r_2}\delta_2) > e$. Put $D(\varepsilon_1\varepsilon_2) = p^t$ with $1 \le t \le 2e$ or $t = +\infty$.

If $t \le e - r_1 - 1$, take $K = \vartheta x_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$ by the same arguments as in Lemma 2, Case(2).

If $t > e - r_1 - 1$, write $\varepsilon_2^{-1}\varepsilon_1 = \xi^2 + \sigma\pi^t$ where ξ and σ are in U.

When $e - r_1$ is odd, there exists u in ϑ such that

$$\lambda + 2\pi^{r_1-e}\xi(\pi^{(e-r_1-1)/2} - \xi) \\ - \sigma\pi^{t+r_1-e} - 2\varepsilon_2^{-1}\pi^{-e}u + (\varepsilon_1\varepsilon_2)^{-1}\pi^{-e-r_1}\delta_1 u^2 = 0$$

for any $\lambda \in \vartheta$ by Hensel's Lemma. Put $z = \pi x_1 + \pi u y_1 + (\pi^{(e-r_1-1)/2} - \xi)x_2 \in L$. So $Q(z) = \varepsilon_2\pi^{e+1}(1+\lambda\pi)$ and τ_z is in $O(L)$. Therefore $\theta(O^+(L)) \supseteq U\dot{F}^2$.

When $e-r_1$ is even, there exists u in ϑ such that

$$\lambda+2\pi^{r_1+1-e}\xi(\pi^{(e-r_1-2)/2}-\xi)-\sigma\pi^{t+r_1+1-e}-2\varepsilon_2^{-1}\pi^{-e}u + (\varepsilon_1\varepsilon_2)^{-1}\pi^{-r_1-e-1}\delta_1 u^2=0$$

for any λ in ϑ by Hensel's Lemma provided $\operatorname{ord}(-\varepsilon_1^{-1}\pi^{-r_1}\delta_1)\ge e+2$. Put $z=\pi x_1+uy_1+(\pi^{(e-r_1-2)/2}-\xi)x_2\in L$. So $Q(z)=\varepsilon_2\pi^e(1+\lambda\pi)$ and τ_z is in $O(L)$. Therefore $\theta(O^+(L))\supseteq U\dot{F}^2$.

Now we treat the case of $\operatorname{ord}(-\varepsilon_1^{-1}\pi^{-r_1}\delta_1)=e+1$. For any λ in U, write

$$\lambda(-(\varepsilon_1\varepsilon_2)^{-1}\pi^{-r_1-e-1}\delta_1)^{-1}=\alpha^2+\beta\pi^d$$

with $\alpha,\beta\in U$ and $d\ge 1$. By Hensel's Lemma, there exists u in ϑ such that

$$((\varepsilon_1\varepsilon_2)^{-1}\pi^{-r_1-1-e}\delta_1)(\beta\pi^d)+(2\varepsilon_2^{-1}\pi^{-e}\alpha-2\pi^{(r_1-e)/2}\xi)u + (\sigma\pi^{r_1+1+t-e}+2\pi^{r_1+1-e}\xi^2)u^2=0\,.$$

Put $z=\pi ux_1+\alpha y_1+(\pi^{(e-r_1-2)/2}-\xi u)x_2\in L$. So $Q(z)=\varepsilon_2\pi^e(1+\lambda\pi)$ and τ_z is in $O(L)$. Therefore $\theta(O^+(L))\supseteq U\dot{F}^2$ by Lemma 1.

(2) $L=L_1\perp L_2$ where L_1 is a binary unimodular lattice and L_2 is a binary p^3-modular lattice. Write $L_1\cong A(\varepsilon_1\pi^{r_1},-\varepsilon_1^{-1}\pi^{-r_1}\delta_1)$ with base $\{x_1,y_1\}$ and $0\le r_1\le e$ and $D(1+\delta_1)=\delta_1\vartheta$. $L_2\cong\pi^3A(\varepsilon_2\pi^{r_2},-\varepsilon_2^{-1}\pi^{-r_2}\delta_2)$ with base $\{x_2,y_2\}$ and $0\le r_2\le e$ and $D(1+\delta_2)=\delta_2\vartheta$. By the arguments similar to Lemma 2, Case (4), and [H, Lemma 1, Prop. C], and considering the dual lattice of L if necessary, we only need to consider the case $0\le r_1$, $r_2\le e-2$; $r_1=r_2+1$ or $r_1+1=r_2$; and $\operatorname{ord}(-\varepsilon_1^{-1}\pi^{-r_1}\delta_1)>e$ and $\operatorname{ord}(-\varepsilon_2^{-1}\pi^{-r_2}\delta_2)>e$.

When $r_1=r_2+1$, we can obtain the results as desired by the same arguments as in the above Case (1).

Now we treat the case $r_2=r_1+1$. Put $D(\varepsilon_1\varepsilon_2)=p^t$ with $1\le t\le 2e$ or $t=+\infty$.

If $t\le e-r_1-2$, take $K=\vartheta x_1\perp\vartheta x_2$. Then $\theta(O^+(L))=\dot{F}$ by the same arguments as in Lemma 2, Case (2).

If $t>e-r_1-2$, write $\varepsilon_2^{-1}\varepsilon_1=\xi^2+\sigma\pi^t$ where ξ and σ are in U.

When $e-r_1$ is even, there exists u in ϑ such that

$$-\lambda-2\pi^{r_1+1-e}(\pi^{(e-r_1-2)/2}-1)+\sigma\pi^{t-e+r_1+1}\xi^{-2}(\pi^{(e-r_1-2)/2}-1)^2 + 2\varepsilon_2^{-1}\pi^{-e}u-\varepsilon_2^{-2}\pi^{-(r_1+1)-e}\delta_2u^2=0$$

for any $\lambda\in\vartheta$ by Hensel's Lemma. Put $z=\pi^2\xi^{-1}(\pi^{(e-r_1-2)/2}-1)x_1+x_2+uy_2\in L$. So $Q(z)=\varepsilon_2\pi^{e+2}(1+\lambda\pi)$ and τ_z is in $O(L)$. Therefore $\theta(O^+(L))\supseteq U\dot{F}^2$.

When $e-r_1$ is odd, $r_1+1=r_2\le e-2$ and $r_1\le e-3$. Then there exists u in ϑ such that

$$\lambda + 2\pi^{-e+r_1+2}(\pi^{(e-r_1-3)/2} - 1) - \sigma\xi^{-2}\pi^{t-e+r_1+2}(\pi^{(e-r_1-3)/2} - 1)^2 \\ -2\pi^{-e}\varepsilon_2^{-1}u + \varepsilon_2^{-2}\pi^{-(r_1+1)}\delta_2\pi^{-e-1}u^2 = 0$$

for any $\lambda \in \vartheta$ provided $\operatorname{ord}(-\varepsilon_2^{-1}\pi^{-r_2}\delta_2) \ge e + 2$. Put $z = \pi^3(\pi^{(e-r_1-3)/2} - 1)\xi^{-1}x_1 + \pi x_2 + uy_2 \in L$. So $Q(z) = \varepsilon_2\pi^{e+3}(1 + \lambda\pi)$ and τ_z is in $O(L)$. Therefore $\theta(O^+(L)) \supseteq U\dot{F}^2$.

Finally, we consider the case of $\operatorname{ord}(-\varepsilon_2^{-1}\pi^{-r_2}\delta_2) = e+1$. For any $\lambda \in U$, write $\lambda(-\varepsilon_2^{-2}\pi^{-(r_1+1)}\pi^{-e-1}\delta_2)^{-1} = \alpha^2 + \beta\pi^d$ with $\alpha, \beta \in U$ and $d \ge 1$. By Hensel's Lemma, there exists u in ϑ such that

$$(\varepsilon_2^{-2}\pi^{-(r_1+1)}\pi^{-e-1}\delta_2)(\beta\pi^d) + \sigma\xi^{-2}\pi^{t-1} \\ + (2\varepsilon_2^{-1}\pi^{-e}\alpha - 2\pi^{(r_1+1-e)/2} - 2\sigma\xi^{-2}\pi^{t+(r_1-e+1)/2})u \\ + (2\pi^{-e+r_1+2} + \sigma\xi^{-2}\pi^{t-e+r_1+2})u^2 = 0\,.$$

Put $z = \pi^3(\pi^{(e-r_1-3)/2} - u)\xi^{-1}x_1 + \pi u x_2 + \alpha y_2 \in L$. So $Q(z) = \varepsilon_2\pi^{e+3}(1+\lambda\pi)$ and τ_z is in $O(L)$. Therefore $\theta(O^+(L)) \supseteq U\dot{F}^2$ by Lemma 1.

By the above lemmas and [X, Theorem 3.1] and the same arguments as in [BD] and by the results in [X] and [EH], we have

THEOREM. *Let L be a regular ϑ lattice with $sL \subseteq \vartheta$ and* $\operatorname{rank} L = n \ge 3$. *If*

$$\operatorname{ord}(dL) < \begin{cases} n(n-2) & \text{if } n \text{ is even,} \\ (n-1)^2 & \text{if } n \text{ is odd,} \end{cases}$$

then $\theta(O^+(L)) \supseteq U\dot{F}^2$.

Remark. The bound on $\operatorname{ord}(dL)$ in the above theorem is the best possible. For $e = 1$ this easily follows from [EH, Theorem 3.14]. Consider the following example for $e \ge 2$:

$$L = \begin{cases} A(1,\pi^{2e-1})\perp\pi^4 A(1,\pi^{2e-1})\perp\ldots\perp\pi^{4(n/2-1)}A(1,\pi^{2e-1}) \\ \quad \text{with base } \{x_1, y_1; x_2, y_2; \ldots; x_{n/2}, y_{n/2}\} \quad \text{if } n \text{ is even,} \\ A(1,\pi^{2e-1})\perp\pi^4 A(1,\pi^{2e-1})\perp\ldots\perp\pi^{4((n-1)/2-1)}A(1,\pi^{2e-1}) \\ \quad \perp\langle\pi^{2(n-1)}\rangle \\ \quad \text{with base } \{x_1, y_1; x_2, y_2; \ldots; x_{(n-1)/2}, y_{(n-1)/2}; z\} \quad \text{if } n \text{ is odd.} \end{cases}$$

We will show that $\theta(O^+(L)) \subset U\dot{F}^2$.

First, by the same arguments as in the above Example when $e \ge 3$, and by the arguments as in [X, Example 4.3] when $e = 2$, we can prove $O(L)$ is generated by the symmetries of L. Next we compute the spinor norms. For convenience, we only treat the case of even n. When n is odd, the arguments are similar.

When $e \geq 3$, we take any maximal vector v of L which gives rise to a symmetry of L. Write $v = \sum_{i=1}^{n/2}(a_i x_i + b_i y_i)$. Then

$$(*) \qquad \operatorname{ord}(Q(v)) = \operatorname{ord}\Big(\sum_{i=1}^{n/2} \pi^{4(i-1)}(a_i^2 + 2a_i b_i + b_i^2 \pi^{2e-1})\Big) \leq e + \min_{1\leq i\leq n/2}\{4(i-1) + \operatorname{ord} a_i, 4(i-1) + \operatorname{ord} b_i\}.$$

We choose the largest k such that

$$\min\{4(k-1) + \operatorname{ord} a_k, 4(k-1) + \operatorname{ord} b_k\} = \min_{1\leq i\leq n/2}\{4(i-1) + \operatorname{ord} a_i, 4(i-1) + \operatorname{ord} b_i\}.$$

If $\operatorname{ord} a_k \leq 1$, then

$$\operatorname{ord}(\pi^{4(i-1)}(a_i^2 + 2a_i b_i + b_i^2 \pi^{2e-1})) - \operatorname{ord}(\pi^{4(k-1)}(a_k^2 + 2a_k b_k + b_k^2 \pi^{2e-1})) \geq 2$$

for all $i \neq k$ by $(*)$.

If $\operatorname{ord} a_k \geq 2$, note that

$$Q(v) = \Big(\sum_{i=1}^{n/2} \pi^{2(i-1)} a_i\Big)^2 - 2\sum_{1\leq s<t\leq n/2} \pi^{2(s-1)+2(t-1)} a_s a_t + \sum_{i=1}^{n/2} b_i^2 \pi^{4(i-1)+2e-1} + 2\sum_{i=1}^{n/2} a_i b_i \pi^{4(i-1)}.$$

We have

$$\begin{aligned}&\operatorname{ord}(-2\pi^{2(s-1)+2(t-1)} a_s a_t) - \operatorname{ord} Q(v)\\ &\quad \geq e + 2(s-1) + 2(t-1) + \operatorname{ord} a_s + \operatorname{ord} a_t - (e + 4(s-1) + \operatorname{ord} a_s)\\ &\quad = 2(t-s) + \operatorname{ord} a_t \geq 2\end{aligned}$$

for any $1 \leq s < t \leq n/2$ by $(*)$, and

$$\begin{aligned}&\operatorname{ord}(b_i^2 \pi^{4(i-1)+2e-1}) - \operatorname{ord} Q(v)\\ &\quad \geq 2\operatorname{ord} b_i + 4(i-1) + (2e-1) - (4(i-1) + \operatorname{ord} b_i + e)\\ &\quad = \operatorname{ord} b_i + (e-1) \geq 2\end{aligned}$$

for any $1 \leq i \leq n/2$ by $(*)$; also $\operatorname{ord} a_i \geq 2$ for any $i \leq k$ by the choice of k. So

$$\begin{aligned}&\operatorname{ord}(2a_i b_i \pi^{4(i-1)}) - \operatorname{ord} Q(v)\\ &\quad \geq e + 4(i-1) + \operatorname{ord} a_i + \operatorname{ord} b_i - (e + 4(i-1) + \operatorname{ord} b_i) = \operatorname{ord} a_i \geq 2\end{aligned}$$

for any $i \leq k$ by $(*)$.

Suppose there exists $j > k$ such that $\operatorname{ord} a_i = \operatorname{ord} b_i = 0$ and

$$4(j-1) = \min\{4(k-1) + \operatorname{ord} a_k, 4(k-1) + \operatorname{ord} b_k\} + 1\,.$$

Then

$$\operatorname{ord}(\pi^{4(i-1)}(a_i^2 + 2a_ib_i + b_i^2\pi^{2e-1})) - \operatorname{ord}(\pi^{4(j-1)}(a_j^2 + 2a_jb_j + b_j^2\pi^{2e-1})) \geq 2$$

for any $i \neq j$ by $(*)$. Otherwise,

$$\operatorname{ord}(2a_ib_i\pi^{4(i-1)}) - \operatorname{ord} Q(v) \geq 4(i-1)+e+\operatorname{ord} a_i+\operatorname{ord} b_i-(e+\min\{4(k-1)+\operatorname{ord} a_k, 4(k-1)+\operatorname{ord} b_k\}) \geq 2$$

for any $i > k$ by the choice of k.

Therefore we obtain $\theta(O^+(L)) \subseteq (1+p^2)\dot{F}^2$ by [H, Prop. D].

When $e = 2$, the above arguments are still in force except

$$\begin{aligned}\operatorname{ord}(b_i^2\pi^{2e-1}\pi^{4(i-1)}) - \operatorname{ord} Q(v) &\geq 2\operatorname{ord} b_i + (2e-1) + 4(i-1) - (4(i-1) + \operatorname{ord} b_i + e)\\ &= e - 1 + \operatorname{ord} b_i \geq e - 1 = 1\,.\end{aligned}$$

Note that

$$\begin{aligned}Q(v) = {}&\Big(\sum_{i=1}^{n/2}\pi^{2(i-1)}a_i\Big)^2 + 2\Big(\sum_{i=1}^{n/2}\pi^{2(i-1)}a_i\Big)\Big(\sum_{i=1}^{n/2}\pi^{2(i-1)}b_i\Big)\\ &+ \Big(\sum_{i=1}^{n/2}\pi^{2(i-1)}b_i\Big)^2\pi^{2e-1} - 2\sum_{1\leq s<t\leq n/2}\pi^{2(s-1)+2(t-1)}a_sa_t\\ &- 2\sum_{1\leq s<t\leq n/2}\pi^{2(s-1)+2(t-1)}b_sb_t\pi^{2e-1}\\ &- 2\sum_{1\leq s\neq t\leq n/2}\pi^{2(s-1)+2(t-1)}a_sb_t\,.\end{aligned}$$

We have

$$\operatorname{ord}(2\pi^{2(s-1)+2(t-1)}b_sb_t\pi^{2e-1}) - \operatorname{ord} Q(v) \geq 2$$

and

$$\operatorname{ord}(2\pi^{2(s-1)+2(t-1)}a_sb_t) - \operatorname{ord} Q(v) \geq 2$$

for any $s \neq t$ by $(*)$. So we obtain $\theta(O^+(L)) = U\dot{F}^2 \cap Q([1, \dot{\pi}^3 - 1])$ by [X_0] and [X, Remark 1].

By the above theorem, we can improve [BD, Prop. 4.1], in fact, we can modify $s_p(n)$ appearing there as follows:

$$s_p(n) = \begin{cases} n(n-2)/2 & \text{if } p \text{ is nondyadic,}\\ s(n) & \text{if } p \text{ is dyadic,}\end{cases}$$

where

$$s(n) = \begin{cases} n(n-2) & \text{if } n \text{ is even,} \\ (n-1)^2 & \text{if } n \text{ is odd.} \end{cases}$$

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Received on 26.3.1991
and in revised form on 9.8.1991 and 3.8.1992 (2128)