An improved estimate concerning $3n + 1$ predecessor sets

by

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Introduction. Consider the following operator on the set of integers:

$$T(n) := \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even,} \\ \frac{1}{2}(3n + 1) & \text{if } n \text{ is odd.} \end{cases}$$

Now choose a starting number $x \in \mathbb{N}$, and look at its $3n + 1$ trajectory \( \{T^k(x) : k \geq 0\} \), where $T^k = T \circ \ldots \circ T$ denotes the $k$-fold iterate of $T$ for $k \geq 1$, and $T^0(x) = x$. The famous and unsolved $3n + 1$ conjecture says that any $3n + 1$ trajectory eventually hits 1, for any starting number $x \in \mathbb{N}$.

There is an extensive literature on associated problems and generalizations of this conjecture (see [3] and [4]).

This paper proves an estimate on the functions

$$\vartheta_a(x) := |\{n \in P(a) : n \leq ax\}|$$

where $P(a)$ denotes the $3n + 1$ predecessor set of $a \in \mathbb{Z}$, that is,

$$P(a) := \{n \in \mathbb{Z} : T^k(n) = a \text{ for some } k \geq 0\}.$$

The investigation of the set $P(1)$ began with Crandall [1] who succeeded in proving

$$\vartheta_1(x) \geq x^\beta$$

for some $\beta > 0$ and large $x$, where the exponent has been computed to be $\beta \approx 0.057$. In 1987, Sander [5] improved Crandall’s technique to show $\beta = \frac{1}{4}$ in (3). In 1989, Krasikov [2] introduced another technique to prove $\beta = \frac{3}{7}$. Here we extend Krasikov’s method to obtain the estimate

$$\vartheta_a(x) \geq x^{0.48}$$

for large $x$, if $a$ is not divisible by 3.

Starting out from the set of Krasikov’s inequalities given here in (7) it might be possible to get a further improvement of this exponent.
The improvement of Krasikov’s estimate. For a given positive integer \( v \) and a given positive real number \( x \), consider the set

\[
G(v, x) := \left\{ n \in \mathbb{N} : T^k(n) = v \text{ for some } k \geq 0 \right\}.
\]

In his paper [2], Krasikov defines a function \( f \) by

\[
f(v, x) = |G(v, x)|.
\]

Then he puts \( \Phi^m_n(y) := \inf \{ f(v, 2^ny) : v \text{ is noncyclic and } v \equiv m \mod 3^n \} \)

(an integer \( v \) is called noncyclic if \( T^k(v) \neq v \) for each \( k \geq 1 \)), which gives immediately the equation

\[
\Phi^m_{n-1}(y) = \min \{ \Phi^m_n(y), \Phi^{m+3^{n-1}}_n(y), \Phi^{m+2\cdot3^{n-1}}_n(y) \},
\]

and he proves the following set of inequalities:

\[
\begin{cases}
\Phi^m_n(y) \geq \Phi^{4m}_n(y-2) + \Phi^{(4m-2)/3}_n(y + \alpha - 2) & \text{if } m \equiv 2 \mod 9, \\
\Phi^m_n(y) \geq \Phi^{4m}_n(y-2) & \text{if } m \equiv 5 \mod 9, \\
\Phi^m_n(y) \geq \Phi^{4m}_n(y-2) + \Phi^{(2m-1)/3}_n(y + \alpha - 1) & \text{if } m \equiv 8 \mod 9
\end{cases}
\]

with the constant \( \alpha = \log_2 3 = 1.5849 \). Note that (5) implies \( \Phi^m_n(y) = 0 \) for \( y < 0 \), and that \( \Phi^m_n(y) \) is a nondecreasing function of \( y \). In addition, we have \( \Phi^m_n(0) \geq 1 \) by the fact that \( v \in G(v, v) \) gives \( f(v, 2^0v) \geq 1 \) for each integer \( v > 0 \).

Since \( G(a, ax) \subset \{ n \in \mathcal{P}(a) : n \leq x \} \), there is an obvious inequality between the functions \( \vartheta_a \) defined in (2) and the \( \Phi^m_n \), provided \( a \) is noncyclic:

\[
\vartheta_a(x) \geq \Phi^m_n(\log_2 x) \quad \text{if } a \equiv m \mod 3^n.
\]

Krasikov uses the set (7) of inequalities for \( n = 2 \) to prove \( \beta = \frac{3}{2} \) in the estimate (3), but he does not deal with \( n \geq 3 \). The following lemma provides the key to extract information out of (7) for the case \( n = 3 \).

**Lemma 1.**

\[
\Phi^2_2(y) \geq \sum_{k=0}^{\infty} \Phi^2_2(y-2 + k(\alpha - 4)).
\]

**Proof.** An immediate consequence of (7) is

\[
\Phi^2_2(y) \geq \Phi^2_2(y-2) + \Phi^2_1(y + \alpha - 2).
\]

Moreover, we have, like Krasikov in his proof of Theorem 1 in [2],

\[
\Phi^2_1(y) = \min \{ \Phi^2_2(y), \Phi^2_2(y), \Phi^2_2(y) \} \geq \Phi^2_2(y-2)
\]
since $\Phi_2^8(y) \geq \Phi_2^3(y - 2)$ by (7), and $\Phi_2^8(y) \geq 1 + \Phi_2^i(y + \alpha - 1) > \Phi_2^i(y)$, if $y \geq 2$. If $y < 2$ then (10) is obvious. (9) and (10) combine to give inductively

$$\Phi_2^i(y) \geq \sum_{k=0}^{n} \Phi_2^i(y - 2 + k(\alpha - 4)) + \Phi_2^i((y - 2 + n(\alpha - 4)) + \alpha - 2).$$

In what follows, the transcendental function

$$g(\lambda) := \lambda^{-12} + \lambda^{\alpha - 7} + \lambda^{\alpha - 6} + \frac{\lambda^{\alpha - 16} + \lambda^{\alpha - 5}}{1 - \lambda^{\alpha - 4}}$$

will play an essential rôle. $g(\lambda)$ is a decreasing function of $\lambda$ on the positive real axis, so there is a unique $\lambda_1 > 1$ with $g(\lambda_1) = 1$. This number $\lambda_1$ will be responsible for the exponent $\beta = 0.48 < \log_2 \lambda_1$ in the estimate (4).

**Proposition 2.** Let the real number $\lambda_0 > 1$ be given such that $g(\lambda_0) > 1$. Then $\Phi_2^8(y) \geq \lambda_0^y$ if $y$ is sufficiently large.

**Proof.** If we fix arbitrary numbers $\lambda > 1$ and $\tilde{y} > 0$, the facts that $\Phi_2^8$ is nondecreasing and $\Phi_2^8(0) \geq 1$ imply that there is a constant $c = c(\lambda, \tilde{y}) > 0$ such that

$$\Phi_2^8(y) \geq c\lambda^y \quad \text{for } 0 \leq y \leq \tilde{y}.$$ 

Now the idea is to show—using Krasikov’s inequalities (7)—that the condition $g(\lambda) > 1$ suffices to prolong the inequality (12) to all $y \geq 0$. Having done this prolongation, the claim follows by decreasing $\lambda$ slightly to get rid of the constant $c$, while restricting the range to all sufficiently large $y$.

The system (7) reads for $n = 3$:

$$\begin{align*}
\Phi_3^2(y) &\geq \Phi_3^3(y - 2) + \Phi_3^2(y + \alpha - 2), \\
\Phi_3^5(y) &\geq \Phi_3^2(y - 2), \\
\Phi_3^8(y) &\geq \Phi_3^5(y - 2) + \Phi_3^5(y + \alpha - 1), \\
\Phi_3^{11}(y) &\geq \Phi_3^{17}(y - 2) + \Phi_3^5(y + \alpha - 1), \\
\Phi_3^{14}(y) &\geq \Phi_3^2(y - 2), \\
\Phi_3^{17}(y) &\geq \Phi_3^{14}(y - 2) + \Phi_3^2(y + \alpha - 1), \\
\Phi_3^{20}(y) &\geq \Phi_3^{26}(y - 2) + \Phi_3^5(y + \alpha - 2), \\
\Phi_3^{23}(y) &\geq \Phi_3^{11}(y - 2), \\
\Phi_3^{26}(y) &\geq \Phi_3^{23}(y - 2) + \Phi_3^5(y + \alpha - 1).
\end{align*}$$

Since the functions $\Phi_n^m$ are nondecreasing, and because $\alpha > 1$ and $\Phi_3^0(0) \geq 1$, the last line of (13) implies $\Phi_3^{26}(y) \geq 1 + \Phi_3^5(y + \alpha - 1) > \Phi_3^5(y)$, provided $y \geq 2$. Hence we conclude by (6)

$$\Phi_2^8(y) = \min\{\Phi_3^8(y), \Phi_3^{17}(y)\} \quad \text{for } y \geq 2.$$
Starting with the third line of system (13) and running through this system, one arrives at the inequality
\[
\Phi_3^8(y) \geq \Phi_3^{17}(y - 12) + \Phi_2^8(y + \alpha - 1) + \Phi_2^5(y + \alpha - 6) \\
+ \Phi_2^8(y + \alpha - 7) + \Phi_2^5(y + \alpha - 12).
\]
By (7) and Lemma 1, one infers \(\Phi_2^8(y) \geq \sum_{k=0}^{n} \Phi_2^8(y - 4 + k(\alpha - 4))\) for any given integer \(n \geq 0\). If we put
\[
G_n(y) := \Phi_2^8(y - 12) + \Phi_2^8(y + \alpha - 6) + \Phi_2^5(y + \alpha - 7) \\
+ \sum_{k=0}^{n} \Phi_2^8(y + \alpha - 16 + k(\alpha - 4)) \\
+ \Phi_2^5(y + \alpha - 5 + k(\alpha - 4)),
\]
we come—using (14)—to the inequality
\[
\Phi_3^8(y) \geq G_n(y) \quad \text{for any } n \in \mathbb{N}.
\]
An inspection of (15) shows that \(G_n(y)\) needs the values of \(\Phi_2^8(x)\) only at points in the range
\[
y - 12 - (n + 1)(\alpha - 4) \leq x \leq y - (5 - \alpha).
\]
Fixing an arbitrary \(n \geq 0\) and a sufficiently large \(\bar{y}\), and calculating a constant \(c(\lambda, \bar{y})\) according to (12), we have
\[
G_n(y) \geq c(\lambda, \bar{y}) \lambda^{g_n(\lambda)} \quad \text{if } 12 + (n + 1)(4 - \alpha) \leq y \leq \bar{y} + (5 - \alpha),
\]
where
\[
g_n(\lambda) := \lambda^{-12} + \lambda^\alpha - 7 + \lambda^{\alpha - 6} + \sum_{k=0}^{n} (\lambda^{\alpha - 16 + k(\alpha - 4)} + \lambda^{\alpha - 5 + k(\alpha - 4)}).
\]
Analogously, chasing through the system (13) starting at the sixth line and using (14) gives
\[
\Phi_3^{17}(y) \geq \Phi_3^5(y - 6) + \Phi_2^8(y + \alpha - 6) + \Phi_2^5(y + \alpha - 1).
\]
As before, put
\[
H_n(y) := \Phi_2^5(y - 6) \\
+ \sum_{k=0}^{n} (\Phi_2^8(y + \alpha - 8 + k(\alpha - 4)) + \Phi_2^5(y + \alpha - 3 + k(\alpha - 4))),
\]
to get the inequality
\[
\Phi_3^{17}(y) \geq H_n(y) \quad \text{for any } n \in \mathbb{N}.
\]
Again we see that \(H_n(y)\) needs the values of \(\Phi_2^8(x)\) only at points in the range
\[
y - 4 - (n + 1)(\alpha - 4) \leq x \leq y - (3 - \alpha),
\]
and we have

\[ H_n(y) \geq c(\lambda, y) \lambda^y h_n(\lambda) \quad \text{if} \quad 4 + (n + 1)(4 - \alpha) \leq y \leq \bar{y} + (3 - \alpha), \]

with the abbreviation

\[ h_n(\lambda) := \lambda^{-6} + \sum_{k=0}^{n}(\lambda^{\alpha - 8 + k(\alpha - 4)} + \lambda^{\alpha - 3 + k(\alpha - 4)}). \]

Now the limiting functions

\[ g(\lambda) = \lim_{n \to \infty} g_n(\lambda) \quad \text{and} \quad h(\lambda) := \lim_{n \to \infty} h_n(\lambda) = \lambda^{-6} + \frac{\lambda^{\alpha - 8} + \lambda^{\alpha - 3}}{1 - \lambda^{\alpha - 4}} \]

are clearly decreasing in the range \( \lambda > 1 \). Hence, there are unique numbers \( \lambda_1, \lambda_2 > 1 \) with \( g(\lambda_1) = h(\lambda_2) = 1 \). A simple numerical calculation shows that \( \lambda_2 > \lambda_1 \).

Given a number \( \lambda_0 > 1 \) satisfying \( g(\lambda_0) > 1 \) as in the assumption of Proposition 2, we know that \( \lambda_0 < \lambda_1 \). Choose \( \lambda' \) with \( \lambda_0 < \lambda' < \lambda_1 \) and \( n' \) with the property

\[ g_n(\lambda') > 1 \quad \text{and} \quad h_n(\lambda') > 1 \quad \text{for} \quad n \geq n', \]

which is possible by (20). Moreover, put

\[ y_0 := 12 + (n' + 1)(4 - \alpha). \]

By the definition of \( c(\lambda', y_0) \) above (12), we have

\[ \Phi_2(y) \geq c(\lambda', y_0)(\lambda')^y \quad \text{for} \quad 0 \leq y \leq y_0. \]

Combine (14), (16), and (18) to get

\[ \Phi_2(y) = \min\{\Phi_3^5(y), \Phi_3^{17}(y)\} \geq \min\{G_{n'}(y), H_{n'}(y)\}. \]

This gives using (17) and (19)

\[ \Phi_2(y) \geq c(\lambda', y_0)(\lambda')^y \min\{g_{n'}(\lambda'), h_{n'}(\lambda')\} \quad \text{for} \quad y_0 \leq y \leq y_0 + (3 - \alpha) \]

\[ \geq c(\lambda', y_0)(\lambda')^y \]

where the last inequality is due to (21). Using in addition inequality (22), the claim \( \Phi_2(y) \geq c(\lambda', y_0)(\lambda')^y \) can be proved inductively on the intervals

\[ 0 \leq y \leq y_0 + k(3 - \alpha), \]

which completes the proof of Proposition 2.

\[ \text{Theorem 3.} \quad \text{For any integer} \quad a > 0 \quad \text{which is not divisible by} \quad 3, \quad \text{we have} \]

\[ \vartheta_a(x) \geq x^{0.48} \quad \text{if} \quad x \quad \text{is sufficiently large}. \]

\[ \text{Proof.} \quad \text{If} \quad a \equiv 8 \quad \text{mod} \quad 3^2, \quad \text{the result follows from} \quad (8) \quad \text{and Proposition 2:} \]

\[ \vartheta_a(x) \geq \Phi_2^5(\log_2 x) \geq x^{\log_2 \lambda_0} \quad \text{if} \quad x \quad \text{is sufficiently large}, \]

where \( \lambda_0 \) satisfies \( g(\lambda_0) > 1 \). The number \( \lambda_1 \) with \( g(\lambda_1) = 1 \) and its \( \log_2 \) are approximately (with an error \( < 10^{-3} \)) given by \( \lambda_1 \approx 1.397 \) and \( \log_2 \lambda_1 \approx 0.482 \), whence the result.
If, more generally, we have only $a \not\equiv 0 \mod 3$, it is easy to see that there is a noncyclic predecessor $b \in \mathcal{P}(a)$ satisfying $b \equiv 8 \mod 3^2$. But this means $T^k(b) = a$ for some $k$, whence

$$\vartheta_a(x) \geq \vartheta_b\left(\frac{ax}{b}\right) \geq \left(\frac{a}{b}\right)^\beta x^\beta \quad \text{if } x \text{ is sufficiently large}.$$ 

Applying the remarks following (12) to this inequality completes the proof. 

References


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Received on 20.11.1990
and in revised form on 12.11.1992