ACTA ARITHMETICA
LXIII.2 (1993)

On the zeros of $\zeta'(s) - a$

by

R. Balasubramanian (Madras) and K. Ramachandra (Bombay)

1. Introduction and notation. As usual we write $s = \sigma + it$. The main object of this paper is to prove the following

Theorem 1. *Let a be any complex constant and μ any real constant satisfying $0 < \mu < 1$. Then there exists a constant $\delta = \delta(a, \mu)$ (> 0) depending only on a and μ such that the number of distinct zeros of $\zeta'(s) - a$ in $(\sigma \geq 1/2 + \delta,\ T \leq t \leq T + T^{\mu})$ exceeds a positive constant multiple of T^{μ}.* (Hereafter we write $\gg T^{\mu}$ to mean this.)

Remark. The history of the theorem on the zeros of $\zeta'(s) - a$ is connected with the names B. C. Berndt [6] (see p. 287 of [8]) and N. Levinson and H. L. Montgomery [7] (see pp. 287–289 of [8]). See also Theorem 11.5(C) on p. 298 of [8] for the earlier history. But our Theorem 1 deals with the distribution of the zeros of $\zeta'(s) - a$ in "short t-slabs" in the "right half of the critical strip" (i.e. the region $\sigma \geq 1/2 + \delta$), as compared with earlier results. Another point about our proof of Theorem 1 is its novelty and its generality.

We will be concerned with proving a similar theorem for more general functions $G(s)$ defined for large σ by a convergent series $1 - \sum_{n=2}^{\infty} b_n \mu_n^{-s}$ where $\mu_2, \mu_3, \ldots$ are real numbers and $b_2, b_3, \ldots$ are complex numbers restricted by the following conditions:

(i) Put $\mu_1 = 1$. Then for $n = 1, 2, 3, \ldots$, $C_1^{-1} \leq \mu_{n+1} - \mu_n \leq C_1$ where C_1 (≥ 1) is any fixed constant,

(ii) $G(s)$ can be continued analytically in $(\sigma \geq 1/2,\ T \leq t \leq T + T^{\mu})$ and there the maximum of $|G(s)|$ is $\leq T^{C_2}$ where C_2 (> 0) is any fixed constant, and

(iii) $|b_n| < n^{C_3}$ where C_3 (> 0) is any constant.

Theorem 1 is a special case of

Theorem 2. *Let $\mu_n = (n_0 + n - 1) n_0^{-1}$ where n_0 is any integer constant ≥ 1. Let $\sum_{n=2}^{\infty} |b_n|^2 n^{-1-\varepsilon}$ be convergent for every $\varepsilon > 0$ and $\to \infty$ as $\varepsilon \to 0$.*

Let further

$$\lim_{\varepsilon\to 0}\inf\min_{q\ge 2}\Big\{q^2\Big(\sum_{n=2}^{\infty}|b_n|^2 n^{-1-\varepsilon}\Big)^{1/q}\Big({\sum}'|b_n|^2 n^{-1-\varepsilon}\Big)^{-1}\Big\}=0$$

where the accent denotes the sum over all integers $\ge n_0+1$ *for which* n_0+n-1 *is prime. Then* $G(s)$ *has* $\gg T^{\mu}$ *distinct zeros in* ($\sigma\ge 1/2+\delta$, $T\le t\le T+T^{\mu}$) *for every fixed* μ *with* $0<\mu<1$ *and a suitable constant* δ (>0) *depending only on* μ, C_1, C_2 *and* C_3.

Remark 1. Theorem 1 follows from the observations $\zeta'(s)=-2^{-s}(\log 2)G(s)$ for a suitable $G(s)$ and if $a\ne 0$, $\zeta'(s)-a=-aG(s)$ for a suitable $G(s)$.

Remark 2. Obviously we can deduce similar theorems for higher derivatives of $\zeta(s)$. We can also get similar theorems for the derivatives of ζ and L-functions and for ζ-functions of ray classes in any algebraic number field. However, we have neater theorems for $\zeta(s)-a$ ($a\ne 0$) and certain more general functions (compared with $\zeta(s)-a$ ($a\ne 0$)). But these need a different treatment and are dealt with in [5].

For general μ_n we need a conjecture (see Section 2) which is true for example for $\mu_n=n+\sqrt{2}$. It is also true for more general situations. Thus we can state

THEOREM 3. *Let* $b_2,b_3,\ldots$ *be non-negative real numbers and* $\mu_n=n+\sqrt{2}$ ($n\ge 2$). *Then subject to the only conditions that* $\sum_{n=2}^{\infty}|b_n|^2 n^{-1-\varepsilon}$ *shall be convergent for every* $\varepsilon>0$ *and that it shall tend to infinity as* $\varepsilon\to 0$, $G(s)$ *has* $\gg T^{\mu}$ *distinct zeros in* ($\sigma\ge 1/2+\delta$, $T\le t\le T+T^{\mu}$) *for every constant* μ ($0<\mu<1$) *and a suitable positive constant* δ *depending only on* μ, C_1, C_2 *and* C_3.

Remark 1. This theorem is true for all those sequences μ_n for which the conjecture in Section 2 is true.

Remark 2. This theorem includes functions like

$$1-\sum_{n=1}^{\infty}d(n)(n+\sqrt{2})^{-s},\qquad 1-\sum_{r}(N(r)+\sqrt{2})^{-s}$$

where r runs over all ideals in a ray class of an algebraic number field and so on.

As a closing remark in this section we mention that our method enables us to prove something stronger. For example in Theorems 1 and 3 we can prove that the number of zeros counted with multiplicity is either $\gg T^{\mu}\log T$ or there exist $\gg T^{\mu}$ distinct zeros of odd orders. For this we have to replace Lemma 1 of Section 3 by our Theorem 3 of [1].

2. Some preliminaries. In this section we collect together some results of our earlier papers and state a Local Convexity Theorem [3], a lower bound for the mean-value of Titchmarsh series [4] and two simple lemmas from [2]. We number these results with an asterisk thus: Theorems 1^*, 2^*, 3^* and 4^* and Lemmas 1^* and 2^*. Also we state a conjecture.

(Local Convexity) Theorem 1^*. *Suppose $f(s)$ is an analytic function of $s = \sigma + it$ defined in the rectangle*

$$R : \{a \le \sigma \le b,\ t_0 - H \le t \le t_0 + H\}$$

where a and b are constants with $a < b$. Let the maximum of $|f(s)|$ taken over R be $\le M$. Let $a \le \sigma_0 < \sigma_1 < \sigma_2 \le b$ and let A be any large positive constant. Let r be any positive integer, $0 < D < H$ and $s_1 = \sigma_1 + it_0$. Then

$$|2\pi f(s_1)| \le 2\{I_0^{\sigma_2-\sigma_1}(I_2 + M^{-A})^{\sigma_1-\sigma_0}\}^{(\sigma_2-\sigma_0)^{-1}} E_0^r$$
$$+2M^{A+2}(\sigma_2 - \sigma_0)\left(2\left(1 + \left(\log\left(\frac{D}{\sigma_1 - \sigma_0}\right)\right)^*\right)\right)^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}} \left(\frac{2E_0}{D}\right)^r,$$

where

$$E_0 = \exp\left(\frac{(\sigma_2 - \sigma_1)(\sigma_1 - \sigma_0)}{\sigma_2 - \sigma_0}\right),$$
$$I_0 = \int_{|v|\le D} \left| f(\sigma_0 + it_0 + iv)\frac{dv}{\sigma_0 - \sigma_1 + iv}\right|,$$
$$I_2 = \int_{|v|\le D} \left| f(\sigma_2 + it_0 + iv)\frac{dv}{\sigma_2 - \sigma_1 + iv}\right|,$$

and we have written $x^* = \max(0, x)$ for any real number x.

Remark. We have borrowed the result with $C = 1$ from our paper [3].

In the rest of this section we state a conjecture which gives a lower bound for the mean-value of Titchmarsh series. We then collect some special results where the conjecture is proved. We borrow Sections 2 and 3 of [2]. We believe that the following conjecture is true (at least in a modified form). We stipulate that certain constants shall be integers only for a technical reason which is not serious.

Conjecture. *Let $1 = \mu_1 < \mu_2 < \ldots$ be any sequence of real numbers with $C^{-1} \le \mu_{n+1} - \mu_n \le C$, where C (≥ 1) is an integer constant and $n = 1, 2, 3, \ldots$ Let us form the sequence $1 = \lambda_1 < \lambda_2 < \ldots$ of all possible distinct finite power products of $1 = \mu_1 < \mu_2 < \ldots$ with non-negative integral exponents. Let $s = \sigma + it$, H (≥ 10) a real parameter and $\{a_n\}$ $(n = 1, 2, 3, \ldots)$ with $a_1 = 1$ be any sequence of complex numbers (possibly depending on H) such that $F(s) = \sum_{n=1}^{\infty} a_n\lambda_n^{-s}$ is absolutely convergent* *at $s = B$ where B (≥ 3) is an integer constant. Suppose $F(s)$ can be continued analytically in $(\sigma \geq 0,\ 0 \leq t \leq H)$ and that there exist T_1, T_2, with $0 \leq T_1 \leq H^{3/4}$, $H - H^{3/4} \leq T_2 \leq H$, such that for some K (≥ 30)*

$$\max_{\sigma \geq 0}(|F(\sigma + iT_1)| + |F(\sigma + iT_2)|) \leq K\,.$$

Finally, let $\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B} \leq H^A$ where A (≥ 1) is an integer constant. Then there exists a δ_1 (> 0) (depending only on A, B, C) such that for all $H \geq H_0(A, B, C)$

$$\frac{1}{H}\int_0^H |F(it)|^2 dt \geq \frac{1}{2}\sum_{\lambda_n \leq H^{\delta_1}} |a_n|^2$$

provided $H^{-1}\log\log K$ does not exceed a small positive constant.

Remark 1. We have used the symbol δ_1 (in place of δ) so that it should not clash with the δ already introduced. Also we conjecture that $1/2$ can be replaced by a quantity ~ 1 provided $H^{-1}\log\log K \to 0$ as $H \to \infty$. Whenever we have succeeded in proving this conjecture we have proved it in this stronger form.

Remark 2. We need this conjecture only for a_n defined (for large σ) by

$$F(s) = (G(1/2 + \delta + s))^{1/q} = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

and for any fixed $\delta > 0$ and a suitable rectangle $(\sigma \geq 0,\ T \leq t \leq T + H)$ where $G(1/2 + \delta + s)$ has no zeros. We choose q to be any integer (≥ 2).

We now quote the corollaries to the main theorem of [4].

Theorem 2*. *Let $\mu_n = n$. Then the conjecture is true.*

Theorem 3*. *Let n_0 (≥ 2) be any integer constant and*

$$\mu_n = (n_0 + n - 1)n_0^{-1}\,.$$

Then the conjecture is true.

Theorem 4*. *Let β (> 0) be any algebraic constant and*

$$\mu_n = (n + \beta)(1 + \beta)^{-1}\,.$$

Then the conjecture is true (the conjecture is also true for the choice $\mu_1 = 1$ and $\mu_n = n + \beta - 1$ for $n > 1$).

Remark. It is possible to state a more general corollary than Theorem 4*. But we do not state it since our ambition is to prove a sufficiently general result.

We now record two important observations as Lemmas 1* and 2*.

Lemma 1*. *Let $\mu_n = (n_0 + n - 1)n_0^{-1}$ where n_0 (≥ 1) is an integer constant and $G(s) = 1 - \sum_{n=2}^{\infty} b_n \mu_n^{-s}$ be absolutely convergent for some complex s. Then we have, for any integer q (> 0) and all σ large enough, and any $\delta > 0$,*

$$G(1/2 + \delta + s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

where the λ's are formed as in the conjecture, $a_1 = 1$ and further whenever $n_0 + n - 1$ is prime, $|a_n| = q^{-1}|b_n|\mu_n^{-1/2-\delta}$ and so

$$\sum_{\lambda_n \leq H^{\delta_1}} |a_n|^2 \geq q^{-2} \sideset{}{'}\sum_{\mu_n \leq H^{\delta_1}} |b_n|^2 \mu^{-1-2\delta}$$

where the accent denotes the restriction of the sum to those n for which $n_0 + n - 1$ is prime.

P r o o f. See Lemma 1 of [2].

Lemma 2*. *Let $G(s) = 1 - \sum_{n=2}^{\infty} b_n \mu_n^{-s}$ where the b_n are non-negative and the sum involved converges for some complex s. Then for any integer q (> 0) and for all large σ and any $\delta > 0$*

$$(G(1/2 + \delta + s))^{1/q} = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

where the λ's are formed as in the conjecture, $a_1 = 1$ and further for $n \geq 2$, $a_n \leq 0$ and $-a_n \geq b_n \mu_n^{-1/2-\delta} q^{-1}$ whenever $\lambda_n = \mu_n$ and so

$$\sum_{\lambda_n \leq H^{\delta_1}} |a_n|^2 \geq q^{-2} \sum_{\mu_n \leq H^{\delta_1}} b_n^2 \mu_n^{-1-2\delta}\,.$$

P r o o f. Trivial.

3. An outline of the method. Instead of giving a detailed proof of Theorems 1, 2 and 3 we give a rough sketch of the proof. We begin with

Lemma 1. *Let $t_0 \geq 100$ and let $D(s)$ be any function analytic in ($\sigma \geq 1/2 + \delta$, $|t - t_0| \leq C(\delta)$) where δ is any positive constant and $C(\delta)$ is a large positive constant depending on δ and D_0 to follow. In this region let the maximum of $|D(s)|$ be $\leq M$ (≥ 30) and also $D(s) \neq 0$. Suppose further that for all σ exceeding a constant D_0 we have $|\log D(s)| \leq 1/2$. Then $\log D(s) = O(\log M)$ in ($\sigma \geq 1/2 + 3\delta/2$, $|t - t_0| \leq C(\delta)/2$) and $\log D(s) = O((\log M)^{\theta})$, with a θ (< 1) not depending on t_0 in ($\sigma \geq 1/2+2\delta$, $|t - t_0| \leq C(\delta)/3$). Here the O-constants depend on δ and D_0.*

R e m a r k. For the purposes of the present paper the conclusion $\log D(s) = o(\log M)$ will do in place of $O((\log M)^{\theta})$. But we have stated the lemma in this form since we will need it in a later paper [5].

P r o o f. The proof is essentially due to J. E. Littlewood. See pages 336–337 of [8] for a proof which can be easily generalised to give this lemma.

We introduce small positive constants ε_1, ε_2, ε_3, ε and δ (all $< 1/100$) and these will be fixed in a suitable way. The only free parameters will be T and H (H will be fixed to be a large constant in the end).

LEMMA 2. *Let* $A(s) = \sum_{n\le T^\nu} b_n\mu_n^{-s}$ *where* $b_1 = -1$, $\mu_1 = 1$ $\nu = \mu/2$ *and* b_n *as before. Then, for* $\sigma \ge 1/2+\delta$,

$$\int_T^{T+T^\mu} |A(s)|^2\,dt \le T^\mu(1+\varepsilon_2)\sum_{n=1}^{\infty}|b_n|^2\mu_n^{-2\sigma}\,,$$

for arbitrary $\varepsilon_2 > 0$ *and all large* T.

P r o o f. The lemma follows by standard arguments.

Divide the interval $[T, T+T^\mu]$ into N abutting t-intervals J (ignoring a bit at one end) each of length H where $N \sim T^\mu H^{-1}$. With any interval J associate the interval J_1 obtained by deleting intervals of length $H^{1/2}$ at both ends. If the maximum of $|G(s)|$ taken over the rectangle ($\sigma \ge 1/2+2\delta$, $t\in J_1$) is $\ge T^\varepsilon$, then by Lemma 1 (applied to $G(s)$ and by taking M to be a large positive constant power of T) the rectangle ($\sigma \ge 1/2+\delta$, $t \in J$) must contain a zero of $G(s)$. Otherwise we are easily led to the contradiction $T^\varepsilon = O(\exp((\log T)^\theta))$. If there are $\ge \varepsilon_1 N$ ($\varepsilon_1 > 0$ a small constant) such rectangles ($\sigma \ge 1/2+2\delta$, $t \in J_1$) where $\max|G(s)| \ge T^\varepsilon$ then there is nothing to prove, since each rectangle ($\sigma \ge 1/2+\delta$, $t\in J$) will contain a zero and the total number of zeros would be $\ge \varepsilon_1 N \gg T^\mu H^{-1}$. Hence we assume that the number of rectangles ($\sigma \ge 1/2+2\delta$, $t \in J_1$) where $\max|G(s)| \ge T^\varepsilon$ is $< \varepsilon_1 N$. Hence for $\ge (1-\varepsilon_1)N$ ($\sim (1-\varepsilon_1)T^\mu H^{-1}$) such rectangles ($\sigma \ge 1/2+2\delta$, $t\in J_1$) $\max|G(s)| < T^\varepsilon$. Denote the set of these rectangles by S. We prove that a (positive) constant proportion of these rectangles must contain a zero of $G(s)$. Denote by J_2 the t-intervals J_1 of S with intervals of length $H^{1/2}$ deleted from both ends. For $t_0 \in J_2$ we now apply convexity Theorem 1* taking $A = 1$ and M to be a large (positive) constant power of $T; r = [\varepsilon \log T], H$ large enough, $a = 1/2+2\delta, b$ a large (positive) constant independent of ε and $\delta, \sigma_0 = a, \sigma_2 = b, b/2 \ge \sigma_1 \ge 1/2+3\delta, D = \sqrt{H}$ and $f(s) = (G(s))^2 - (A(s))^2$ where $A(s)$ is as in Lemma 2. It follows that

$$|2\pi f(s_1)| \le 2\{I_0^{\sigma_2-\sigma_1}(I_2+T^{-3})^{\sigma_1-\sigma_0}\}^{(\sigma_2-\sigma_0)^{-1}}T^\varepsilon + T^{-3}\,, \tag{1}$$

where

$$I_0 = \int_{|v|\le\sqrt{H}} \left| f(\sigma_0+it_0+iv)\frac{dv}{\sigma_0-\sigma_1+iv}\right| \tag{2}$$

and

$$(3) \qquad I_2 = \int_{|v|\le\sqrt{H}} \left| f(\sigma_2 + it_0 + iv)\frac{dv}{\sigma_2 - \sigma_1 + iv}\right| .$$

Notice that for rectangles of S we have $|G(s)|^2 < T^{2\varepsilon}$. We now integrate (1) with respect to $t_0 \in J_2$. Now as t_0 varies over J_2, $t_0 + v$ runs over at most J_1. Thus we get from (1)

$$\frac{2\pi}{|J_2|}\int_{t_0\in J_2} |f(s_1)|\,dt_0$$
$$\le 2\biggl\{\biggl(\frac{1}{|J_2|}\int_{t_0\in J_2} I_0\,dt_0\biggr)^{\sigma_2-\sigma_1}\biggl(\frac{1}{|J_2|}\int_{t_0\in J_2} I_2\,dt_0 + T^{-3}\biggr)^{\sigma_1-\sigma_0}\biggr\}^{(\sigma_2-\sigma_0)^{-1}} + T^{-3}$$
$$= O\biggl(\biggl(\frac{1}{H}\int_{t_0\in J_1} |f(\sigma_0+it_0)|\,dt_0\biggr)^{\sigma_2-\sigma_1}$$
$$\times\biggl(\frac{1}{H}\int_{t_0\in J_1} |f(\sigma_2+it_0)|\,dt_0 + T^{-3}\biggr)^{\sigma_1-\sigma_0}\biggr)^{(\sigma_2-\sigma_0)^{-1}} + T^{-3}\,.$$

Hence

$$(4) \qquad \frac{1}{H}\int_{t_0\in J_2} |f(s_1)|\,dt_0$$
$$= O\biggl(\biggl\{\biggl(\frac{1}{H}\int_{t_0\in J_1} |A(\sigma_0+it_0)|^2\,dt_0 + T^{2\varepsilon}\biggr)^{\sigma_2-\sigma_1}$$
$$\times (T^{C_0-\mu\sigma_2/2} + T^{-3})^{\sigma_1-\sigma_0}\biggr\}^{(\sigma_2-\sigma_0)^{-1}}\biggr) + O(T^{-3})\,.$$

By Lemma 2 we have

$$\sum_{J_1}\int_{t_0\in J_1} |A(\sigma_0+it_0)|^2\,dt_0 < T^{\mu}(1+\varepsilon_2)\sum_{n=1}^{\infty}|b_n|^2\mu_n^{-1-4\delta}\,,$$

and so

$$\int_{t_0\in J_1} |A(\sigma_0+it_0)|^2\,dt_0 > H(1+\varepsilon_2)(1+3\varepsilon_2)\sum_{n=1}^{\infty}|b_n|^2\mu_n^{-1-4\delta}\,,$$

for at most $\le (1+3\varepsilon_2)^{-1}T^{\mu}H^{-1}$ intervals J_1. Hence for $\ge (1-2\varepsilon_1)T^{\mu}H^{-1} - (1+3\varepsilon_2)^{-1}T^{\mu}H^{-1} \gg T^{\mu}H^{-1}$ (provided $\varepsilon_2 = 10\varepsilon_1$) rectangles of S we have

$|G(s)|^2 < T^{2\varepsilon}$ and also $\int_{t_0 \in J_1} |A(\sigma_0 + it_0)|^2 \, dt_0 = O(H)$. Thus from (4) we obtain (for $\gg T^{\mu} H^{-1}$ out of these rectangles of S)

$$\frac{1}{H} \int_{t_0 \in J_2} |f(s_1)| \, dt_0 = O((T^{2\varepsilon(\sigma_2 - \sigma_1)} T^{-3(\sigma_1 - \sigma_0)})^{(\sigma_2 - \sigma_0)^{-1}} + T^{-3})$$

by choosing σ_2 such that $C_0 - \mu\sigma_2/2 \leq -3$. Now choose ε such that

$$2\varepsilon(\sigma_2 - \sigma_1) \leq 3(\sigma_1 - \sigma_0) - \varepsilon \, .$$

Then we have

$$\frac{1}{H} \int_{t_0 \in J_2} |f(s_1)| \, dt_0 = O(T^{-\varepsilon(\sigma_2 - \sigma_0)^{-1}}) \, .$$

Again since $|G(s_1)|^2 \leq |f(s_1)| + |A(s_1)|^2$ we have

$$\frac{1}{H} \int_{t_0 \in J_2} |G(s_1)|^2 \, dt_0 \leq (1 + 10\varepsilon_2) \sum_{n=1}^{\infty} |b_n|^2 \mu_n^{-2\sigma_1} \, , \tag{5}$$

by Lemma 2. Since the absolute value of an analytic function at a point is majorised by its mean-value taken over a disc of radius $\delta/10$ about that point as centre, we see that the maximum of $|G(s)|$ taken over the rectangle ($\sigma \geq 1/2 + 3\delta$, $t \in J_3$) is $\leq H^2$ (J_3 being J_2 with the intervals of length $H^{1/2}$ deleted at both ends). Hence by Theorems 2^*, 3^* and 4^* and Lemmas 1^* and 2^* we have

$$\frac{1}{H} \int_{t_0 \in J_3} |G(1/2 + 3\delta + it_0)|^{2/q} \, dt_0 \geq q^{-2}(1 - \varepsilon_3) \sideset{}{'}\sum_{\mu_n \leq H^{\delta_1}} |b_n|^2 \mu_n^{-1-6\delta}$$

where the accent indicates the inclusion of only those n for which $|a_n| \geq q^{-1}|b_n|\mu_n^{-1/2-3\delta}$. Also ε_3 is an arbitrarily small constant which is fixed. (We have applied the theorems and lemmas with δ replaced by 3δ.) We compare this with the upper bound

$$\Big\{(1 + 10\varepsilon_2)\Big(\sum_{n=1}^{\infty} |b_n|^2 \mu_n^{-1-6\delta}\Big)\Big\}^{1/q}$$

obtained from (5) by Hölder's inequality. This leads to a contradiction (for $\gg T^{\mu} H^{-1}$ out of the rectangles of S) to our assumption of Theorem 2 (resp. Theorem 3). This completes the proof of Theorems 1, 2 and 3 stated in the introduction.

Acknowledgements. The authors are indebted to Professors M. Jutila, A. Ivić, D. R. Heath-Brown and B. C. Berndt for encouragement.

References

[1] R. Balasubramanian and K. Ramachandra, *On the zeros of a class of generalised Dirichlet series. III*, J. Indian Math. Soc. 41 (1977), 301–315.

[2] —, —, *On the zeros of a class of generalised Dirichlet series. XI*, to appear.

[3] —, —, *Some local convexity theorems for the zeta-function-like analytic functions*, Hardy–Ramanujan J. 11 (1988), 1–12.

[4] —, —, *Proof of some conjectures on the mean-value of Titchmarsh series. III*, Proc. Indian Acad. Sci. Math. Sci. 102 (1992), 83–91.

[5] —, —, *On the zeros of* $\zeta(s) - a$, Acta Arith., to appear.

[6] B. C. Berndt, *On the number of zeros of* $\zeta^{(k)}(s)$, J. London Math. Soc. (2) 2(1970), 577–580.

[7] N. Levinson and H. L. Montgomery, *Zeros of the derivative of the Riemann zeta-function*, Acta Math. 133 (1974), 49–65.

[8] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed. (revised and edited by D. R. Heath-Brown), Clarendon Press, Oxford 1986.

MATSCIENCE
THARAMANI P.O.
MADRAS 600 113
INDIA

SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
COLABA
BOMBAY 400 005
INDIA

Received on 8.5.1992 (2258)