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Simultaneous diophantine approximation with square-free numbers

by

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1. Introduction. The well known theorems of Dirichlet and Kronecker in the theory of diophantine approximation have been generalized in many directions. We mention here a result of Heath-Brown [6]. Let α be irrational. Then there are infinitely many pairs n, m of square-free numbers such that

$$|n\alpha - m| < n^{\varepsilon - 2/3}\,. \tag{1}$$

Here and throughout ε is an arbitrarily small but positive real number. In particular, writing $\|\gamma\|$ for the distance of γ to the nearest integer we deduce from (1) that $\|\alpha n\| < n^{\varepsilon-2/3}$ has infinitely many square-free solutions.

In the present paper we investigate simultaneous approximations by square-free numbers. For a given set of real numbers $\alpha_1, \ldots, \alpha_s$ we wish to prove that $\|\alpha_1 n\|, \ldots, \|\alpha_s n\|$ are all small for infinitely many square-free integers n. This is certainly not possible without a further hypothesis on $\alpha_1, \ldots, \alpha_s$. It is obviously necessary that whenever $l_1, \ldots, l_s$ are integers such that

$$\sum_{j=1}^{s} l_j \alpha_j = \frac{u}{v} \in \mathbb{Q}, \qquad (u, v) = 1\,, \tag{2}$$

then v must be square-free. A set of real numbers satisfying this condition will be called *weakly compatible.*

THEOREM 1. *Let $\alpha_1, \ldots, \alpha_s$ be a set of weakly compatible algebraic numbers such that $1, \alpha_1, \ldots, \alpha_s$ span a linear space of dimension $d \ge 2$ over $\mathbb{Q}$. Then for any $A < 1/d(d-1)$ there are infinitely many square-free numbers n satisfying*

$$\|\alpha_j n\| < n^{-A} \qquad (j = 1, \ldots, s)\,. \tag{3}$$

Moreover, if $d = 2$, any $A < 2/3$ is admissible.

A similar result with square-free numbers replaced by primes has been established recently by Harman [5], improving on results of Balog and Friedlander [2]. In the case of primes, the range for A is shorter by a factor 2 (even worse when $d = 2$), and a stronger compatibility condition is required. Balog and Friedlander called a set $\alpha_1, \ldots, \alpha_s$ of real numbers *compatible* if (2) implies that $v = 1$.

The question arises whether weak compatibility is sufficient to prove a result of the form (3). It is not difficult to see that

$$\liminf_{n\to\infty,\, \mu^2(n)=1} \max_{1\le j\le s} \|\alpha_j n\| = 0 \tag{4}$$

whenever $\alpha_1, \ldots, \alpha_s$ form a weakly compatible set. The next theorem, which sharpens Theorem 2 of Harman [5], shows that (4) is best possible.

THEOREM 2. *Let $f(n)$ be any function tending to zero as n tends to infinity. Then there are uncountably many pairs of real numbers α, β such that $1, \alpha, \beta$ are linearly independent over $\mathbb{Q}$, but*

$$\max(\|\alpha n\|, \|\beta n\|) < f(n) \tag{5}$$

has at most finitely many square-free solutions n.

Next we state a more general version of Theorem 1.

THEOREM 3. *Let $\alpha_1, \ldots, \alpha_s$ be a weakly compatible set of real numbers contained in a linear space of dimension d over $\mathbb{Q}$ spanned by $1, \alpha_1, \ldots, \alpha_{d-1}$. Write*

$$r = \sup\Big\{\gamma : \liminf_{N\to\infty} N^{\gamma} \min_{0<|\mathfrak{l}|\le N} \Big\| \sum_{j=1}^{d-1} \alpha_j l_j \Big\| = 0\Big\}. \tag{6}$$

Then for any $A < ((d-1)(r+1))^{-1}$, there are infinitely many solutions to (3) *in square-free integers n.*

By Schmidt's theorem on linear forms with algebraic coefficients ([7], Theorem 7C) Theorem 3 implies Theorem 1, at least when $d \ge 3$. Subject to a stronger hypothesis it is also possible to prove an inhomogeneous version of Theorem 3, with the n restricted to arithmetic progressions.

THEOREM 4. *Let $\alpha_1, \ldots, \alpha_s$ be a set of real numbers such that $1, \alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Q}$, and define r as in Theorem* 3 (*note that now $d = s+1$*). *Let real numbers $\beta_1, \ldots, \beta_s$ be given. Suppose that g, G are integers with (g, G) square-free. Then for any $A < (s(r+1))^{-1}$ there are infinitely many square-free integers $n \equiv g \bmod G$ with*

$$\max \|n\alpha_j + \beta_j\| < n^{-A}.$$

These results should be compared with Theorems 3 and 4 of Harman [5].

Our main task is proving Theorem 4. We begin with a lemma on exponential sums in §2 which is then used in §3 to establish Theorem 4. We deduce Theorem 3 from Theorem 4 in §4, and then prove Theorem 1 in the special case $d = 2$. Finally, we prove Theorem 2; our method closely follows that of [5], Theorem 2.

2. An exponential sum. In this section we suppose that N is a large real number, $D \le N^{1/2}$, and we write $M = Nd^{-2}$ throughout. Moreover, G is a fixed natural number. We shall be concerned with the exponential sum

$$S = S(\alpha, D) = \sum_{d\sim D} \sum_{h=1}^{G} \Big| \sum_{m\sim M} e\Big(\Big(\alpha + \frac{h}{G}\Big) d^2 m\Big)\Big| . \tag{7}$$

Here the notation $x \sim X$ indicates the condition $X < x \le 2X$.

LEMMA 1. *Suppose* $1 \le B \le N^{1/3-\varepsilon}$. *Then either* $S \le NB^{-1}\log^2 N$, *or there exists a natural number* q *such that*

$$q \le N^{\varepsilon} \min(D^2, B), \qquad \|q\alpha\| \le BN^{\varepsilon-1} .$$

Proof. Suppose that $S \ge NB^{-1}\log^2 N$. Then for some h with $1 \le h \le G$ we must have $S_h \ge G^{-1}NB^{-1}\log^2 N$; here S_h denotes the contribution to (7) with a fixed h.

First suppose that $D \le B^{1/2}$. We put $X = G^{-1}N^{1-\varepsilon}B^{-1}$. By Dirichlet's theorem, there is a natural number t and an integer b with $t \le X$ and $|t(\alpha + h/G) - b| \le X^{-1}$. Now, by a standard argument,

$$NB^{-1}\log^2 N \ll S_h \ll \sum_{d\sim D} \Big\| \Big(\alpha + \frac{h}{G}\Big) d^2 \Big\|^{-1} .$$

If we had $t > G^{-1}D^2N^{\varepsilon}$ then $4D^2t^{-2} < (2t)^{-1}$, and hence the right hand side here would be bounded by

$$\ll \sum_{d\sim D} \Big\| \frac{bd^2}{t} \Big\|^{-1} \ll t\log N \ll X\log N .$$

This is a contradiction. Hence $t \le G^{-1}N^{\varepsilon}D^2$, and the lemma follows with $q = tG$.

Now suppose that $B^{1/2} < D \le GB$, and pick t and b as before. By Lemma 3.2 of Baker [1],

$$\begin{aligned} S_h &\ll \sum_{d\sim D} \min\Big(M, \Big\| \Big(\alpha + \frac{h}{G}\Big) d^2 \Big\|^{-1}\Big) \\ &\ll \sum_{u\le 4D^2} \min\Big(ND^{-2}, \Big\| \Big(\alpha + \frac{h}{G}\Big) u \Big\|^{-1}\Big) \end{aligned}$$

$$\ll (ND^{-2}+t)(D^2t^{-1}+1)\log N\,.$$

We immediately deduce that $t \le G^{-1}N^{\varepsilon}B$ whenever $S_h \gg NB^{-1}\log^2 N$, and the proof is completed as before.

If $D > GB$ we may use the trivial bound $S \le GND^{-1} \le NB^{-1}$ to complete the proof of the lemma.

3. Proof of Theorem 4. When $s = 1$ Theorem 4 can be proved by a simple adjustment of the argument used to establish Theorem 2 of Harman [4]. When $s \ge 2$ we prove Theorem 4 by contradiction. Observe that for (g, G) square-free the interval $[N, 2N]$ contains $\gg N$ square-free numbers $n \equiv g \bmod G$. Now put $L = (2N)^A$ and suppose that there are no square-free solutions to

$$\|\alpha_1 n\| < L^{-1}, \quad \ldots, \quad \|\alpha_s n\| < L^{-1}$$

with $n \sim N$ and $n \equiv g \bmod G$. Then, writing

$$\phi(\boldsymbol{k}) = \sum_{j=1}^{s} \alpha_j k_j\,, \tag{8}$$

$$T(\alpha) = \sum_{n\sim N,\, n\equiv g \bmod G} \mu^2(n)e(\alpha n)\,, \tag{9}$$

a familiar argument shows that

$$\sum_{0<|\mathsf{k}|\le LN^{\varepsilon}} |T(\phi(\boldsymbol{k}))| \gg N\,. \tag{10}$$

From the identity

$$\mu^2(n) = \sum_{d^2\,|\,n} \mu(d)$$

we deduce that

$$\begin{aligned} T(\alpha) &= \frac{1}{G}\sum_{n\sim N}\sum_{d^2|n}\mu(d)e(\alpha n)\sum_{h=1}^{G} e\biggl(\frac{h(n-g)}{G}\biggr) \\ &= \frac{1}{G}\sum_{h=1}^{G} e\biggl(-\frac{hg}{G}\biggr)\sum_{d\le N^{1/2}}\mu(d)\sum_{m\sim Nd^{-2}} e\biggl(\biggl(\alpha+\frac{h}{G}\biggr)d^2m\biggr). \end{aligned}$$

By a splitting up argument and (10) there is a D with $1 \le D \le N^{1/2}$ and

$$\sum_{d\sim D}\sum_{h=1}^{G}\sum_{0<|\mathsf{k}|\le LN^{\varepsilon}}\biggl|\sum_{m\sim Nd^{-2}} e\biggl(\biggl(\phi(\boldsymbol{k})+\frac{h}{G}\biggr)md^2\biggr)\biggr| \gg N(\log N)^{-1}\,.$$

Using the notation from the previous section we can rewrite this as

$$\sum_{0<|\mathbf{k}|\le LN^\varepsilon} S(\phi(\boldsymbol{k}),D) \gg N(\log N)^{-1}\,.$$

Another splitting up argument shows that there is a $K \ll (LN^\varepsilon)^s$ and a set of points $\mathcal{K}\subset\mathbb{Z}^s$ with $|\mathcal{K}|=K$, such that $|\boldsymbol{k}|\le LN^\varepsilon$ and

$$S(\phi(\boldsymbol{k}),D) \gg NK^{-1}(\log N)^{-1} \tag{11}$$

for all $\boldsymbol{k}\in\mathcal{K}$. Note that for $s\ge 2$ we have $L^s < N^{1/3-2s\varepsilon}$. Moreover, the left hand side of (11) is $\ll ND^{-1}$ by a trivial estimate. This shows that $K \gg D(\log N)^{-1}$. We use Lemma 1 to infer that for any $\boldsymbol{k}\in\mathcal{K}$ there is a natural number $q=q(\boldsymbol{k})$ such that $q \ll N^\varepsilon \min(D^2,K)$ and

$$\|q(\boldsymbol{k})\phi(\boldsymbol{k})\| \ll N^{\varepsilon-1}K\,.$$

Now $q\phi(\boldsymbol{k})=\phi(q\boldsymbol{k})$. By a familiar divisor argument we deduce that there are $\gg KN^{-\varepsilon}$ points $\boldsymbol{n}$ in the region $|\boldsymbol{n}|\le KLN^{2\varepsilon}$ with

$$\|\phi(\boldsymbol{n})\| \ll N^{\varepsilon-1}K\,. \tag{12}$$

By the pigeon hole principle we find an $\boldsymbol{n}$ satisfying (12) with

$$|\boldsymbol{n}| \ll K^{1-1/s}LN^{2\varepsilon}\,.$$

By the definition of r in Theorems 3 and 4, we must have

$$\|\phi(\boldsymbol{n})\| \gg (K^{1-1/s}LN^{2\varepsilon})^{-r-\varepsilon}\,,$$

so that by (12),

$$(K^{1-1/s}LN^{2\varepsilon})^{-r-\varepsilon} \ll N^{\varepsilon-1}K\,. \tag{13}$$

Recall that $K \ll L^s N^{s\varepsilon}$. Now (13) produces a contradiction if ε is sufficiently small.

4. Proof of Theorem 3. In view of Theorem 4 we may suppose that $s\ge d$. There are integers $D\ge 1, a_{ij}, k_i$ such that

$$D\alpha_i = \sum_{j=1}^{d-1} a_{ij}\alpha_j + k_i, \qquad i=d,\ldots,s\,.$$

Let $t_d,\ldots,t_s$ be integers; then

$$D(\alpha_d t_d+\ldots+\alpha_s t_s) = \sum_{j=1}^{d-1}\alpha_j\Big(\sum_{i=d}^{s} a_{ij}t_i\Big) + \sum_{i=d}^{s} t_i k_i\,. \tag{14}$$

Because $\alpha_1,\ldots,\alpha_s$ are weakly compatible, it is clear from (14) that

$$\gcd\Big(p^h, \sum_{i=d}^{s} a_{i,1}t_i,\ldots,\sum_{i=d}^{s} a_{i,d-1}t_i\Big) \,\Big|\, p\sum_{i=d}^{s} t_i k_i \tag{15}$$

for any prime p having $p^h \| D$. By Lemma 3 of Harman [5] we may infer from (15) that there is a solution in integers $b_1, \ldots, b_{d-1}$ to the set of congruences

$$\sum_{j=1}^{d-1} a_{ij} b_j \equiv p k_i \bmod p^h, \qquad i = d, \ldots, s.$$

Now let g be an integer satisfying

$$g \equiv p \bmod p^h \qquad \text{for all } p^h \| D. \tag{16}$$

By the Chinese remainder theorem there are integers $b_1, \ldots, b_{d-1}$ satisfying

$$\sum_{j=1}^{d-1} a_{ij} b_j \equiv g k_i \bmod D, \qquad i = d, \ldots, s. \tag{17}$$

From (16) we see that (g, D) is square-free. According to Theorem 4, there are infinitely many square-free numbers n satisfying

$$\left\| \frac{n\alpha_j + b_j}{D} \right\| < n^{-A-\varepsilon} \qquad (1 \le j \le d-1) \tag{18}$$

and

$$n \equiv g \bmod D, \tag{19}$$

providing A is as in Theorem 3 and ε is sufficiently small. We see at once that

$$\|n\alpha_j\| < D n^{-A-\varepsilon} < n^{-A} \qquad (1 \le j \le d-1)$$

for all large n satisfying (18) and (19). Now let $d \le i \le s$. From (17)–(19), any such n satisfies

$$\begin{aligned} \|n\alpha_i\| &= \Big\| \sum_{j=1}^{d-1} n a_{ij} \alpha_j D^{-1} + n k_i D^{-1} \Big\| \\ &= \Big\| \sum_{j=1}^{d-1} a_{ij} \Big(\frac{n\alpha_j + b_j}{D} \Big) + \frac{1}{D} \Big(n k_i - \sum_{j=1}^{d-1} a_{ij} b_j \Big) \Big\| \\ &\le \sum_{j=1}^{d-1} |a_{ij}| \left\| \frac{n a_j + b_j}{D} \right\| < n^{-A}. \end{aligned}$$

This completes the proof of Theorem 3.

We can now sketch a proof of (4). If $\alpha_1, \ldots, \alpha_s$ are linearly independent over $\mathbb{Q}$ then the exponential sum estimates in §§2–3 are readily modified to show that $T(\phi(\boldsymbol{k})) = o(N)$ for any $\boldsymbol{k} \in \mathbb{Z}^s$, $\boldsymbol{k} \ne 0$. Thus the vectors $(\alpha_1 n, \ldots, \alpha_s n)$ with $n \equiv g \bmod G$, $\mu^2(n) = 1$, are uniformly distributed in the s-dimensional unit cube, providing we have chosen g and G with

(g, G) square-free. This establishes (4). The general case then follows by the argument used to prove Theorem 3.

5. Theorem 1 when $d = 2$**.** Theorem 1 follows from Theorem 3 when $d \geq 3$. When $d = 2$, however, Theorems 4 and 3 yield an admissible range $A < 1/2$ only. We now show how to enlarge this range to $A < 2/3$. A careful inspection of the work in the previous section shows that all what is required is (18) with (19) when $d = 2$ and $A < 2/3$. We simplify the notation from §4 to $\alpha_1 = \alpha$, $b_1 = b$. Hence it remains to prove:

PROPOSITION. *Suppose that* b, g, D *are fixed integers with* (g, D) *square-free, and* α *irrational. Then there are infinitely many square-free numbers* n *satisfying*

$$\left\| \frac{n\alpha + b}{D} \right\| < n^{\varepsilon - 2/3}, \qquad n \equiv g \bmod D\,. \tag{20}$$

Note that (20) is equivalent to $|n\alpha + b + tD| < Dn^{\varepsilon - 2/3}$ for some $t \in \mathbb{Z}$. Hence it suffices to show that there are infinitely many pairs $(m, n) \in \mathbb{Z}^2$ with

$$|n\alpha - m| < n^{\varepsilon - 2/3}, \quad \mu^2(n) = 1, \quad n \equiv g \bmod D, \quad m \equiv b \bmod D\,. \tag{21}$$

This can be established by closely following the argument from §3 of Heath-Brown [6]. We may suppose that $\alpha > 0$. Let a/q be any convergent to α so that $|q\alpha - a| < q^{-1}$. Let $0 < \theta < 2/3$ and define

$$N = q^{2/(1+\theta)}, \qquad L = Nq^{-1}(\log q)^{-1}\,,$$

$$\mathcal{S} = \{(l, m, n) \in \mathbb{Z}^3 : 1 \leq l \leq L, n \sim N, n \equiv g \bmod D,$$
$$m \equiv b \bmod D, an - qm = l\}.$$

For $(l, m, n) \in \mathcal{S}$ we have $|n\alpha - m| \leq 8n^{-\theta}$ so that it suffices to bound

$$R = \sum_{(l,m,n) \in \mathcal{S}} \mu^2(n)$$

from below. Let $z = \log q$ and P be the product of all primes not exceeding z. Then define

$$f(n) = \sum_{d^2 | n,\, d | P} \mu(d)\,.$$

As in Heath-Brown [6, (13)] we have

$$R \geq A - \sum_{p > z} C_p \tag{22}$$

where

$$A = \sum_{(l,m,n) \in \mathcal{S}} f(n), \qquad C_p = \sum_{(l,m,n) \in \mathcal{S}, p^2 | n} 1\,.$$

Note that the C_p defined in Heath-Brown [6] are no smaller than our C_p so that we may quote the bound

$$\sum_{p>z} C_p \ll \frac{NL}{q\log\log q} \tag{23}$$

from [6], p. 344. It now suffices to show that $A \gg NLq^{-1}$.

We begin the evaluation of A by writing $n = e^2v$ and obtain

$$A = \sum_{e\mid P} \mu(e)\#\{(l,m,v)\in\mathbb{Z}^3 : 1\le l\le L, v\sim Ne^{-2}, m\equiv b \bmod D, \\ e^2v\equiv g \bmod D, ae^2v - qm = l\}\,.$$

Here we put $m = b + m'D$ to see that

$$\begin{aligned} A &= \sum_{e\mid P} \mu(e)\#\{(l,m',n) : qb < l \le L+qb,\ v\sim Ne^{-2}, \\ &\qquad\qquad e^2v\equiv g \bmod D,\ ae^2v - qDm' = l\} \\ &= \sum_{\delta\mid D}\ \sum_{\substack{e\mid P\\ (e^2,D)=\delta}} \mu(e)A_e, \qquad \text{say}\,. \end{aligned}$$

Note that $e^2v \equiv g \bmod D$ gives $\delta\,|\,g$, otherwise $A_e = 0$. In particular, $\delta\,|\,(D,g)$ which implies that δ is square-free whence $\delta\,|\,e$. The congruence reduces to

$$\delta^{-1}e^2v \equiv g\delta^{-1} \bmod D\delta^{-1}\,.$$

This fixes a certain congruence class, $g' \bmod D\delta^{-1}$ say, in which v must lie. We write $v = g' + uD\delta^{-1}$ and find that

$$\begin{aligned} A_e &= \#\{(l,m,u) : qb < l \le L+qb,\ g'+uD\delta^{-1}\sim Ne^{-2}, \\ &\qquad\qquad ae^2(g'+uD\delta^{-1}) - qDm = l\} \\ &= \#\{(l,u) : qb - ae^2g' < l \le L+qb-ae^2g', \\ &\qquad Ne^{-2}-g' < D\delta^{-1}u \le 2Ne^{-2}-g',\ ae^2D\delta^{-1}u \equiv l \bmod qD\}\,. \end{aligned}$$

We now write $\Delta = (ae^2D\delta^{-1}, qD)$; then $\Delta\,|\,l$. We set $l = \Delta k$ and deduce that

$$\begin{aligned} A_e &= \#\bigg\{(k,u) : (qb-ae^2g')\Delta^{-1} < k \le (L+qb-ae^2g')\Delta^{-1}, \\ &\qquad \frac{\delta}{D}(Ne^{-2}-g') < u \le \frac{\delta}{D}(2Ne^{-2}-g'),\ \frac{ae^2D}{\delta\Delta}u \equiv k \bmod \frac{qD}{\Delta}\bigg\} \\ &= \bigg(\frac{L}{\Delta}+O(1)\bigg)\bigg(\frac{\delta}{D}N\frac{\Delta}{qDe^2}+O(1)\bigg) = \frac{LN\delta}{qe^2D^2} + O\bigg(L+\frac{N}{q}\bigg)\,. \end{aligned}$$

The number of $e \mid P$ is $O(q^{\varepsilon})$. We conclude that

$$A = \frac{LN}{qD^2} \sum_{\delta \mid (D,g)} \sum_{\substack{e \mid P \\ (e^2,D)=\delta}} \mu(e)\frac{\delta}{e^2} + O(Lq^{\varepsilon} + Nq^{\varepsilon-1}) . \tag{24}$$

The error term is $o(LNq^{-1})$ as q tends to infinity. Recall that the summation conditions imply that $\delta \mid e$. Write $e = \delta d$; then

$$\sum_{\delta \mid (D,g)} \sum_{\substack{e \mid P \\ (e^2,D)=\delta}} \mu(e)\frac{\delta}{e^2} = \sum_{\delta \mid (D,g)} \frac{\mu(\delta)}{\delta} \sum_{\substack{d \mid P\delta^{-1} \\ (d^2\delta, D\delta^{-1})=1}} \frac{\mu(d)}{d^2} .$$

The summation condition gives $(d,\delta) = 1$. The previous line now becomes

$$\sum_{\substack{\delta \mid (D,g) \\ (\delta, D\delta^{-1})=1}} \frac{\mu(\delta)}{\delta} \sum_{\substack{d \mid P\delta^{-1} \\ (d, D\delta^{-1})=1}} \frac{\mu(d)}{d^2} = \prod_{\substack{\pi \mid (D,g) \\ \pi^2 \nmid D}} \left(1 - \frac{1}{\pi}\right) \prod_{\substack{p \le z \\ p \nmid D}} \left(1 - \frac{1}{p^2}\right).$$

Here p and π denote primes. As $z \to \infty$ this product converges to a positive number $c = c(g, D)$. The Proposition follows from (22)–(24).

6. Proof of Theorem 2. We use standard notation and results on continued fractions. For definitions and proofs we refer to Hardy and Wright [3], Chapter 10. We shall determine uncountably many sequences $(a_j), (b_j)$ of natural numbers such that (5) can have at most a finite number of solutions if α, β are given by

$$\alpha = [1, a_1, a_2, \ldots], \qquad \beta = [1, b_1, b_2, \ldots] .$$

We write

$$[1, a_1, \ldots, a_t] = \frac{p_t}{q_t}, \qquad [1, b_1, \ldots, b_t] = \frac{r_t}{s_t} .$$

Then we have

$$\left|\alpha - \frac{p_t}{q_t}\right| < \frac{1}{q_t q_{t+1}}, \qquad \left|\beta - \frac{r_t}{s_t}\right| < \frac{1}{s_t s_{t+1}} \tag{25}$$

and

$$\begin{gathered} q_0 = s_0 = 1, \qquad q_1 = a_1, \qquad s_1 = b_1, \\ q_t = a_t q_{t-1} + q_{t-2}, \qquad s_t = b_t s_{t-1} + s_{t-2} \qquad (t \ge 2) . \end{gathered} \tag{26}$$

As in [5] it suffices to prove the theorem for functions f which are non-increasing and satisfy $f(n) < 1/2$ for all n.

Let (ε_j) be an arbitrary sequence of zeros and ones. Let a_1 be the smallest integer with $a_1 \ge 2$, $f(a_1) < 1/4$. For $j \ge 1$ let b_j be the least integer with

$$b_j \ge 2q_j + \varepsilon_j, \qquad f(b_j - \varepsilon_j) < (4q_j)^{-1} ,$$

and, for $j \geq 2$,

$$b_j s_{j-1} \equiv -s_{j-2} \bmod p_j^2 \,. \tag{27}$$

Here p_j is the least prime exceeding s_{j-1}.

For $j \geq 2$, let a_j be the smallest integer satisfying the conditions

$$a_j \geq 2s_{j-1}, \quad f(a_j) < (4s_{j-1})^{-1}, \quad a_j q_{j-1} \equiv -q_{j-2} \bmod P_j^2. \tag{28}$$

Here P_j is the least prime exceeding q_{j-1}.

For $j \geq 2$ this gives, by (26), $s_j > 2q_j s_{j-1}$ and $q_j > q_{j-1}s_{j-1}$, and in particular, $s_j \geq 2q_j$ and $q_j \geq 2s_{-1}j$.

Now let n be a square-free number with $s_j/2 < n \leq q_{j+1}/2$ for some $j \geq 2$. By (28) and (26) we have $q_j \nmid n$. Hence, by (25),

$$\|n\alpha\| \geq \left\|\frac{np_j}{q_j}\right\| - n\left|\alpha - \frac{p_j}{q_j}\right| \geq \frac{1}{q_j} - \frac{n}{q_j q_{j+1}} \geq \frac{1}{2q_j} > f(b_j) \geq f(s_j/2) \geq f(n)\,.$$

A similar argument shows that $\|n\beta\| > f(n)$ whenever n is square-free and lies in the range $q_j/2 < n \leq s_j/2$ for some $j \geq 3$. This shows that any square-free solution to (5) has $n \leq q_3/2$. Of course, different choices of (ε_j) produce different numbers (α, β). As on p. 412 of Harman [5] it can be shown that $1, \alpha, \beta$ are linearly independent over the rationals.

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