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On a conjecture of A. Schinzel and H. Zassenhaus

by

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1. Introduction. Let α be a non-zero algebraic integer of degree d and $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ its conjugates. Write

$$\overline{|\alpha|} = \max_{1\le i\le d} |\alpha_i|, \qquad M(\alpha) = \prod_{t=1}^{d} \max(1, |\alpha_t|).$$

Suppose that α is not a root of unity. In 1933 Lehmer [6] posed the following question: is it true that there exists an absolute constant $\delta > 0$ such that $M(\alpha) > 1 + \delta$? In 1965 Schinzel and Zassenhaus [8] conjectured that $\overline{|\alpha|} > 1 + c/d$, $c > 0$. It is known (see [9]) that if α is not reciprocal and θ is the real zero of the polynomial $z^3 - z - 1$, then $M(\alpha) \ge \theta$. Hence $\overline{|\alpha|} \ge M(\alpha)^{1/d} > 1 + c/d$.

Now let α be reciprocal, i.e. $d = 2m$, $m \in \mathbb{N}$, $\alpha_{m+1} = 1/\alpha_1$, $\alpha_{m+2} = 1/\alpha_2, \ldots, \alpha_{2m} = 1/\alpha_m$. Various estimates from below for $M(\alpha)$ and $\overline{|\alpha|}$ were obtained by Blanksby and Montgomery [1], Stewart [10], Dobrowolski [3]. In 1979 Dobrowolski [4] proved that

$$M(\alpha) > 1 + (1-\varepsilon)\left(\frac{\log\log d}{\log d}\right)^3, \qquad d > d_1(\varepsilon). \tag{1}$$

Since $M(\alpha) \le \overline{|\alpha|}^{\,d/2}$, (1) implies

$$\overline{|\alpha|} > 1 + (2-\varepsilon)\left(\frac{\log\log d}{\log d}\right)^3 \frac{1}{d}, \qquad d > d_2(\varepsilon). \tag{2}$$

Later, Cantor and Straus [2] replaced the constant $1-\varepsilon$ by $2-\varepsilon$ in (1) and $2-\varepsilon$ by $4-\varepsilon$ in (2) respectively. In 1983 Louboutin [7] obtained $\frac{9}{4}-\varepsilon$ in (1). Thus (2) holds with the constant $\frac{9}{2}-\varepsilon$ instead of $2-\varepsilon$. In the present paper the following theorem is proved.

Theorem. *If α is a non-zero algebraic integer of degree d and α is not a root of unity, then for any $\varepsilon > 0$ there exists an effective constant $d_0(\varepsilon)$*

such that for $d > d_0(\varepsilon)$

$$\overline{|\alpha|} > 1 + \left(\frac{64}{\pi^2} - \varepsilon\right)\left(\frac{\log\log d}{\log d}\right)^3 \frac{1}{d}\,. \tag{3}$$

We shall need the following lemma.

Lemma. *If*

$$D = |a_{ij} x_j^{i-1}|_{i,j=1,2,\ldots,n}, \tag{4}$$

where $a_{ij},\ x_j \in \mathbb{C}$ *and* $|x_1| \geq |x_2| \geq \ldots \geq |x_n|$, *then*

$$|D| \leq |x_1|^{n-1}|x_2|^{n-2}\ldots|x_{n-1}| \prod_{j=1}^{n}\Big(\sum_{i=1}^{n}|a_{ij}|^2\Big)^{1/2}. \tag{5}$$

In the case of Vandermonde's determinant $a_{ij} = 1$ this lemma was proved in [5].

2. Proof of the Lemma. If $x_j = 0$, where $1 \leq j \leq n-1$, then $x_{j+1} = x_{j+2} = \ldots = x_n = 0$ and $D = 0$. Hence, let $x_j \neq 0$, $1 \leq j \leq n-1$, and $y_1 = x_2/x_1$, $y_2 = x_3/x_2, \ldots, y_{n-1} = x_n/x_{n-1}$. We express the determinant (4) in the form

$$D = \sum_{\sigma}(-1)^{\sigma} a_{\sigma(1),1} a_{\sigma(2),2} \ldots a_{\sigma(n),n} x_1^{\sigma(1)-1} x_2^{\sigma(2)-1} \ldots x_n^{\sigma(n)-1}\,,$$

where σ is a permutation of the set $\{1, 2, \ldots, n\}$. Since

$$\begin{aligned}
\prod_{j=1}^{n} x_j^{\sigma(j)-1} &= \prod_{j=1}^{n} x_j^{n-j} \prod_{j=1}^{n} x_j^{\sigma(j)+j-n-1} \\
&= \prod_{j=1}^{n} x_j^{n-j} \prod_{j=1}^{n}\Big(x_1 \prod_{t=1}^{j-1} y_t\Big)^{\sigma(j)+j-n-1} \\
&= \prod_{j=1}^{n} x_j^{n-j} x_1^{\Sigma_{j=1}^{n}(\sigma(j)+j-n-1)} \prod_{t=1}^{n-1} y_t^{\Sigma_{j=t+1}^{n}(\sigma(j)+j-n-1)} \\
&= \prod_{j=1}^{n} x_j^{n-j} \prod_{t=1}^{n-1} y_t^{\Sigma_{j=t+1}^{n}(\sigma(j)+j-n-1)}
\end{aligned}$$

and

$$\sum_{j=t+1}^{n}(\sigma(j)+j-n-1)$$

$$= \sum_{j=t+1}^{n} \sigma(j) + \frac{n(n+1) - t(t+1)}{2} - (n-t)(n+1)$$

$$\geq 1 + 2 + \ldots + (n-t) - \frac{(n-t)(n-t+1)}{2} = 0\,,$$

we obtain

$$D = \prod_{j=1}^{n} x_j^{n-j} P(y_1, y_2, \ldots, y_{n-1})\,. \tag{6}$$

Here P is a polynomial in $y_1, y_2, \ldots, y_{n-1}$.

On the other hand,

$$D = \prod_{j=1}^{n} x_j^{n-j} \left| a_{ij} \frac{x_j^{i-1}}{x_{n-i+1}^{i-1}} \right|_{i,j=1,2,\ldots,n}$$
$$= \prod_{j=1}^{n} x_j^{n-j} \left| a_{ij} \left(\frac{x_1 y_1 y_2 \ldots y_{j-1}}{x_1 y_1 y_2 \ldots y_{n-i}} \right)^{i-1} \right|_{i,j=1,2,\ldots,n}.$$

Using the maximum modulus principle and the inequalities $|y_j| = |x_{j+1}/x_j| \leq 1$, $j = 1, 2, \ldots, n-1$, we have

$$|P(y_1, y_2, \ldots, y_{n-1})| \leq |P(y_1^0, y_2^0, \ldots, y_{n-1}^0)|\,, \tag{7}$$

where $|y_1^0| = |y_2^0| = \ldots = |y_{n-1}^0| = 1$. From (6), (7) and Hadamard's inequality we find

$$|P(y_1^0, y_2^0, \ldots, y_{n-1}^0)| \leq \prod_{j=1}^{n} \Big(\sum_{i=1}^{n} |a_{ij}|^2 \Big)^{1/2}.$$

Hence the inequality (5) holds.

3. Proof of the Theorem. Define

$$h_0(z) = h(z) = \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{n-1} \end{pmatrix},$$

$$h_k(z) = \frac{z^k}{k!} \, \frac{d^k h(z)}{d^k z} = \begin{pmatrix} 0 \\ \vdots \\ \binom{n-2}{k} z^{n-2} \\ \binom{n-1}{k} z^{n-1} \end{pmatrix}, \qquad k \in \mathbb{N}\,.$$

Consider the determinant

$$D_1 = |h_{u_r}(\alpha_j^{p_r})|\,, \tag{8}$$

where D_1 consists of $n = (k_0 + k_1 + \ldots + k_s)d$ columns, $u_r = 0, 1, \ldots, k_r - 1$; $j = 1, 2, \ldots, d$. Here p_r is rth prime number ($p_0 = 1$, $p_1 = 2$, $p_2 = 3, \ldots$),

$$s = \left[\frac{\pi^2}{16}\left(\frac{\log d}{\log\log d}\right)^2\right], \qquad k_0 = \left[\frac{\pi^3}{64}\left(\frac{\log d}{\log\log d}\right)^3\right],$$

$$k_r = \left[s\cos\frac{\pi(r-1)}{2s}\right], \qquad 1 \le r \le s\,.$$

Recall that α is reciprocal. Therefore the determinant (8) can be expressed in the form

$$D_1 = \sum_{(v)} c_v \alpha_1^{v_1}\alpha_2^{v_2}\ldots\alpha_m^{v_m}\,,$$

where $v_i \in \mathbb{Z}$, $i = 1, 2, \ldots, m$; $m = d/2$. Let $\varrho = \lceil\overline{\alpha}\rceil$. Then $1/\varrho \le |\alpha_i| \le \varrho$, $i = 1, 2, \ldots, m$. By the maximum modulus principle we obtain $|D_1| \le |D_2|$, where

$$D_2 = \sum_{(v)} c_v \beta_1^{v_1}\beta_2^{v_2}\ldots\beta_m^{v_m} = |h_{u_r}(\beta_j^{p_r})|\,,$$

$$|\beta_j| \in \{1/\varrho, \varrho\}, \quad j = 1, 2, \ldots, m, \quad |\beta_{m+1}| = 1/|\beta_1|, \ \ldots, \ |\beta_{2m}| = 1/|\beta_m|\,.$$

Assume without loss of generality that $|\beta_1| = |\beta_2| = \ldots = |\beta_m| = \varrho$, $|\beta_{m+1}| = \ldots = |\beta_{2m}| = 1/\varrho$. Let also $x_u = \varrho^{p_{s-r}}, x_{n+1-u} = \varrho^{-p_{s-r}}$, where the indices u are defined as follows:

$$m\sum_{i=0}^{r-1} k_{s-i} < u \le m\sum_{i=0}^{r} k_{s-i}, \qquad r = 0, 1, 2, \ldots, s\,.$$

Now using the lemma we find

$$|D_2| \le \varrho^A \Big(d\sum_{i=0}^{s} k_i\Big)^{\frac{d}{2}\Sigma_{i=0}^{s} k_i^2}\,. \tag{9}$$

Here

$$\begin{aligned} A &= \sum_{r=0}^{s} p_{s-r}\Big(\sum_{j=1}^{k_{s-r}m}\Big(n - m\sum_{i=0}^{r-1}k_{s-i} - j\Big) \\ &\quad - \sum_{j=1}^{k_{s-r}m}\Big(m\sum_{i=0}^{r-1}k_{s-i} + j - 1\Big)\Big) \\ &= \sum_{r=0}^{s} p_{s-r}\sum_{j=1}^{k_{s-r}m}\Big(n - 2m\sum_{i=0}^{r-1}k_{s-i} - 2j + 1\Big) \\ &= \sum_{r=0}^{s} p_{s-r}\sum_{j=1}^{k_{s-r}m}\Big(2m\sum_{i=0}^{s-r}k_i - 2j + 1\Big) \end{aligned} \tag{10}$$

$$= \sum_{r=0}^{s} p_{s-r}\Big(2k_{s-r}m^2 \sum_{i=0}^{s-r} k_i - k_{s-r}^2 m^2\Big)$$
$$< 2m^2 \sum_{r=0}^{s} p_{s-r}k_{s-r} \sum_{i=0}^{s-r} k_i = \frac{d^2}{2} \sum_{i=0}^{s} p_i k_i \sum_{j=0}^{i} k_j\,.$$

Since $|D_1| \ge \prod_{r=1}^{s} p_r^{k_r k_0 d}$ (see [7]), from (9), (10) we get

$$k_0 \sum_{r=1}^{s} k_r \log p_r \le \tfrac{1}{2} d \log \varrho \sum_{i=0}^{s} p_i k_i \sum_{j=0}^{i} k_j + \tfrac{1}{2} \log\Big(d \sum_{i=0}^{s} k_i\Big) \sum_{i=0}^{s} k_i^2\,.$$

For d tending to infinity the following asymptotic formulas hold:

$$k_0 \sum_{r=1}^{s} k_r \log p_r \sim \frac{\pi^6}{4096} \frac{(\log d)^7}{(\log\log d)^6}\,,$$
$$\frac{1}{2} \sum_{i=0}^{s} p_i k_i \sum_{j=0}^{i} k_j \sim \frac{\pi^8}{1048576} \frac{(\log d)^{10}}{(\log\log d)^9}\,,$$
$$\frac{1}{2} \log\Big(d \sum_{i=0}^{s} k_i\Big) \sum_{i=0}^{s} k_i^2 \sim \frac{3\pi^6}{16384} \frac{(\log d)^7}{(\log\log d)^6}\,.$$

If $d > d_0(\varepsilon)$, then

$$\overline{|\alpha|} = \varrho > 1 + \log \varrho > 1 + \left(\frac{64}{\pi^2} - \varepsilon\right)\left(\frac{\log\log d}{\log d}\right)^3 \frac{1}{d}\,.$$

The proof of the inequality (3) is thus complete.

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