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Points at rational distance from the vertices of a triangle

by

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The following will be proved:

Theorem. *Let ABC be a triangle such that the length of at least one side is rational and the squares of the lengths of all sides are rational. Then the set of points P whose distances PA, PB, PC to the vertices of the triangle are rational is dense in the plane of the triangle.*

Almering [1] established the theorem for a triangle with all sides of rational length. According to [3, p. 34] J. Leech has an elementary proof of this case. The proof of the theorem given here uses, as does Almering's proof of the special case, results from the theory of elliptic curves, but otherwise involves only elementary projective geometry.

Explicit constructions of points at rational distance from the vertices of equilateral and isosceles right triangles can be found in [2]–[5]. It was the study of points at rational distance from the vertices of an isosceles right triangle in [2], which suggested the present proof.

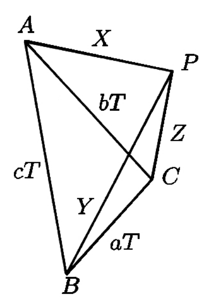

Fig. 1

P r o o f o f t h e t h e o r e m. The notation is illustrated in Figure 1. As usual, a, b, c denote the lengths of the sides CB, AC, AB. In order to have homogeneous equations, we introduce the scale factor T and consider a triangle of lengths aT, bT, cT. Then the necessary and sufficient condition that three non-negative reals X, Y, Z be the distances PA, PB, PC for some point P is

$$\begin{aligned}(1)\quad & a^2X^4+b^2Y^4+c^2Z^4+a^2b^2c^2T^4 \\ &= (a^2+b^2-c^2)(X^2Y^2+c^2Z^2T^2)+(b^2+c^2-a^2)(Y^2Z^2+a^2X^2T^2) \\ &\quad +(c^2+a^2-b^2)(X^2Z^2+b^2Y^2T^2)\,.\end{aligned}$$

This relation was known in the last century. Necessity can be established by applying the cosine formula to triangles APB, PCA, ABC, and eliminating the angles $\angle CAB$, $\angle CAP$. A far neater derivation, by Cayley, is indicated in [8, p. 134, Ex. 4]. It is clear from (1) that the condition that the squares of the lengths of the sides be rational is necessary for the theorem to hold.

We view equation (1) as defining a quartic surface S in complex projective 3-space with homogeneous coordinates (X, Y, Z, T). It turns out that S is a Kummer surface, that is, S has exactly 16 singular points, all nodes. This is established directly by finding the singularities of S by calculating explicitly the simultaneous solutions of the equations obtained from (1) by taking partial derivatives of both sides. The result is the set of 16 solutions given in Table 1. (The calculation was computer-assisted.) The most famous property of Kummer surfaces (cf. [6]) is the existence of the 16_6 configuration: there are 16 planes, called the singular tangent planes of S; each node of S lies on six singular tangent planes, and each plane contains six nodes. The singular tangent planes are given in Table 2, and Table 3 gives the incidence diagram of the configuration: column i names the nodes that lie on the singular tangent plane i, or dually names the singular tangent planes passing through node i.

Under the hypotheses of the theorem to be proved, S is defined over $\mathbb{Q}$ and we shall prove that $S(\mathbb{Q})$ is dense in $S(\mathbb{R})$. This clearly implies the theorem. (Here as usual $X(K)$ denotes the set of points of the variety X whose coordinates lie in the field K.) However, in the following two propositions the arguments are entirely geometric and independent of any hypotheses on the numbers a, b, c. Thus, in order to keep the structure of the arguments clear, we do not yet introduce any restrictions on a, b, c — until further notice they can be any non-zero complex numbers.

Table 1. The nodes of S

1. $(1\ \ {-1}\ 1\ 0)$	5. $(b\ \ {-a}\ 0\ 1)$	9. $(c\ 0\ \ {-a}\ 1)$	13. $(0\ c\ \ {-b}\ 1)$
2. $(-b\ a\ 0\ 1)$	6. $(-1\ 1\ 1\ 0)$	10. $(0\ c\ b\ \ {-1})$	14. $(c\ 0\ a\ \ {-1})$
3. $(-c\ 0\ a\ 1)$	7. $(0\ c\ b\ 1)$	11. $(1\ 1\ 1\ 0)$	15. $(b\ a\ 0\ \ {-1})$
4. $(0\ \ {-c}\ b\ 1)$	8. $(c\ 0\ a\ 1)$	12. $(b\ a\ 0\ 1)$	16. $(1\ 1\ \ {-1}\ 0)$

Table 2. The singular tangent planes of S

1. $X + Y = cT$	5. $aX + bY = -cZ$	9. $Y + Z = -aT$	13. $X - Z = bT$
2. $aX + bY = cZ$	6. $X + Y = -cT$	10. $X + Z = -bT$	14. $Y - Z = aT$
3. $-Y + Z = aT$	7. $-X + Z = bT$	11. $-X + Y = cT$	15. $-aX + bY = cZ$
4. $X + Z = bT$	8. $Y + Z = aT$	12. $aX - bY = cZ$	16. $X - Y = cT$

Table 3. The 16_6 configuration

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11

Each singular tangent plane cuts out doubly on S a conic, called a *trope*. For example, $X+Y=cT$ cuts out the conic whose equations in 3-space are

$$(2) \qquad X + Y = cT\,, \qquad c(X^2 - Z^2 + b^2T^2) + (a^2 - b^2 - c^2)XT = 0\,.$$

This can be seen by eliminating Y between $X+Y=cT$ and equation (1). The result is the second equation of (2), squared. We shall denote by C_i the trope cut out by singular tangent plane i.

An intuitive hold on S can be obtained by means of the classical representation of a Kummer surface as a double plane branched over six lines tangent to a conic. The representation is obtained by projecting the surface away from a node onto a plane. This means the following. Let N be a node of S, and let Π be any plane not passing through N. Define $p : S - N \to \Pi$ by sending $P \in S - N$ to the intersection of the line PN with Π. A line has in general four intersections with S, but a node counts as two, so that p is a generically 2:1 map from $S - N$ to Π. Each trope through N projects into a line, namely the intersection of the plane of the trope with Π. Since the plane of the trope cuts out the trope doubly, its image on Π is part of the branch locus. This gives six branch lines coming from the six tropes containing N, and one sees without much difficulty that this is all the branch locus, and that the six branch lines are tangent to the conic which is the intersection of the Zariski tangent cone at N with Π. (We shall not need this last statement.) In our case we take N as node 11 and Π as $Z = 0$. In coordinates, the projection is given by $p(X,Y,Z,T) = (X-Z, Y-Z, 0, T)$. The branch locus consists of the lines $X = \pm bT$, $Y = \pm aT$, $-X+Y = cT$, $X - Y = cT$, in the plane $Z = 0$.

Now let us consider the pencil of curves $\{E_\lambda\}$ cut out on S by planes passing through nodes 1 and 11. To be quite specific, let E_λ denote the curve cut out on S by the plane $X = Z + \lambda T$, where $\lambda \in \mathbb{P}^1 = \mathbb{C} \cup \infty$. Each E_λ is a binodal plane quartic (except for finitely many values of λ when it acquires extra singularities or is reducible), and as such it has genus 1, by the usual genus formula for plane curves. On a non-singular model of S, the inverse images of the E_λ form a pencil of curves of genus 1, without base points (once the fixed components which are the inverse images of nodes 1 and 11 are removed). We shall apply some known theorems on such pencils. For a useful summary of the theory, and complete bibliography, see [9]. In what follows, however, to avoid introducing extra notation, we abuse language by suppressing mention of the non-singular model. Specifically, the term "fibre" will always refer to a member of the pencil on the non-singular model, and if a curve of S is referred to as a section it is understood that it is the proper transform of the curve which is the section (i.e. it is unisecant to the fibres). Recall that if an elliptic pencil admits sections, then once one section is chosen as the zero — call it, temporarily, C_0 — the set of all sections has the structure of a group. Moreover, when the set of non-singular points on any particular fibre is given the structure of a group by taking as zero the intersection of the fibre with C_0, then the map from the set of sections to the points of the fibre given by intersection is a group homomorphism.

For fixed λ, let L_λ be the line in $\mathbb{P}^3$ with equations $Z = 0$, $X = \lambda T$. Points on L_λ can be given as $(\lambda, Y, 0, 1)$ so we consider Y as a non-homogeneous coordinate on L_λ.

Proposition 1. *For any λ such that E_λ is a binodal quartic, the projection p, restricted to E_λ , represents E_λ as the double cover of L_λ branched over points $Y = \pm a, \lambda \pm c$.*

Proof. Since E_λ lies in a plane which passes through the vertex of projection, the image of E_λ is the intersection of its plane with the plane of projection; this is the line L_λ. Since $p : S - N \to \Pi$ represents S as a branched double cover of Π, the restriction of p to E_λ has degree at most 2. It cannot have degree 1 since then it would give a birational equivalence of E_λ with L_λ, contradicting the hypothesis that E_λ has genus 1. Thus the restriction of p represents E_λ as a branched double cover of L_λ. By the general theory of double covers, the branch points of the restriction of p to E_λ are the simple points of the intersections of L_λ with the branch locus of p (so the point $(0, 1, 0, 0)$ which is on $X = bT$ and $X = -bT$ is a double point of the branch locus of p and does not contribute). Calculation of the intersections of L_λ with the branch lines gives the stated branch points.

Proposition 2. (a) *Any trope which passes through one of nodes* 1 *and* 11 *but not both is a section of the pencil* $\{E_\lambda\}$.

(b) *If one of the tropes through node* 11 *but not node* 1 *is taken as the* *zero for the group law on the sections*, *then the tropes through node* 1 *but not* 11 *have infinite order in the group of sections.*

Proof. The proof will only be sketched, since a similar argument appears in [2] for the case of the isosceles right triangle. Assertion (a) can most elegantly be established by calculating intersections on a non-singular model of S but it also follows from Proposition 1. Indeed, away from node 11, the intersections of the tropes through node 11 but not 1 with an E_λ are ramification points of the projection p, and by Proposition 1 there are 4 of these, one on each trope. Thus each trope has just one intersection with E_λ away from node 11, so that its proper transform is a section. This proves that the nodes through 11 but not 1 are sections. A similar argument using the projection from node 1 proves that tropes through 1 but not 11 are sections.

(b) It is an immediate consequence of the definition of the group law that if an elliptic curve is represented as a double cover of the line and one branch point is chosen as the zero of the group law, then the remaining branch points are the 2-torsion elements of the group. Thus it follows from the proof of (a) that when one trope through node 11 and not 1 is taken as the zero for the group law on the sections, then the remaining tropes through node 11 but not 1 are 2-torsion elements. Assertion (b) will therefore be established if it is shown that the torsion subgroup of the group of sections has order 4.

The torsion in the group of sections injects into the torsion of the group of non-singular points on any fibre, so it is enough to find a fibre whose group of non-singular points has torsion of order 4. Now the singular tangent planes 7 and 13 pass through nodes 1 and 11 and so cut out members of $\{E_\lambda\}$. On S the planes cut out tropes; it is easily seen that the corresponding fibres on the minimal non-singular model of S have Kodaira type I_0^*; indeed, on resolving singularities by blowing up the nodes each node becomes a $\mathbb{P}^1$ with self-intersection -2 and the trope another such $\mathbb{P}^1$ transversal to the inverse images of the nodes. The inverse images of nodes 1 and 11 are to be discarded, and what remains is an I_0^*. The group of non-singular points on an I_0^* is known to have torsion $\mathbb{Z}/2 \times \mathbb{Z}/2$; thus (b) is proved.

Remark. Proposition 2 gives as much information on the group of sections of $\{E_\lambda\}$ as is needed for present purposes. The entire structure of the group can, however, be elucidated. This is much easier than it would be for a general elliptic surface, or even for a general K_3 surface, because one can call into play the Abelian surface A which is a double cover of the Kummer surface. For example, noting that the plane $X = 0$ cuts out on S a pair of conics through four nodes, it follows that A is isogeneous to the product $E_1 \times E_2$ where the E_i are elliptic curves defined as the double covers of the two conics ramified over the nodes. A calculation shows that E_1 and

E_2 are isogeneous, so that finally A is isogeneous to $E \times E$ where E is a known elliptic curve. It follows that $\varrho(A)$, the rank of the Neron–Severi group of A, is 3 or 4 according as E does or does not have complex multiplication. (Both cases can occur. For the surface arising from an isosceles right triangle there is no complex multiplication, but for the equilateral triangle there is.) By a theorem of Shioda, the rank of the Neron–Severi group of S is $16 + \varrho(A)$ and from this the rank of the group of sections of E_λ can be determined, using further results of Shioda. Pushing these ideas further one can calculate generators for the Neron–Severi group of S and the group of sections of $\{E_\lambda\}$. This program is carried through in [2] for the surface S arising from an isosceles right triangle.

Now we introduce the hypotheses of the theorem to be proved and assume that a, b, c are lengths of the sides of a triangle ABC, and that $a^2, b^2, c \in \mathbb{Q}$. Then the surface S, the projection p and the pencil $\{E_\lambda\}$ are all defined over $\mathbb{Q}$, as are the tropes C_{11}, C_1, C_6 (cf. Table 2; it is here that the hypothesis that $c \in \mathbb{Q}$ is used). From now on we fix C_{11} as the zero for the group law on the group of sections of $\{E_\lambda\}$. Since C_{11} is defined over $\mathbb{Q}$ we obtain a group law on E_λ defined over $\mathbb{Q}$ for any $\lambda \in \mathbb{Q}$, whose zero is $C_{11} \cap E_\lambda$. We shall show that $S(\mathbb{Q})$ is dense in $S(\mathbb{R})$, by showing that, for almost all rational values of λ, the set $E_\lambda(\mathbb{Q})$ is dense in $E_\lambda(\mathbb{R})$. By the theorem of Hurwitz and Poincaré (cf. [10, p. 78]) if E is an elliptic curve defined over $\mathbb{Q}$ which has a rational point on each connected component of $E(\mathbb{R})$ and a rational point of infinite order, then $E(\mathbb{Q})$ is dense in $E(\mathbb{R})$.

We now show that, for almost all $\lambda \in \mathbb{Q}$, the elliptic curve E_λ satisfies the hypotheses of the theorem of Hurwitz and Poincaré. We first deal with the existence of points of infinite order on E_λ, $\lambda \in \mathbb{Q}$. Since C_1 is defined over $\mathbb{Q}$ and passes through node 1 but not 11 we have, by Proposition 2(b), an element of infinite order, defined over $\mathbb{Q}$, in the group of sections of $\{E_\lambda\}$. But, by a theorem of Silverman [7, p. 306], if C is any element of infinite order, defined over $\mathbb{Q}$, in the group of sections, then for almost all $\lambda \in \mathbb{Q}$ the intersection of C with E_λ has infinite order in $E_\lambda(\mathbb{Q})$. Thus with finitely many exceptions, if $\lambda \in \mathbb{Q}$ then $E_\lambda \cap C_1$ is a point of infinite order in $E_\lambda(\mathbb{Q})$.

We next show that the rational points on E_λ given by the intersections of C_{11}, C_{16} and C_1 with E_λ never lie all on the same connected component of $E_\lambda(\mathbb{R})$. This will follow from examination of the relative positions of the projections of these points on L_λ. The projections of the first two are calculated in Proposition 1 and we proceed to calculate the projection of the third. The trope C_1 is given by equations (2) above. On the other hand, if (X, Y, Z, T) in the plane $X + Y = cT$ projects under p to $(X^*, Y^*, 0, T^*)$ then

$$X = cT^* + X^* - Y^*\,, \quad Y = cT^* - X^* + Y^*\,, \quad Z = cT^* - X^* - Y^*\,, \quad T = 2T^*$$

as may easily be calculated. Substituting in (2) for X, Y, Z, T and dropping the asterisks, we find that the image of trope C_1 under p is the conic

$$Z = 0\,,\quad c((cT + X - Y)^2 - (cT - X - Y)^2 + 4b^2T^2) \\ +2(a^2 - b^2 - c^2)(cT + X - Y)T = 0\,.$$

Calculating the intersections of this with the line L_λ, whose equations are $Z = 0$, $X = \lambda T$, we find $(0, 1, 0, 0)$ and $(\lambda, Y(\lambda), 0, 1)$, where

$$Y(\lambda) = \frac{\lambda(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2)}{2\lambda c + (a^2 - b^2 - c^2)}$$

and this point is the image under p of the rational point which is the intersection of C_1 with E_λ.

PROPOSITION 3. *If* $c - a \le \lambda \le c + a$ *then* $|Y(\lambda)| \ge a$. *If* $\lambda \le c - a$ *or* $\lambda \ge a + c$ *then* $|Y(\lambda)| \le a$.

Proof. Direct calculation shows that $Y(c - a) = -a$ and $Y(c + a) = a$. It follows from the cosine formula in triangle ABC that

$$Y(\lambda) = a\left(\frac{\lambda \cos B + b \cos C}{\lambda - b \cos A}\right),$$

whence

$$\lim_{\lambda\to\infty} |Y(\lambda)| = |a \cos B| \le a\,, \quad \frac{dY(\lambda)}{d\lambda} = \frac{-\sin A \sin B}{(\lambda - \cos B)^2} < 0\,.$$

Here A, B, C denote as usual the angles $\angle BAC$, $\angle CBA$, $\angle ACB$, and the final inequality holds since all angles are $< \pi$. The proposition follows from these inequalities by elementary calculus. Incidentally, the only point at which we use the fact that a, b, c are the sides of a triangle is in the proof of this proposition.

The proof of the theorem may now be completed. A curve given as the double cover of the line branched over real points α, β, γ, δ with $\alpha < \beta < \gamma < \delta$ has two real connected components — a finite loop lying over $[\beta, \gamma]$ and two infinite parts, which join at infinity, lying over $[-\infty, \alpha]$ and $[\delta, \infty]$. Three points of the curve lie all on the same real component if and only if their projections onto the branch line lie all in the finite interval $[\beta, \gamma]$ or all in the infinite interval $[-\infty, \alpha] \cup [\delta, \infty]$. By Proposition 1 the branch points of $p : E_\lambda \to L_\lambda$ are $\pm a$, $\lambda \pm c$. The points of intersection of C_{11} and C_6 with E_λ project onto the branch points $\lambda \pm c$, while the intersection of C_1 with E_λ projects onto $Y(\lambda)$, where as before we use the Y-coordinate as non-homogeneous coordinate on L_λ. Thus $E_\lambda(\mathbb{R})$ has two components (if $\lambda \in \mathbb{R}$). Moreover, the branch points $\lambda \pm c$ themselves lie one on the finite and one on the infinite interval defined by the branch points, unless one or other of the inequalities $-a < \lambda - c < \lambda + c < a$ or $\lambda - c < -a < a < \lambda + c$ is satisfied. But if either of these inequalities is satisfied then λ satisfies the corresponding inequality appearing in Proposition 3, and Proposition 3 then shows that $Y(\lambda)$ lies on the interval to which the points $\lambda \pm c$ do not belong. Thus if $\lambda \in \mathbb{Q}$ then E_λ has rational points on both real components. There are finitely many $\lambda \in \mathbb{Q}$ for which either E_λ is not elliptic (i.e. the fibre corresponding to E_λ is singular or reducible) or $C_1 \cap E_\lambda$ is torsion. For all but these finitely many $\lambda \in \mathbb{Q}$, the Hurwitz–Poincaré theorem applies to E_λ and $E_\lambda(\mathbb{Q})$ is dense in $E_\lambda(\mathbb{R})$. Therefore $S(\mathbb{Q})$ is dense in $S(\mathbb{R})$, and the theorem follows.

We remark finally that the annoying hypothesis that one side of the triangle be rational cannot be totally eliminated. Elementary arguments show that the surface S corresponding to $a^2 = 2$, $b^2 = 3$, $c^2 = 5$ has no rational points except for the nodes lying in $T = 0$. Thus there are no points at all at rational distance from the vertices of the right triangle with sides of lengths $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$.

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