ACTA ARITHMETICA
LXII.3 (1992)

Discrepancy estimates for a class of normal numbers

by

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To the memory of Gerold Wagner

1. Introduction. Let $r \geq 2$ be a fixed integer and let $\theta = 0.a_1a_2\ldots$ be the r-adic expansion of a real number θ with $0 < \theta < 1$. Let $N(\theta; b_1 \ldots b_l; n)$ denote the number of a given block $b_1 \ldots b_l \in \{0, 1, \ldots, r-1\}^l$ appearing in the first n digits $a_1a_2 \ldots a_n$. Then θ is said to be *normal to the base* r if, for every fixed $l \geq 1$,

$$R_n(\theta) = R_{n,l}(\theta) = \sup_{b_1\ldots b_l} \left| \frac{1}{n} N(\theta; b_1 \ldots b_l; n) - \frac{1}{r^l} \right| = o(1) \tag{1}$$

as $n \to \infty$, where the supremum is taken over all $b_1 \ldots b_l \in \{0, 1, \ldots, r-1\}^l$. Historical surveys on the study of normal numbers can be found in, e.g., [6].

Let $g(t)$ be a polynomial of t with real coefficients such that $g(t) > 0$ for $t > 0$. We define a real number

$$\theta_r = \theta_r(g) = 0.a_{11}a_{12}\ldots a_{1k(1)}a_{21}a_{22}\ldots a_{2k(2)}a_{31}\ldots$$

to be the infinite r-adic fraction obtained from the r-adic expansion $[g(n)] = a_{n1}a_{n2}\ldots a_{nk(n)}$ of the integral part of $g(n)$, which will be written simply as

$$\theta_r = 0.[g(1)][g(2)][g(3)]\ldots$$

Let $N(g(n); b_1 \ldots b_l)$ denote the number of a given block $b_1 \ldots b_l$ appearing in the r-adic expansion of $[g(n)]$.

If $g(t)$ is a nonconstant polynomial with *rational* coefficients all of whose values for $t = 1, 2, 3, \ldots$ are positive integers, Davenport and Erdős [3] proved that $R_n(\theta_{10}(g)) = o(1)$, namely, $\theta_{10}(g)$ is normal to the base 10. They did not give explicit estimates for $R_n(\theta_r(g))$. Schoißengeier [11] showed that $R_n(\theta_r(g)) = O((\log\log n)^{4+\varepsilon}/\log n)$. Later, Schiffer [10] improved it by giving the best possible result $R_n(\theta_r(g)) = O(1/\log n)$. In the case of polynomials with *real*, but not necessarily *rational*, coefficients, we proved in [9] that $R_n(r_r(g)) = O((\log\log n)/\log n)$, which will be replaced in this paper by $O(1/\log n)$.

Theorem. *Let $g(t)$ be any nonconstant polynomial with real coefficients such that $g(t) > 0$ for all $t > 0$. Then for any block $b_1 \ldots b_l \in \{0, 1, \ldots \ldots, r-1\}^l$, we have*

$$\sum_{n \le x} N(g(n); b_1 \ldots b_l) = \frac{1}{r^l} x \log_r g(x) + O(x)$$

as $x \to \infty$, where the implied constant depends possibly on g, l, and r.

Noting that the number of digits in the r-adic expansion of $0.[g(1)][g(2)] \ldots [g(n)]$ is

$$n \log_r g(n) + O(n) \gg\ll n \log n \tag{2}$$

with $\log_r y = \log y / \log r$, we obtain

Corollary. *For any $g(t)$ as in the theorem, we have*

$$R_n(\theta_r(g)) = O\left(\frac{1}{\log n}\right) \tag{3}$$

as $n \to \infty$. In particular, $\theta_r(g)$ is normal to the base r.

Remark 1. Let us consider a more general function of the following form:

$$h(t) = \alpha t^{\beta} + \alpha_1 t^{\beta_1} + \ldots + \alpha_d t^{\beta_d}, \tag{4}$$

where α's and β's are real numbers with $\beta > \beta_1 > \ldots > \beta_d \ge 0$. We assume that $h(t) > 0$ for $t > 0$. If $h(t)$ is not a polynomial, we proved in [8] that $R_n(\theta_r(h)) = O(1/\log n)$. Combining this with our result in the present paper, we have $R_n(\theta_r(h)) = O(1/\log n)$ for all functions $h(t)$ given above; in particular, the number $\theta_r(h)$ is normal to the base r for all $h(t)$.

Remark 2. Our method of the proof in [9], which is quite different from that of Schiffer [10], made use of an estimate of Weyl sums in a somewhat unusual manner and of simple remarks on diophantine approximation. In this paper, we further develop this method by employing inductive arguments and we obtain the improved results. As for the proof of the result in [8], tricky estimates for exponential sums of Vinogradov type were used.

2. Lemmas

Lemma 1 ([9], Corollary of Lemma). *Let $p(t)$ be a polynomial with real coefficients and the leading term γt^k, where $\gamma \ne 0$ and $k \ge 1$. Let $Q \ge 2$ and let A/B be a rational number with $(A, B) = 1$ such that*

$$(\log Q)^h \ll B \ll Q^k (\log Q)^{-h}, \tag{5}$$

and

$$|\gamma - A/B| \le B^{-2}\,,$$

where $h \ge (k-1)^2 + 2^k G$ *with* $G > 0$. *Then*

$$\Big|\sum_{1\le n\le Q} e(p(n))\Big| \ll Q(\log Q)^{-G}\,,$$

where $e(t) = e^{2\pi it}$.

LEMMA 2. *Let* $f(t)$ *be a polynomial of the form*

$$f(t) = \beta_0 t^{k_0} + \beta_1 t^{k_1} + \ldots + \beta_d t^{k_d}\,,$$

where $k_0 > k_1 > \ldots > k_d \ge 1$ *and* $\beta_0, \ldots, \beta_d$ *are nonzero real numbers. Let* $G > 0$ *be any constant and* $X \ge 2$. *Let* s *be an integer with* $0 \le s \le d$, *let* H_i, K_i $(i = 0, 1, \ldots, s-1)$ *be any positive constants, and let* H_s^*, K_s^* *be constants such that*

$$H_s^* \ge 2^{k_s+1}(G + \max_{0\le i<s} H_i + 1) + k_s \sum_{i=0}^{s-1} K_i\,,$$

$$K_s^* \ge 2^{k_s+1}(G + \max_{0\le i<s} H_i + 1) + 2k_s \sum_{i=0}^{s-1} K_i\,.$$

Suppose that there are rational numbers A_i/B_i $(0 \le i < s)$ *such that*

$$1 \le B_i \le (\log X)^{K_i} \quad \text{and} \quad \Big|\beta_i - \frac{A_i}{B_i}\Big| \le \frac{(\log X)^{H_i}}{B_i X^{k_i}} \quad (0 \le i < s)$$

and that there is no rational number A_s/B_s *with* $(A_s, B_s) = 1$ *such that*

$$1 \le B_s \le (\log X)^{K_s^*} \quad \text{and} \quad \Big|\beta_s - \frac{A_s}{B_s}\Big| \le \frac{(\log X)^{H_s^*}}{B_s X^{k_s}}\,.$$

Then, for any real P *and* Q *with* $|P| \ll Q \le X$,

$$\Big|\sum_{P<n\le P+Q} e(f(n))\Big| \ll X(\log X)^{-G}\,.$$

Proof. We may assume $P = 0$ and

$$X(\log X)^{-G} \le Q \le X\,. \tag{6}$$

If $s = 0$, the inequality follows immediately from Lemma 1. We put $p(t) = f(t)$, so that $\gamma = \beta_0$ and $k = k_0$. Since $s = 0$, $\max_{0\le i<s} H_i = \sum_{i=0}^{s-1} K_i = 0$. We choose, by the well-known argument, a rational number A/B with $(A, B) = 1$ such that

$$1 \le B \le \frac{X^k}{(\log X)^{H_0^*}} \quad \text{and} \quad \Big|\gamma - \frac{A}{B}\Big| \le \frac{(\log X)^{H_0^*}}{BX^k} \quad (\le B^{-2})\,,$$

where $H_0^*, K_0^* \geq 2^{k+1}(G+1)$. Then by the assumption, we have $B \geq (\log X)^{K_0^*}$. These inequalities as well as (6) imply (5) with $h = (k-1)^2 + 2^k G$. Therefore we obtain

$$\Big| \sum_{1 \leq n \leq Q} e(f(n)) \Big| \ll Q(\log Q)^{-G} \ll X(\log X)^{-G}.$$

Let $s \geq 1$. We denote by D the least common multiple of $B_0, \dots, B_{s-1}$ and by N the integer defined by $DN \leq Q < D(N+1)$, so that

$$1 \leq D \leq (\log X)^K \quad \text{with } K = \sum_{i=0}^{s-1} K_i$$

and by (6)

$$X(\log X)^{-(G+K)} \ll N \gg\ll Q/D \leq X/D.$$

It follows that

$$\sum_{1 \leq n \leq Q} e(f(n)) = \sum_{\lambda=0}^{D-1} \sum_{\nu=1}^{N} e(f(\lambda + D\nu)) + O((\log X)^K). \tag{7}$$

We put

$$f_\lambda(y) = \sum_{i=0}^{s-1} \Omega_i(\lambda + Dy)^{k_i}, \qquad \Omega_i = \beta_i - A_i/B_i,$$

$$\varphi_\lambda(y) = \sum_{i=s}^{d} \beta_i(\lambda + Dy)^{k_i}, \qquad T_\lambda(\nu) = \sum_{n=1}^{\nu} e(\varphi_\lambda(n)).$$

Then we have

$$\begin{aligned}
&\sum_{\lambda=0}^{D-1} \sum_{\nu=1}^{N} e(f(\lambda + D\nu)) \\
&\quad = \sum_{\lambda=0}^{D-1} e\Big(\sum_{i=0}^{s-1} \frac{A_i}{B_i}\lambda^{k_i}\Big) \sum_{\nu=1}^{N} e(f_\lambda(\nu))(T_\lambda(\nu) - T_\lambda(\nu-1)) \\
&\quad = \sum_{\lambda=0}^{D-1} e\Big(\sum_{i=0}^{s-1} \frac{A_i}{B_i}\lambda^{k_i}\Big) \Big\{ e(f_\lambda(N+1))T_\lambda(N) \\
&\qquad + \sum_{\nu=1}^{N} (e(f_\lambda(\nu)) - e(f_\lambda(\nu+1)))T_\lambda(\nu) \Big\} \\
&\quad \ll \sum_{\lambda=0}^{D-1} \Big(|T_\lambda(N)| + \sum_{\nu=1}^{N} |e(f_\lambda(\nu)) - e(f_\lambda(\nu+1))|\, |T_\lambda(\nu)| \Big).
\end{aligned}$$

Here we have, using the mean-value theorem,

$$|e(f_\lambda(\nu)) - e(f_\lambda(\nu+1))| \ll D\sum_{i=0}^{s-1}|\Omega_i|Q^{k_i-1} \ll D\frac{(\log X)^H}{X}$$

with

$$H = \max_{0\le i<s} H_i\,.$$

Therefore we obtain

$$\sum_{\lambda=0}^{D-1}\sum_{\nu=1}^{N} e(f(\lambda+D\nu)) \ll \sum_{\lambda=0}^{D-1}\Big(|T_\lambda(N)| + D\frac{(\log X)^H}{X}\sum_{\nu=1}^{N}|T_\lambda(\nu)|\Big). \tag{8}$$

We next prove that

$$|T_\lambda(\nu)| = \Big|\sum_{n=1}^{\nu} e(\varphi_\lambda(n))\Big| \ll \frac{X}{D(\log X)^{G+H}} \tag{9}$$

for all ν with $1\le\nu\le N$. For this, we may assume that

$$\frac{X}{D(\log X)^{G+H}} \ll \nu \quad (\le N \le X/D)\,. \tag{10}$$

We put $p(t) = \varphi_\lambda(t)$ in Lemma 1, so that the leading coefficient is $\gamma = D^{k_s}\beta_s$. Suppose first that there is a rational number A/B with $(A,B)=1$ such that

$$(\log X)^{H'} \le B \le X^{k_s}(\log X)^{-H'} \tag{11}$$

and

$$|\gamma - A/B| \le B^{-2}\,,$$

where $H' = 2^{k_s+1}(G+H+1)+k_sK$. Then (11) together with (10) implies

$$(\log\nu)^{h'} \le B \le \nu^{k_s}(\log\nu)^{-h'}\,,$$

where $h' = (k_s-1)^2 + 2^{k_s}(G+H)$. Hence we have by Lemma 1

$$|T_\lambda(\nu)| \ll \nu(\log\nu)^{-(G+H)} \ll \frac{X}{D(\log X)^{G+H}}\,.$$

If there is no such rational number, we can choose a rational number A'/B' with $(A',B')=1$ such that

$$1\le B' \le (\log X)^{H'} \quad\text{and}\quad \Big|\gamma - \frac{A'}{B'}\Big| \le \frac{(\log X)^{H'}}{B'X^{k_s}}\,.$$

Then we have

$$D^{k_s}B' \le (\log X)^{H'+k_sK} \le (\log X)^{K_s^*}$$

and

$$\left|\beta_s - \frac{A'}{D^{k_s}B'}\right| \le \frac{(\log X)^{H_s^*}}{D^{k_s}B'X^{k_s}},$$

which contradicts the assumption on β_s.

Combining (7), (8), and (9), we obtain

$$\Big|\sum_{1\le n\le Q} e(f(n))\Big| \ll (\log X)^H + \sum_{\lambda=0}^{D-1}\left(1 + DN\frac{(\log X)^H}{X}\right)\frac{X}{D(\log X)^{G+H}}$$
$$\ll X(\log X)^{-G},$$

and the proof is complete.

3. Preliminaries of the proof of theorem. Let $g(t)$ be as in the theorem. Let j_0 be an integer chosen sufficiently large. Then, for each $j \ge j_0$, there is a positive integer n_j such that $r^{j-2} \le g(n_j) < r^{j-1} \le g(n_j+1) < r^j$. It follows that $n_j < n \le n_{j+1}$ if and only if $r^{j-1} \le g(n) < r^j$ and that

$$n_j \gg\ll r^{j/k}, \quad n_{j+1} - n_j \gg\ll r^{j/k},$$

where $k \ge 1$ is the degree of the polynomial $g(t)$. Let $x > r^{j_0}$ and let J be a positive integer such that $n_J < x \le n_{J+1}$, so that

$$J = \log_r g(x) + O(1) = O(\log x). \tag{12}$$

Put $X_J = x - n_J$ and $X_j = n_{j+1} - n_j$ for $(j_0 \le)\ j \le J-1$. We write $N(g(n)) = N(g(n); b_1 \dots b_l)$. Then

$$\sum_{n\le x} N(g(n)) = \sum_{j_0\le j\le J}\ \sum_{n_j<n\le n_j+X_j} N(g(n)) + O(1).$$

Defining the periodic function $I(t)$ with period 1 by

$$I(t) = \begin{cases} 1 & \text{if } \displaystyle\sum_{h=1}^{l}\frac{b_h}{r^h} \le t - [t] < \sum_{h=1}^{l}\frac{b_h}{r^h} + \frac{1}{r^l}, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$\sum_{n_j<n\le n_j+X_j} N(g(n)) = \sum_{l\le m\le j}\ \sum_{n_j<n\le n_j+X_j} I\left(\frac{g(n)}{r^m}\right).$$

Let j be any integer with $j_0 \le j \le J$ and let C be a constant chosen sufficiently large.

In this section, we treat those m with $C\log j \le m \le j - C\log j$. There are, for each j, functions $I_-(t)$ and $I_+(t)$, periodic with period 1, such that

$I_-(t) \le I(t) \le I_+(t)$, having Fourier expansion of the form

$$I_\pm(t) = r^{-l} \pm j^{-1} + \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} A_\pm(\nu) e(\nu t)$$

with $|A_\pm(\nu)| \ll \min(|\nu|^{-1}, j|\nu|^{-2})$ (cf. [14]).

We shall estimate the exponential sums

$$S(j,m,\nu) = \sum_{n=n_j+1}^{n_j+X_j} e\biggl(\frac{\nu}{r^m} g(n)\biggr),$$

where $J \ge j \ge j_0$, $j - C\log j \ge m \ge C\log j$, and $1 \le \nu \le j^2$. Here the leading coefficient of $\nu r^{-m} g(t)$ is $\nu r^{-m}\alpha$. Assume first that $j < J$. For any pair (m,ν) for which there is a rational number a/q such that

$$(13) \qquad \begin{gathered} (a,q) = 1, \qquad \biggl|\frac{\nu}{r^m}\alpha - \frac{a}{q}\biggr| \le \frac{1}{q^2}, \\ (\log X_j)^H \le q \le X_j^b (\log X_j)^{-H} \end{gathered}$$

with $G = 3$ and H as in Lemma 1, we have

$$|S(j,m,\nu)| \ll X_j(\log X_j)^{-3} \ll X_j j^{-3}$$

by Lemma 1. Hence, denoting by $\sum'$ the sum over all pairs (m,ν) having this property, we have the following estimates:

$$\sum_m \sum_\nu{}' \min(\nu^{-1}, J\nu^{-2})|S(j,m,\nu)| \ll j\log j \cdot X_j j^{-3} \ll X_j \ll r^{j/b}.$$

If $j = J$, there are two cases. Assume first that $X_J = O(r^{J/b}J^{-3})$. Then we have trivial estimates

$$\sum_{m=l}^{J} \sum_{\nu=1}^{J^2}{}' \min(\nu^{-1}, J\nu^{-2})|S(J,m,\nu)| \ll r^{J/b}J^{-1}.$$

Otherwise, namely if $X_J \gg r^{J/b}J^{-3}$, then $\log X_J \gg\ll J$, so that we can repeat the same argument as for $j < J$. In any case, we get

$$\sum_m \sum_\nu{}' \min(\nu^{-1}, j\nu^{-2})|S(j,m,\nu)| \ll r^{j/b}$$

for $(j_0 \le)\ j \le J$ (see [9; p. 208]).

On the other hand, if $(j \ge)\ m \ge (j/\beta)(\beta - \delta)$ with a small positive constant δ, we can appeal to Lemmas 4.2 and 4.8 of [12], with $f(t) = \nu r^{-m} g(t)$. Then, for these m and $\nu \le j^2$, we have, with positive constants c_0 and c_1,

$$0 < c_0 \nu r^{-m+j(1-1/\beta)} < f'(t) < c_1 \nu r^{-m+j(1-1/\beta)} < 1/2$$

throughout the interval $[n_j, n_j+X_j]$, since

$$j\left(1-\frac{1}{\beta}\right)-m \le j\left(1-\frac{1}{\beta}\right)-j\left(1-\frac{\delta}{\beta}\right) < \frac{\delta-1}{\beta} < 0\,.$$

Hence by the lemmas cited,

$$|S(j,m,\nu)| = O\left(\frac{1}{\nu}r^{j/\beta+m-j}\right)$$

provided $(j/\beta)(\beta-\delta) \le m \le j$ and $1 \le \nu \le j^2$ (see [8; p. 26]).

Thus it is proved that

$$\sum_{C\log j\le m\le j}\ \sum_{n_j<n\le n_j+X_j}\left(I\left(\frac{g(n)}{r^m}\right)-\frac{1}{r^l}\right) = O(r^{j/k})\,.$$

Therefore, if we can prove the inequality

$$\sum_{l\le m\le C\log j}\ \sum_{n_j<n\le n_j+X_j}\left(I\left(\frac{g(n)}{r^m}\right)-\frac{1}{r^l}\right) = O(r^{j/k})\,, \tag{14}$$

we shall have obtained

$$\sum_{l\le m\le j}\ \sum_{n_j<n\le n_j+X_j} I\left(\frac{g(n)}{r^m}\right) = \frac{1}{r^l}jX_j + O(r^{j/k})\,,$$

which leads to

$$\sum_{n\le x} N(g(n)) = \frac{1}{r^l}xJ + O(r^{J/k}) = \frac{1}{r^l}x\log_r g(x) + O(x)\,,$$

which is the assertion of our theorem. Thus it remains to show (14).

4. Proof of the inequality (14). In this section, we shall prove (14) for those j for which at least one of the coefficients of $g(t)$ has no rational approximations with *small* denominators in the sense stated in Lemma 2.

To estimate the sum

$$\sum_{n_j<n\le n_j+X_j} I\left(\frac{g(n)}{r^m}\right)$$

in (14), we approximate the function $I(t)$ by functions $I_-(t)$ and $I_+(t)$ periodic with period 1, such that $I_-(t) \le I(t) \le I_+(t)$, having Fourier expansion of the form

$$I_\pm(t) = \frac{1}{r^l} \pm \frac{1}{j} + \sum_{\nu\in\mathbb{Z},\nu\ne0} A_\pm(\nu)e(\nu t)$$

with $|A_\pm(\nu)| \ll \min(|\nu|^{-1}, j\nu^{-2})$, where the constant implied is absolute (cf. [14]). Then we have

$$\sum_{n_j<n\le n_j+X_j} I\biggl(\frac{g(n)}{r^m}\biggr) = \frac{X_j}{r^l} + O\biggl(\frac{X_j}{j}\biggr) + O\biggl(\sum_{\nu=1}^{j^2}\frac{1}{\nu}\biggl|\sum_{n_j<n\le n_j+X_j} e\biggl(\frac{\nu}{r^m}g(n)\biggr)\biggr|\biggr).$$

We shall evaluate

$$\biggl|\sum_{n_j<n\le n_j+X_j} e\biggl(\frac{\nu}{r^m}g(n)\biggr)\biggr|$$

with $l\le m\le C\log j$ and $1\le\nu\le j^2$, by making use of Lemma 2 inductively.

Let the polynomial $g(t)$ be of the form

$$g(t)=\alpha_0t^{k_0}+\alpha_1t^{k_1}+\ldots+\alpha_dt^{k_d},$$

where $k=k_0>k_1>\ldots>k_d\ge 0$ and $\alpha_0,\ldots,\alpha_d$ are nonzero real numbers. We may assume $k_d\ge 1$ in estimating the exponential sum above. We put in Lemma 2

$$f(t)=r^{-m}\nu g(t)$$

so that

$$\beta_i=r^{-m}\nu\alpha_i \quad (0\le i\le d).$$

We choose a constant $c>0$ such that $cr^{j/k}\ge X_j$ for all $j\le J$, and define a parameter X by

$$X=X(j)=cr^{j/k} \quad (j_0\le j\le J).$$

Then $\log X=j^{1+o(1)}$, as $j\to\infty$, so that

$$r^m\le(\log X)^{C\log r+o(1)}, \quad \nu\le(\log X)^{2+o(1)},$$

since $m\le C\log j$ and $\nu\le j^2$.

Case 0. Let j be an integer with $j_0\le j\le J$ for which there is no rational number a_0/b_0 with $(a_0,b_0)=1$ such that

$$1\le b_0\le(\log X)^{2h_0} \quad\text{and}\quad \biggl|\alpha_0-\frac{a_0}{b_0}\biggr|\le\frac{(\log X)^{h_0}}{b_0X^{k_0}},$$

where

$$h_0=H_0^*+C\log r+1, \quad H_0^*=2^{k_0+1}(G+1).$$

The set of all j with this property will be denoted by $\mathbb{J}_0$. If $j\in\mathbb{J}_0$, there is no rational number A_0/B_0 with $(A_0,B_0)=1$ such that

$$1\le B_0\le(\log X)^{2H_0^*} \quad\text{and}\quad \biggl|\beta_0-\frac{A_0}{B_0}\biggr|\le\frac{(\log X)^{H_0^*}}{B_0X^{k_0}},$$

since, if there is such a rational number A_0/B_0, we shall have

$$1 \le \nu B_0 \le (\log X)^{2H_0^*+3} \le (\log X)^{2h_0}$$

and

$$\left|\alpha_0 - \frac{r^m A_0}{\nu B_0}\right| \le \frac{(\log X)^{H_0^*+C\log r+1}}{\nu B_0 X^{k_0}} \le \frac{(\log X)^{h_0}}{\nu B_0 X^{k_0}}\,,$$

which contradicts the assumptions in this case. Hence we can apply Lemma 2 with $s = 0$ and obtain

$$\left|\sum_{n_j<n\le n_j+X_j} e\left(\frac{\nu}{r^m} g(n)\right)\right| \ll \frac{X}{(\log X)^G} \tag{15}$$

for all $j \in \mathbb{J}_0$.

Case s. Let $1 \le s \le d$. We put

$$H_0^* = 2^{k_0+1}(G+1)\,, \qquad H_0 = H_0^* + 2^{k_0+1}(G+1)$$

and define H_i^* and H_i $(1 \le i \le d)$ inductively by

$$H_i^* = 2^{k_i+1}(G + H_{i-1} + 1) + 2k_i(H_0 + \ldots + H_{i-1})\,,$$
$$H_i = H_i^* + 2(C\log r + 1)\,.$$

Also we write

$$h_i = H_i^* + C\log r + 1 \qquad (0 \le i \le d)\,.$$

Let j be an integer with $j_0 \le j \le J$ for which there are rational numbers $a_0/b_0, \ldots, a_{s-1}/b_{s-1}$ such that

$$1 \le b_i \le (\log X)^{2h_i} \quad \text{ and } \quad \left|\alpha_i - \frac{a_i}{b_i}\right| \le \frac{(\log X)^{h_i}}{b_i X^{k_i}} \qquad (0 \le i < s)\,,$$

but there is no rational number a_s/b_s with $(a_s, b_s) = 1$ such that

$$1 \le b_s \le (\log X)^{2h_s} \quad \text{ and } \quad \left|\alpha_s - \frac{a_s}{b_s}\right| \le \frac{(\log X)^{h_s}}{b_s X^{k_s}}\,.$$

The set of all j with this property will be denoted by $\mathbb{J}_s$. If $j \in \mathbb{J}_s$, we have

$$1 \le r^m b_i \le (\log X)^{2H_i} \quad \text{ and } \quad \left|\beta_i - \frac{\nu a_i}{r^m b_i}\right| \le \frac{(\log X)^{H_i}}{r^m b_i X^{k_i}}$$

for $0 \le i < s$, but there is no rational number A_s/B_s with $(A_s, B_s) = 1$ such that

$$1 \le B_s \le (\log X)^{2H_s^*} \quad \text{ and } \quad \left|\beta_s - \frac{A_s}{B_s}\right| \le \frac{(\log X)^{H_s^*}}{B_s X^{k_s}}\,,$$

since otherwise we have a contradiction as in Case 0. Hence, by Lemma 2 with these H_i, H_s^* and $K_i = 2H_i$, $K_s^* = 2H_s^*$, we have again (15) for all $j \in \mathbb{J}_s$.

Choosing $G = 3$ in (15), we get

$$\Big| \sum_{n_j < n \le n_j + X_j} e\Big(\frac{\nu}{r^m} g(n)\Big)\Big| \ll \frac{r^{j/k}}{j^2},$$

for all $(l \le)\ m \le C\log j$, $(1 \le)\ \nu \le j^2$, and $j \in \mathbb{J}_0 \cup \ldots \cup \mathbb{J}_d$, and hence by (14)

$$\sum_{l \le m \le C\log j}\ \sum_{n_j < n \le n_j + X_j} \Big(I\Big(\frac{g(n)}{r^m}\Big) - \frac{1}{r^l}\Big) = O\Big(\frac{r^{j/k}}{j}\Big)$$

for all $j \in \mathbb{J}_0 \cup \ldots \cup \mathbb{J}_d$.

It remains to prove (14) for $j \notin \mathbb{J}_0 \cup \ldots \cup \mathbb{J}_d$ with $j_0 \le j \le J$, which will be done in the next section.

5. Proof of the inequality (14). Continued. Let $\mathbb{J}_{d+1}$ be the set of all integers j with $j_0 \le j \le J$ for which there are rational numbers a_i/b_i with $(a_i, b_i) = 1$ such that

$$1 \le b_i \le (\log X)^{2h_d} \quad \text{and} \quad \Big|\alpha_i - \frac{a_i}{b_i}\Big| \le \frac{(\log X)^{h_d}}{b_i X^{k_i}}$$

for all $i = 0, 1, \ldots, d$, where h_d is defined in Section 4. Then by definition

$$\{j_0, j_0 + 1, \ldots, J\} = \mathbb{J}_0 \cup \ldots \cup \mathbb{J}_d \cup \mathbb{J}_{d+1}.$$

In the rest of this paper, we shall prove (14) for all $j \in \mathbb{J}_{d+1}$ by a method different from that used in the preceding section. We assume $k_d \ge 1$. The proof is valid also in the case of $k_d = 0$.

Let $j \in \mathbb{J}_{d+1}$. We denote by a_* the greatest common divisor of $a_0, \ldots, a_d$ and by b^* the least common multiple of $b_0, \ldots, b_d$. Then $(a_*, b^*) = 1$ and

$$1 \le b^* \le j^h, \quad 1 \le a_* \ll j^h,$$

where $h = 2(d+1)h_d + 1$. We then define integers $c_0, \ldots, c_d$ by

$$\frac{a_i}{b_i} = \frac{a_* c_i}{b^*}$$

so that $(b^*, a_* c_0, \ldots, a_* c_d) = 1$. We write for brevity $L_1 = \log j$ and $L_w = \log L_{w-1}$ $(2 \le w \le w_j)$, where w_j is the greatest integer w for which $L_w \ge 3$.

For a given positive constant C, we have

$$\tag{16} \sum_{l \le m \le C\log j} \Big| \sum_{n_j < n \le n_j + X_j} (I(r^{-m} g(n)) - r^{-l})\Big| \le \sum_{1 \le w \le w_j}\ \sum_{VL_{w+1} < m \le VL_w} \Big| \sum_{n_j < n \le n_j + X_j} (I(r^{-m} g(n)) - r^{-l})\Big| + VX_j$$

where $V \ge C$ is a constant which will be chosen suitably at the end of the proof. For each w, there are functions $I_w^-(t)$ and $I_w^+(t)$, periodic with period 1, such that $I_w^-(t) \le I(t) \le I_w^+(t)$, having Fourier expansion of the form

$$I_w^{\pm}(t) = r^{-l} \pm L_w^{-2} + \sum_{\nu\in\mathbb{Z},\nu\ne 0} A_w^{\pm}(\nu)e(\nu t)\,,$$

with $|A_w^{\pm}(\nu)| \le \min(|\nu|^{-1}, L_w^2\nu^{-2})$ (cf. [14]). Then it follows that

$$\sum_{n_j<n\le n_j+X_j} (I(r^{-m}g(n)) - r^{-l}) \ll X_j L_w^{-2} + \sum_{1\le\nu\le L_w^4} \nu^{-1}\Big|\sum_{n_j<n\le n_j+X_j} e(r^{-m}\nu g(n))\Big|\,. \tag{17}$$

Here we have, for any fixed m with $VL_{w+1} < m \le VL_w$ and ν with $1 \le \nu \le L_w^4$,

$$\begin{aligned}
&\sum_{n_j<n\le n_j+X_j} e(r^{-m}\nu g(n))\\
&= \sum_{0\le\lambda\le r^m b^*} e\biggl(\frac{\nu a_*}{r^m b^*}\sum_{i=0}^d c_i\lambda^{k_i}\biggr) \sum_{\substack{\nu;n=\lambda+r^m b^*\nu\\ n_j<n\le n_j+X_j}} e\biggl(\frac{\nu}{r^m}\sum_{i=0}^d \Omega_i n^{k_i}\biggr)\\
&= \sum_{0\le\lambda\le r^m b^*} e\biggl(\frac{\nu a_*}{r^m b^*}\sum_{i=0}^d c_i\lambda^{k_i}\biggr)\biggl\{\int_{\substack{n_j<n\le n_j+X_j\\ x=\lambda+r^m b^* y}} e\biggl(\frac{\nu}{r^m}\sum_{i=0}^d \Omega_i x^{k_i}\biggr)dy + O(1)\biggr\}\\
&= \sum_{0\le\lambda\le r^m b^*} e\biggl(\frac{\nu a_*}{r^m b^*}\sum_{i=0}^d c_i\lambda^{k_i}\biggr)\frac{1}{r^m b^*}\int_{n_j<n\le n_j+X_j} e\biggl(\frac{\nu}{r^m}\sum_{i=0}^d \Omega_i x^{k_i}\biggr)dx\\
&\quad + O(r^m b^*)\,,
\end{aligned}$$

using a lemma of van der Corput's ([12], Lemma 4.8), where $\Omega_i = \alpha_i - a_i/b_i$. Defining now rational numbers R_i/Q $(0 \le i \le d)$ by

$$\frac{R_i}{Q} = \frac{\nu}{r^m}\frac{a_* c_i}{b^*} \ \Bigl(= \frac{\nu}{r^m}\frac{a_i}{b_i}\Bigr) \quad \text{with } (Q, R_0, R_1, \ldots, R_d) = 1$$

and applying the theorem in [4], Chap. 1, §1, to the exponential sum over λ, we get

$$\begin{aligned}
\sum_{n_j<n\le n_j+X_j} e(r^{-m}\nu g(n)) &\ll \frac{r^m b^*}{Q} Q^{1-9/(10k)}\frac{X_j}{r^m b^*} + r^m b^*\\
&\ll X_j Q^{-9/(10k)} + r^m j^h
\end{aligned}$$

and hence by (17)

$$\text{(18)}\qquad \sum_{VL_{w+1}<m\le VL_w}\Big|\sum_{n_j<n\le n_j+X_j}(I(r^{-m}g(n))-r^{-l})\Big|$$
$$\ll \sum_{VL_{w+1}<m\le VL_w}\Big(X_jL_w^{-2}+X_j\sum_{1\le\nu\le L_w^4}\nu^{-1}Q^{-9/(10k)}+L_{w+1}r^m j^h\Big)$$
$$\ll r^{j/k}L_w^{-1}+X_j\sum_{VL_{w+1}<m\le VL_w}\ \sum_{1\le\nu\le L_w^4}\nu^{-1}Q^{-9/(10k)}\,.$$

Therefore it follows from (16) and (18) that

$$\text{(19)}\qquad \sum_{l\le m\le C\log j}\Big|\sum_{n_j<n\le n_j+X_j}(I(r^{-m}g(n))-r^{-l})\Big|$$
$$\ll r^{j/k}+r^{j/k}\sum_{1\le w\le w_j}\ \sum_{VL_{w+1}<m\le VL_w}\ \sum_{1\le\nu\le L_w^4}\nu^{-1}Q^{-9/(10k)}\,.$$

But, since $\nu Q=r^mR_ib_i/a_i\gg\ll r^mR_i\alpha_i^{-1}\gg\ll r^mR_i\gg r^m$ by the definition of R_i/Q, we obtain

$$\sum_{1\le w\le w_j}\ \sum_{VL_{w+1}<m\le VL_w}\ \sum_{1\le\nu\le L_w^4}\nu^{-1}Q^{-9/(10k)}$$
$$\ll \sum_{1\le w\le w_j}\ \sum_{VL_{w+1}<m\le VL_w}\ \sum_{1\le\nu\le L_w^4}(r^m)^{-9/(10k)}$$
$$\ll \sum_{1\le w\le w_j}VL_w\cdot L_w^4(r^{VL_{w+1}})^{-9/(10k)}$$
$$\ll V\sum_{1\le w\le w_j}L_w^{5-\frac{9\log r}{10k}V}\ll V\sum_{1\le w\le w_j}L_w^{-1}\ll 1\,,$$

provided that $V\ge\max(C,20k/(3\log r))$. Combining this with (19), we have (14) for all $j\in\mathbb{J}_{d+1}$. Therefore, (14) is proved for any j with $j_0\le j\le J$, and the proof of the theorem is complete.

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Received on 25.7.1991
and in revised form on 3.1.1992 (2161)