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Derrick Henry Lehmer

by

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Derrick Henry Lehmer (DHL) began life in Berkeley, California on February 23, 1905. He was one of the five children of Derrick Norman Lehmer (DNL) and [Clara] Eunice Mitchell. As he was growing up, he was greatly influenced by his father who was a professor of mathematics at the University of California at Berkeley. He learned about computing in mathematics and the making of sieves from his father, who had extensively used manual sieves in producing his two famous books [1] and [2]. His pre-university education was in the Berkeley public school system.

As an undergraduate at U. C., Berkeley, DHL adapted an idea of his father for making a sieving mechanism with paper strips, rubber bands, and a sheet of plywood. In his re-design he replaced the paper strips by bicycle chains hanging on sprockets attached to a shaft (with a counter) that was turned by an electric motor. The machine ran without attention, stopping only when a solution was found. (See paper #9. A list of the publications of DHL is given following this article.) With the aid of this rapid parallel "anding" machine, as well as the other sieves that he and his associates designed and built throughout his lifetime, he solved many problems in number theory that were often far beyond what anyone else could solve, such as the factoring and primality testing of large integers and the discovery of pseudo-squares. The original bicycle-chain sieve no longer exists, though the Computer Museum in Boston, Mass. has a working replica of this machine, as well as the photo-electric number sieve, and two other sieves which were donated to the museum by DHL. For more information about these machines see #165, pp. xxix–xlii, #155, [4], and [5].

Also, while he was an undergraduate, he met a young student, Emma Markovna Trotskaia (born in 1906 in Samara, Russia), who had come to U. C., Berkeley in 1924 from Harbin to study engineering. After a couple of years of undergraduate study, Emma changed to mathematics and began to assist DNL and DHL in their work (see #7), such as the making of stencils for factoring integers [3]. In 1927 DHL received his bachelor's degree in physics from Berkeley, whereupon he went to the University of Chicago for graduate studies with L. E. Dickson.

In 1928 Emma received her bachelor's degree in mathematics at Berkeley and the two were married. Since DHL didn't like working under Dickson and since Brown University offered him an instructorship, the couple moved to Providence, Rhode Island. In his first year at Brown, DHL completed a master's degree and in 1930 he finished his PhD (paper #16) under Tamarkin, who, though he was not a number theorist, had a broad understanding of mathematics.

From 1930 to 1932 DHL was a National Research Fellow at Cal. Tech. and Stanford. In 1933 the photo-electric sieve became operational and was displayed (non-operationally) that summer at the World's Fair in Chicago. In the fall the Lehmers went to the Institute for Advanced Studies at Princeton under another fellowship. The following year DHL obtained a more permanent position at Lehigh University where he stayed until 1940, except for a year's leave in 1938–39 to Cambridge, England on a Guggenheim Fellowship, during which time the Lehmers met Hardy, Littlewood, and the engaging computational mathematician J. C. P. Miller at Cambridge and Mordell, Davenport, Mahler, and Erdős at Manchester. Following this respite, they returned to Lehigh with their two young children on almost the last ship to the United States before the Nazi submarine menace began in the Atlantic.

In 1940, the Lehmers were able to leave Bethlemen, PA to accept a proffered position in the mathematics department at Berkeley, a position DHL held with a few interruptions until he became an emeritus professor in 1972. In the two decades that followed his retirement, the Lehmers continued their mathematical research, as always, with a profound admiration and devotion for each other. DHL passed away quietly in Berkeley on May 22, 1991.

In 1945–46 DHL went to Aberdeen Proving Ground to assist in setting up and running the ENIAC, the first automatic computer in the United States. In 1950, when the loyalty oath uproar occurred at Berkeley, he went to UCLA for two years to head the Bureau of Standards' Institute for Numerical Analysis, where the early computer, the SWAC, was running. He returned to Berkeley when the loyalty oath was declared unconstitutional by the courts.

As a lecturer DHL was much appreciated not only for his classical scholarship in mathematics and number theory, but also for his dry sense of humor and his wit. (In his number theory classes during the period when the U.S. refused to recognize mainland China, he always referred to the Chinese Remainder Theorem as the Taiwan Remainder Theorem.) He was a down-to-earth man who eschewed abstraction for its own sake. He loved and valued formulas and numerically examined the mathematical questions that interested him. He always took a deeply computational or constructive approach to mathematics. It is worth quoting him on this (see paper #108, p. 224):

"When I was young, mathematicians still had a slight feeling of guilt about a proof that establishes only the existence of a solution of a problem, without supplying a constructive procedure for finding the solution. Many times I was told that the existence proof was only the first step and, now that the solution is known to exist, very soon some, presumably less pure, mathematician would come up with a constructive solution. Meanwhile, the thing to do is to generalize the existence proof to Banach space and beyond. In almost all cases, I am still waiting for the second man. Often I am asked the rhetorical question 'How can you even begin to solve a problem until it is known that the solution exists?' The answer to this smug question is 'easily! I have done it many times'."

The breadth of Lehmer's mathematical work is best judged by the 17 subject headings he chose for the 1981 publication of his *Selected Papers* (for details see below). These headings are: Lucas' Functions, Tests for Primality, Continued Fractions, Bernoulli Numbers and Polynomials, Diophantine Equations, Numerical Functions, Matrices, Power Residues, Analytic Number Theory, Partitions, Modular Forms, Cyclotomy, Combinatorics, Sieves, Equation Solving, Computing Techniques, and Miscellaneous. In some of these areas his results are very well known. For example, under Lucas' functions there is his influential thesis (#16) which extended Lucas' work on the divisibility properties of what are called "Lucas sequences" to a special type of fourth order sequence. The terms in these more general sequences are now called "Lehmer numbers" [6].

DHL was also interested in primality testing throughout his life. He clarified and extended Lucas' use of the Fermat congruence in primality testing, making its use well known. Paper #143 with John Selfridge and this writer carried this topic a great deal further. DHL is perhaps best known for his sharp and definitive form of Lucas' primality test (#38) for the Mersenne number M_p, p a prime: Let $S_{n+1} = S_n^2 - 2$, $S_1 = 4$. Then M_p is a prime if and only if M_p divides S_{p-1}.

DHL was involved throughout his life with the theory and practice of factoring integers. Besides the many impressive factorizations he obtained while working by hand or with his sieves, he also published the important paper with Powers (#25) in which they present a new factoring algorithm. Although the attitude in the paper is negative toward the algorithm as a hand method, this was the paper that precedes the development of the continued fraction method by M. Morrison and this writer. It was this method which was powerful enough on a computer to factor the seventh Fermat number in 1970 in 45 minutes.

One matter of particular interest is his work on the density of primes having a given primitive root g. Artin had conjectured that this density would be the same (about 3/8) regardless of what the value of g was. In #103 data are presented that seem to contradict this conjecture, at least for small primes such as -3, 5, and -7, for which the set of primes having one of them as a primitive root has a much larger density than the average. Lehmer's observation was later shown to follow from the Generalized Riemann Hypothesis by C. Hooley. In #104 DHL discusses in the same heuristic fashion the densities of prime values of quadratic polynomials that have a certain g as a primitive root. In this case he presents examples which have quite high densities.

The work of DHL on the partition function $p(n)$ is outstanding and is recognized as such. The first of his 5 papers on this subject (#41) dealt with verifying two cases of Ramanujan's conjecture, viz. that 5^4 divides $p(599)$ and 11^3 divides $p(721)$. These values were computed using the Hardy–Ramanujan series, and they verified the conjectures. However, it was not known that this series converges. In #46 DHL showed that the series actually diverges, but can nonetheless be reliably used to compute values of $p(n)$, because after a determinable number of terms the partial sums differ from the actual value by less than 1/2. His later papers in the series #47, #48, and #56 all deal in an elegant way with the question of estimating the remainder term in the series. He also settles two more divisibility questions about $p(n)$ by computing values using the divergent series.

Another major study concerned the roots of the Riemann zeta function. He was the first person to make an extensive calculation of these roots on an automatic computer (#83). He found that the first 10 000 zeros of the zeta function lie on the critical line. In #87 he extended this calculation and showed that the first 25 000 zeros are on the critical line. In both these calculations he discusses statistics relative to the Gram intervals and points out a "near miss" for the Riemann hypothesis. These two papers were also written to show how the zeros of the zeta function can be effectively computed from a programming point of view.

In addition to the influential mathematical work mentioned above, there are two well-known conjectures due to Lehmer that have yet to be settled: (1) (#69) The Ramanujan tau function never vanishes; (2) (#32) If $\phi(n)$ divides $n-1$, then n is prime. An interesting question he also asked (#34), to which he may well have provided the final answer, is whether for any $\varepsilon > 0$ there exists a monic polynomial with integer coefficients for which the absolute value of the product of its zeros that lie outside the unit circle is less than $1+\varepsilon$. This problem has come to be known in the theory of Diophantine approximations as the "Lehmer problem". It seems that the answer is negative when $1+\varepsilon$ equals the relevant product for his polynomial: $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$.

The algorithms he devised were not "Gedanken" algorithms, but practical algorithms he used to get information about what he was studying at the time. His papers on computing, which usually contain material dealing with his current research, abound in the ideas, methods, insights, and discoveries of a man who feels that computing in its various forms, and the theories that go with it, should be primarily practical. To have an eminent professor on the faculty with this point of view was particularly encouraging to students of a similar frame of mind (like myself) who were trying to survive in the ever-growing, rarified atmosphere of abstraction that was so common in Berkeley during his years there.

As a thinker, DHL was sagaciously independent, not being devoted to dogmas, systems, or rituals. On the other hand, he comfortably cooperated with others but strongly disliked slavery in any form, especially in the professor-student relationship. Many of his graduate students worked by themselves, conferring with DHL only on occasion to show him their progress.

In the some sixty years during which they collaborated, the Lehmers were a research team who personally influenced a large number of people with their knowledge, their courtesy and sociability, and their fine mathematical work. There is little doubt that one of their most enduring contributions to the world of mathematicians is their founding of the West Coast Number Theory Meeting in 1969. This meeting, which has been held in the western United States in December of every year since that time, has provided an informal and often merry environment in which old friends can meet and younger mathematicians can present their work. This meeting will remain a lasting tribute to the Lehmers.

Throughout his life DHL was a very well known number theorist and computational mathematician, whose papers contain fundamental contributions to many parts of mathematics. His published books are items #61 and #165. In 1943 he became one of the first editors of the new journal MTAC (later re-named Mathematics of Computation), a position he held until 1954. In 1975 the editors of that journal honored him on his 70th birthday by dedicating a special issue to him. In 1958 DHL began his long tenure as a member of the Advisory Board of Acta Arithmetica. In 1981 three volumes of his papers were published by The Charles Babbage Research Centre. These volumes, titled *Selected Papers of D. H. Lehmer*, are still available in hard or soft cover from that centre at P.O. Box 272, St. Norbert Postal Station, Winnipeg, Manitoba, Canada R3V 1L6. (A review of the *Selected Papers* by the present author will appear in 1993 in a special issue of Math. Comp. that will be published in his memory.)

DHL is survived by his wife Emma, a son Donald and daughter Laura, four grandchildren, two great-grandchildren, and an older sister Helen. His private papers and 23 personal journals, along with an audiotape of his Gibbs Lecture in Denver in 1965 (paper #119) and a videotape of his lecture on the history of sieves at the History of Computing meeting at Los Alamos in 1974 (paper #155) and a videotape of his lecture at the Computer Museum in Boston in 1982, are housed at the Bancroft Library on the U. C. campus at Berkeley. Dr. Robin Rider of the Bancroft Library has stated she would appreciate receiving any correspondence that anyone has had with DHL to be included in the permanent collection there.

It is appropriate here to repeat the final sentence of his obituary at the University of California: "He is sorely missed by all who knew him".

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The PhD students of D. H. Lehmer

Note that David Singmaster was also a PhD student of R. S. Lehman while DHL was on leave.

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2. Henry Ludwig Alder (1947), *The existence and nonexistence of certain identities in the theory of partitions.*
3. William Haddock Simons (1947), *Modular functions of stufe* 2.
4. Tom Mike Apostol (1948), *A study of Dedekind sums and their generalizations.*
5. Donald Dines Wall (1949), *Normal numbers.*
6. Mark Brimhall Wells (1961), *Simplification of normal form expressions for Boolean functions of many variables.*
7. Nand Kishore (1961), *Arithmetical properties of Bessel functions.*
8. Jonathan David Young (1962), *Application of linear programming to the numerical solutions of linear differential equations.*
9. Ronald Lewis Graham (1962), *On finite sums of rational numbers.*

10. Robert Samuel Spira (1962), *Sums of two squares and Brahmagupta's formula.*

11. Jayanthi Chidambaraswamy (1964), *Divisibility properties of certain factorials.*

12. Harold Mead Stark (1964), *On the tenth complex quadratic field with class-number one.*

13. Alan Zame (1965), *On the distribution of the fractional parts of certain sequences.*

14. David Singmaster (1966), *On means of differences of consecutive integers relatively prime to m.*

15. John David Brillhart (1967), *On the Euler and Bernoulli polynomials.*

16. David Friedman (1967), *Cubic character sums and congruences.*

17. James Brown Herreschoff (1968), *A theorem on character sums.*

18. Richard Paul Stauduhar (1969), *The automatic determination of Galois groups.*

19. Peter Jay Weinberger (1969), *Proof of a conjecture of Gauss on class number two.*

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Publications of D. H. Lehmer

The following list was compiled by DHL. It contains only those publications he considered to be most important. His minor notes that are scattered throughout the literature are not listed.

1. *On the use of the calculating machine for cube and fifth roots*, Amer. Math. Monthly 32 (1925), 377–379.

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5. *A list of errors in tables of the Pell equation*, Bull. Amer. Math. Soc. 32 (1926), 545–550.

6. *Note on the Mersenne number* $2^{139} - 1$, ibid., 522.

7. *Tests for primality by the converse of Fermat's theorem*, ibid. 33 (1927), 327–340.

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11. *On Stern's diatomic series*, Amer. Math. Monthly 36 (1929), 59–67.

12. *On the number* $(10^{23} - 1)/9$, Bull. Amer. Math. Soc. 35 (1929), 349–350.

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