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Generalized Frobenius partitions, k-cores, k-quotients, and cranks

by

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1. Introduction. In 1900 Frobenius [3] introduced a symbol to represent an ordinary partition of an integer n. This symbol was a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

where the entries in each row were distinct nonnegative integers arranged in decreasing order and

$$n = \sum_{i=1}^{r} (a_i + b_i + 1)\,.$$

Each partition has a unique representation as a Frobenius symbol. This representation was obtained from the Ferrar's graph of the partition for n by deleting the r nodes on the diagonal and defining a_i (b_i) as the number of nodes to the right of (below) the ith diagonal node. Frobenius used these symbols in his study of the representations of the symmetric groups.

More than three quarters of a century later Andrews [1] laid the foundation for the study of generalized Frobenius partitions (F-partitions). These combinatorial objects are also two-rowed arrays of nonnegative integers with varying restrictions placed on the entries in the top and bottom rows. The study of F-partitions arose out of the identities associated with Regime III of Baxter's Hard Hexagon Model [2] and have ties to ordinary partitions and their properties [7].

In this paper we will concentrate on F-partitions with k colors where a nonnegative integer can be repeated at most k times in each row as long as each repetition is a different color. We will denote our k available colors numerically by $0, 1, 2, \dots, k-1$ and the color of an integer will be indicated by a numerical subscript. For example, the F-partitions with 2 colors of two are

$$\begin{pmatrix} 0_1 \; 0_0 \\ 0_1 \; 0_0 \end{pmatrix}, \quad \begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \quad \begin{pmatrix} 1_1 \\ 0_0 \end{pmatrix}, \quad \begin{pmatrix} 1_0 \\ 0_1 \end{pmatrix}, \quad \begin{pmatrix} 1_0 \\ 0_0 \end{pmatrix},$$
$$\begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \quad \begin{pmatrix} 0_1 \\ 1_0 \end{pmatrix}, \quad \begin{pmatrix} 0_0 \\ 1_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0_0 \\ 1_0 \end{pmatrix}.$$

A direct connection between F-partitions with k colors and ordinary partitions will be made through a bijection between the set of ordinary partitions and the set of F-partitions with k colors. In fact, it will be shown that an F-partition with k colors is simply a representation of an ordinary partition which can be used to keep track of certain pieces of information associated with the ordinary partition.

2. The bijection. We define a map ϕ from the set of generalized Frobenius partitions with k colors to the set of ordinary partitions as follows:

Let λ be an F-partition with k colors. Each entry a_i (where a is a nonnegative integer and $0 \le i \le k-1$) on the top row is replaced by $ka+i$. Each entry b_j (where b is a nonnegative integer and $0 \le j \le k-1$) on the bottom row is replaced by $k(b+1)-j-1$. The new array is read as the Frobenius symbol representing an ordinary partition, $\phi(\lambda)$.

Clearly this is a bijective map between the set of F-partitions with k colors and the Frobenius symbols representing ordinary partitions (and thus the ordinary partitions themselves). Note that this bijection does not preserve the size of the integer being partitioned and two F-partitions of the same integer may be mapped to ordinary partitions of different integers.

$\begin{pmatrix} 0_4 \\ 0_1 \end{pmatrix}$ and $\begin{pmatrix} 0_3 \\ 0_2 \end{pmatrix}$ are F-partitions of 1 using, for example, $k=7$ colors. We could view these as F-partitions with any $k \ge 5$ colors. Now,

$$\phi\left(\begin{pmatrix} 0_4 \\ 0_1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 5 \end{pmatrix},$$

representing an ordinary partition of 10 while

$$\phi\left(\begin{pmatrix} 0_3 \\ 0_2 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

representing an ordinary partition of 8.

We now look at the generating functions associated with this bijection. The generating function for F-partitions with k colors is the coefficient of z^0 in $\prod_{j=0}^{k-1}\prod_{i=0}^{\infty}(1+zt^jq^i)(1+z^{-1}t^{-j}q^{i+1})$ where the parameter t is used to keep track of the difference in the colors on the top and bottom rows. The bijection defined above can be fulfilled by replacing q by q^k and t by q.

This results in the generating function

$$\prod_{j=0}^{k-1}\prod_{i=0}^{\infty}(1+zq^{ki+j})(1+z^{-1}q^{k(i+1)-j}) = \prod_{n=0}^{\infty}(1+zq^n)(1+z^{-1}q^{n+1})\,.$$

The coefficient of z^0 in this last product is the generating function for the Frobenius symbols representing ordinary partitions.

3. The generating function identity. Let $c\phi_k(n,m)$ denote the number of generalized Frobenius partitions of n with k colors such that $\sum_{j=0}^{k-1} jd(j) = m$ where $d(j)$ is the number of appearances of color j on the top row minus the number of appearances of color j on the bottom row. The discussion in Section 2 implies $\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} c\phi_k(n,m)q^n t^m$ is the coefficient of z^0 in $\prod_{j=0}^{k-1}(-zt^j;q)_\infty(-z^{-1}t^{-j}q;q)_\infty$ where $(A;B)_\infty = \prod_{n=0}^{\infty}(1-AB^n)$.

Using Jacobi's triple product identity we have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} c\phi_k(n,m)q^n t^m = \frac{1}{(q;q)_\infty^k}\prod_{j=0}^{k-1}\sum_{n=-\infty}^{\infty} q^{\binom{m-1}{2}} t^{mj} z^m\,.$$

It is not difficult to see that the coefficient of z^0 is

$$\frac{1}{(q;q)_\infty^k}\sum_{\substack{\mathbf{m}\in\mathbb{Z}^k\\ \mathbf{m}\cdot\mathbf{1}=0}} q^{\frac{1}{2}|\mathbf{m}|^2} t^{\mathbf{b}\cdot\mathbf{m}}$$

where $\mathbf{b} = (0,1,2,\ldots,k-1)$ and $\mathbf{1} = (1,1,1,\ldots,1)$. Replacing q by q^k and t by q we have

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q^k;q^k)_\infty^k}\sum_{\substack{\mathbf{m}\in\mathbb{Z}^k\\ \mathbf{m}\cdot\mathbf{1}=0}} q^{\frac{k}{2}|\mathbf{m}|^2+\mathbf{b}\cdot\mathbf{m}}\,.$$

This is the generating function identity presented in [6] which arises from a bijection between an ordinary partition and its k-core and k-quotient. It would seem reasonable that a generalized Frobenius partition with k colors, λ, is a representation of an ordinary partition, $\phi(\lambda)$, which keeps track of the underlying k-core and k-quotient of $\phi(\lambda)$. In the next section we explore how this information is accounted for in λ.

4. k-cores and k-quotients. Examining the bijections presented in [6] we see that the vector $\mathbf{m}$, where the $(i+1)$st component represents the difference in the number of appearances of color i on the top and bottom rows, is the vector associated with the k-core for $\phi(\lambda)$. The biinfinite words, $w_0, w_1, \ldots, w_{k-1}$, associated with the k-quotient for $\phi(\lambda)$ are given by

(1) for $j > 0$, the jth letter of w_i is E if $(j-1)_i$ is a part on the top row of λ and is N otherwise,

(2) for $j \le 0$, the jth letter of w_i is N if $(-j)_i$ is a part on the bottom row of λ and is E otherwise.

The observation in [6] that a partition is a k-core if and only if each w_i is an infinite sequence of E's followed by an infinite sequence of N's gives us an easy way to determine whether or not $\phi(\lambda)$ is a k-core by looking at λ.

THEOREM 1. *$\phi(\lambda)$ is a k-core if and only if λ satisfies the condition that if a_i appears on one row of λ then $(a-1)_i, \ldots, 1_i, 0_i$ also appear on that row and no entries of color i appear in the other row.*

Thus given a vector $\mathbf{m}$ in $\mathbb{Z}^k$ with $\mathbf{m} \cdot \mathbf{1} = 0$, it is a trivial task to reconstruct the corresponding k-core:

(1) If the $(i+1)$st component of $\mathbf{m}$ is 0, then parts of color i do not appear in λ, the F-partition with k colors representing the ordinary partition which is a k-core.

(2) If the $(i+1)$st component of $\mathbf{m}$ is $n > 0$ then $0_i, 1_i, \ldots, (n-1)_i$ appear on the top row of λ.

(3) If the $(i+1)$st component of $\mathbf{m}$ is $-n < 0$ then $0_i, 1_i, \ldots, (n-1)_i$ appear on the bottom row of λ.

(4) $\phi(\lambda)$ is the k-core associated with $\mathbf{m}$.

We can use the technique outlined in [6] to construct the k-quotient for $\phi(\lambda)$ from the biinfinite words $w_0, w_1, \ldots, w_{k-1}$. However, Olsson's work [8] presents another way of constructing the k-quotient for $\phi(\lambda)$ from λ using what he calls Frobenius symbols associated with cuts in a partition sequence. For a specific color j the $(j+1)$st component of the k-quotient for $\phi(\lambda)$ is the partition with Frobenius symbol associated with a cut, $(X|Y)$, where the elements in the set X are the entries on the top row of λ of color j and the elements in the set Y are the entries on the bottom row of λ of color j.

5. An application to cranks. In [6] Garvan presented a crank for ordinary partitions of $kn+r$ for $(k,r) = (5,4), (7,5)$, and $(11,6)$. A *crank* is a statistic which divides the partitions of $kn+r$ into k classes of equal size. The cranks presented in Theorem 2 of [6] are defined as modulo k linear combinations of the components of the vector $\mathbf{m}$ associated with the k-core of the partition:

$(k=5)$ $4m_0 + m_1 + m_3 + 4m_4$,

$(k=7)$ $4m_0 + 2m_1 + m_2 + m_4 + 2m_5 + 4m_6$,

$(k=11)$ $4m_0 + 9m_1 + 5m_2 + 3m_3 + m_4 + m_6 + 3m_7 + 5m_8 + 9m_9 + 4m_{10}$.

In this paper we have seen that the $(i+1)$st component of the vector $\mathbf{m}$ associated with the k-core of a partition is the difference in the number of appearances of color i on the top and bottom rows of its F-partition with k colors representation. In terms of the Frobenius symbol for the partition this translates into the difference in the number of parts on the top row congruent to i modulo k and the number of parts on the bottom row congruent to $-(1+i)$ modulo k. Because of the symmetry in the cranks given above we have the following new interpretation of the cranks.

THEOREM 2. *A crank statistic for partitions of $kn+r$ for $(k,r)=(5,4)$, $(7,5)$, and $(11,6)$ is given by the modulo k linear combinations*

$$(k=5) \qquad 4a_0 + a_1 + a_3 + 4a_4,$$

$$(k=7) \qquad 4a_0 + 2a_1 + a_2 + a_4 + 2a_5 + 4a_6,$$

$$(k=11) \qquad 4a_0 + 9a_1 + 5a_2 + 3a_3 + a_4 + a_6 + 3a_7 + 5a_8 + 9a_9 + 4a_{10},$$

where a_i equals the difference in the number of parts on the top and bottom rows congruent to i modulo k in the Frobenius symbol for the partition.

In correspondence with Garvan [5] he indicated that the above theorem could be restated as follows.

COROLLARY. *A crank statistic for partitions of $kn+r$ for $(k,r)=(5,4)$, $(7,5)$, and $(11,6)$ is given by the modulo k linear combination $\sum_{j=1}^{(k-1)/2} j^{k-3} b_j$ where b_j equals the difference in the number of parts on the top and bottom rows congruent to $\pm j + ((k-1)/2)$ modulo k in the Frobenius symbol for the partition.*

6. Some other observations.

From the generating function identity presented in Section 3 we have

THEOREM 3. *For $s \ge 1$, $k \ge 2$,*

$$p(s) = \sum_{n,m\in\mathbb{Z},\, kn+m=s} c\phi_k(n,m)\,.$$

In [4] Garvan presented several congruences for $a_k(n)$, the number of ordinary partitions of n which are k-cores (and the coefficient of q^n in $\sum_{\mathbf{m}\in\mathbb{Z}^k,\,\mathbf{m}\cdot\mathbf{1}=0} q^{(k/2)|\mathbf{m}|^2+\mathbf{b}\cdot\mathbf{m}}$), for $5 \le k \le 23$ and k prime. The next theorem looks at the parity of $a_k(n)$.

THEOREM 4. $a_k(n) \equiv 0 \pmod 2$ *unless n has an odd number of representations as $\alpha=[k/2]$ summands of the form*

$$(1)\quad (km_1^2+m_1)+(km_2^2+3m_2)+\ldots+(km_\alpha^2+(k-1)m_\alpha) \quad \textit{for } k \textit{ even},$$

$$(2)\quad (km_1^2+2m_1)+(km_2^2+4m_2)+\ldots+(km_\alpha^2+(k-1)m_\alpha) \quad \textit{for } k \textit{ odd}.$$

Since $\mathbf{m}\cdot\mathbf{1}$ must be 0 in our sum it is easy to see that the value of $(k/2)|\mathbf{m}|^2+\mathbf{b}\cdot\mathbf{m}$ is the same for the vectors $(m_0,m_1,\ldots,m_{k-1})$ and $(-m_{k-1},\ldots,-m_1,-m_0)$ with $\sum_{i=0}^{k-1}m_i=0$. Thus the parity of $a_k(n)$ is the same as the number of ways n can be expressed as $n=(k/2)|\mathbf{m}|^2+\mathbf{b}\cdot\mathbf{m}$ with $\mathbf{m}\cdot\mathbf{1}=0$ and the vector $\mathbf{m}=(m_0,m_1,\ldots,m_{k-1})$ is the same as the vector $(-m_{k-1},\ldots,-m_1,-m_0)$. This translates into the condition stated in Theorem 4.

As an immediate consequence of this theorem we see that $a_2(n)\equiv 0 \pmod 2$ provided n is not of the form $2m^2+m$ for any integer m, and $a_3(n)\equiv 0 \pmod 2$ provided n is not of the form $3m^2+2m$ for any integer m.

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