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Pairs of additive quadratic forms modulo one

by

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1. Introduction. Let $\|x\|$ denote the distance from x to the nearest integer. Let $\varepsilon > 0$. A well-known theorem of Heilbronn [15] states that for $N > C_1(\varepsilon)$, and any real number α, we have

$$\min_{1\le n\le N} \|\alpha n^2\| < N^{-1/2+\varepsilon}\,.$$

Among many possible extensions, the following was considered by Danicic [9]. We seek a positive number $\alpha(s)$ with the following property:

Let $Q(x_1,\ldots,x_s)$ *be a real quadratic form, then for* $N > C_2(s,\varepsilon)$ *we have*

$$\|Q(n_1,\ldots,n_s)\| < N^{-\alpha(s)+\varepsilon} \tag{1.1}$$

for some integers $n_1,\ldots,n_s$,

$$0 < \max(|n_1|,\ldots,|n_s|) \le N\,. \tag{1.2}$$

Danicic was able to take $\alpha(s) = s/(s+1)$. An important step forward occurred when Schinzel, Schlickewei and Schmidt [18] showed the relevance of the following "discrete version" of the problem. We seek the least positive number $B_s(q)$ with the following property.

For any $K_i > 0$ *satisfying*

$$(K_1\ldots K_s)^{1/s} \ge B_s(q)\,,$$

and any quadratic form Q *with integer coefficients*, *the congruence* $Q(\boldsymbol{x}) \equiv 0 \pmod q$ *has a nonzero solution satisfying*

$$|x_i| \le K_i \quad (1\le i\le n)\,.$$

Further work on this problem was done by Baker and Harman [6] and by Heath-Brown [14]. Heath-Brown showed that

$$B_s(q) < C_3(s,\varepsilon)q^{\beta(s)+\varepsilon} \tag{1.3}$$

where $\beta(4) = 5/8$, $\beta(6) = 15/26$, $\beta(8) = 6/11$, $\beta(10) = \beta(11) = 8/15$ and $\beta(s) = 1/2 + 3/s^2$ for even $s \ge 12$. For $s = 3,5,7$ the exponent

$\beta(s) = 1/2 + 1/(2s)$ [6] remains the best known. By arguing as in [6], one can show that the exponent

$$\alpha(s) = \frac{s}{2 + s\beta(s)}$$

is permissible in (1.1), whenever (1.3) holds.

Not surprisingly, one can do better for real additive quadratic forms. It is convenient for applications to seek solutions in a box rather than a cube.

THEOREM 1. *Suppose that* (1.3) *holds. Let* $\sigma(1) = 1/2$, $\sigma(2) = 1$,

$$\sigma(s) = \frac{s}{2 + (s-2)\beta(s)} \qquad (s \ge 3)\,.$$

Let $Q(x_1, \dots, x_s)$ *be an additive quadratic form. Let* $N > C_4(s, \varepsilon)$. *Given positive* $N_1, \dots, N_s$ *with*

$$N_1 \dots N_s \ge N^s \tag{1.4}$$

there exist non-negative integers $n_1, \dots, n_s$ *not all zero satisfying* $n_i \le N_i$ $(i = 1, \dots, s)$ *and*

$$\|Q(n_1, \dots, n_s)\| < N^{-\sigma(s)+\varepsilon}\,.$$

The case $s = 2$ of Theorem 1 is a generalization of a theorem of Cook [7]. For $s \ge 3$, see [13] and [1] for earlier results along the lines of Theorem 1.

In proving Theorem 1 we assume, as we may, that $1/2 \le \beta(s) \le 1/2 + 1/(2s-4)$.

We apply Theorem 1 to pairs of additive forms.

THEOREM 2. *Define* $\sigma(s)$ *as above. Let* $Q_1(x_1, \dots, x_s)$, $Q_2(x_1, \dots, x_s)$ *be additive quadratic forms. Then for* $N > C_5(s, \varepsilon)$ *we have*

$$\max(\|Q_1(\boldsymbol{n})\|, \|Q_2(\boldsymbol{n})\|) < N^{-\tau(s)+\varepsilon} \tag{1.5}$$

for some integers $n_1, \dots, n_s$ *satisfying* (1.2). *Here*

$$\tau(2) = 1/3, \qquad \tau(3) = 3/7, \qquad \tau(4) = 1/2\,;$$

$$\tau(s) = \begin{cases} s\sigma(s)/(8\sigma(s) + 2s - 8) & \text{for } 5 \le s \le 7, \\ \sigma(s)/(1 + \sigma(s)) & \text{for } s \ge 8. \end{cases}$$

Since $\sigma(s)$ has limit 2 as $s \to \infty$, we see that $\tau(s)$ has limit 2/3. However, we can replace $\tau(s)$ by an exponent whose limit is 1; see Baker and Harman [5]. In fact, the method of [5] may be refined to give an improvement of Theorem 2 for $s \ge 24$.

For earlier results in a small number of variables along the lines of Theorem 2, see Liu [17] and Baker and Gajraj [4]. The exponent in [4] is much poorer, namely $-1/5 + \varepsilon$ for $s \ge 2$. This is partly because we now have at our disposal the "lattice method" of Schmidt [19], whose result may be stated as

$$\tau(1) = 1/6\,.$$

Weaker versions of this last result were found earlier by Danicic [8], [10] and Liu [16].

For *arbitrary* pairs of quadratic forms, the first results analogous to (1.5) were given by Danicic [11]. Recently Baker and Brüdern [3] improved these results. For example, the analogue of (1.5) for a pair of binary forms has 1/5 in place of $\tau(2)$. Once again, [5] is stronger for large s.

Throughout the paper, implied constants depend at most on ε, s. We write $e(\theta) = e^{2\pi i\theta}$. The cardinality of a finite set $\mathcal{A}$ is denoted by $|\mathcal{A}|$.

2. Proof of Theorem 1. We require two lemmas from [2].

LEMMA 1. *Let x_j $(j = 1, \ldots, N)$ be real numbers satisfying $\|x_j\| \ge M^{-1}$. Then*

$$\sum_{m\le M} \Big|\sum_{n=1}^{N} e(mx_n)\Big| > N/6\,. \tag{2.1}$$

P r o o f. This is Theorem 2.2 of [2].

LEMMA 2. *Let $\delta > 0$ and $N > C_6(\delta)$. Let α be real. Let L be a natural number such that*

$$L^{\delta} < N\,. \tag{2.2}$$

If

$$\sum_{m=1}^{L} \Big|\sum_{n=1}^{N} e(m\alpha n^2)\Big|^2 > A \tag{2.3}$$

where $A \ge N^{1+\delta}L$, then there exist coprime integers r and s with $r \le LN^{2+\delta}A^{-1}$ and

$$|\alpha r - s| < N^{\delta}A^{-1}\,. \tag{2.4}$$

P r o o f. This is given in all essentials in [2], although the condition (2.2) is weaker than the corresponding inequality in [2].

Our next step is to prove Theorem 1 under the additional restrictions

$$s \ge 2\,; \qquad N_j \ge N^{\varepsilon/4} \qquad (j = 1, \ldots, s)\,. \tag{2.5}$$

LEMMA 3. *The assertion of Theorem* 1 *is true when* (2.5) *holds.*

P r o o f. Suppose that the assertion is false. Then, by Lemma 1,

$$\sum_{m=1}^{M} T_1(m) \ldots T_s(m) \ge C_7(s)N_1 \ldots N_s\,. \tag{2.6}$$

Here $M = 1 + [N^{\sigma(s)-\varepsilon}]$, $Q(\boldsymbol{x}) = \alpha_1 x_1^2 + \ldots + \alpha_s x_s^2$,

$$T_j(m) = \Big|\sum_{n\le N_j} e(m\alpha_j n^2)\Big| .$$

The contribution from those m in (2.6) having $T_j(m) < N^{-2}$ for some index j is negligible. We cover $[N^{-2}, C_7 N_1 \ldots N_s]$ by $O(\log N)$ subintervals of the type $[A, 2A)$. There must exist numbers $A_j \ge N^{-2}$ $(j = 1, \ldots, s)$ and a subset $\mathcal{B}$ of $[1, M) \cap \mathbb{Z}$ having

$$\begin{gathered} A_j \le T_j(m) < 2A_j \quad (j = 1, \ldots, s)\,, \\ \sum_{m\in\mathcal{B}} T_1(m)\ldots T_s(m) \gg N_1 \ldots N_s/(\log N)^s\,. \end{gathered} \tag{2.7}$$

This implies

$$|\mathcal{B}|^2 A_1^2 \ldots A_s^2 \gg N_1^2 \ldots N_s^2 (\log N)^{-2s}\,. \tag{2.8}$$

We may suppose ε is sufficiently small. Writing $\delta = \varepsilon^2$,

$$|\mathcal{B}|^s A_1^2 \ldots A_s^2 \ge |\mathcal{B}|^{s-2} N_1^2 \ldots N_s^2 N^{-\delta}\,. \tag{2.9}$$

Choose j, $1 \le j \le s$. The inequality

$$|\mathcal{B}| A_j^2 \le M N_j^{1+\delta} \tag{2.10}$$

must be satisfied. Otherwise,

$$\sum_{m=1}^{M} T_j(m)^2 \ge M N_j^{1+\delta}\,.$$

Now $M^\delta \le N^{\varepsilon/4} \le N_j$. Since N_j is large, Lemma 2 yields a natural number r such that

$$\begin{gathered} r \le M N_j^{2+\delta} (M N_j^{1+\delta})^{-1} = N_j\,, \\ \|\alpha_j r^2\| \le r\|\alpha_j r\| < N_j^{1+\delta} (M N_j^{1+\delta})^{-1} = M^{-1}\,, \end{gathered}$$

contradicting our hypothesis. This proves (2.10).

From (2.10),

$$|\mathcal{B}|^s A_1^2 \ldots A_s^2 \le M^s (N_1 \ldots N_s)^{1+\delta}\,. \tag{2.11}$$

Suppose first that $s = 2$. Then

$$|\mathcal{B}|^2 A_1^2 A_2^2 \le M^2 (N_1 N_2)^{1+\delta}\,.$$

Combining this with (2.8), (1.4), we have

$$\begin{gathered} N_1^2 N_2^2 (\log N)^{-4} \ll M^2 (N_1 N_2)^{1+\delta}\,, \\ M^2 \gg N^{2-2\delta} (\log N)^{-4}\,. \end{gathered}$$

This contradicts the definition of M, and Lemma 3 is proved for $s = 2$.

Suppose now $s > 2$. We combine (2.9) and (2.11) to obtain an upper bound for $|\mathcal{B}|$:

$$|\mathcal{B}|^{s-2}(N_1 \ldots N_s)^2 N^{-\delta} \le M^s (N_1 \ldots N_s)^{1+\delta},$$

$$|\mathcal{B}|^{s-2} \le M^s (N_1 \ldots N_s)^{-1+\delta} N^{\delta} \le (MN^{-1})^s N^{(s+1)\delta} \tag{2.12}$$

from (1.4).

Choose any $m \in \mathcal{B}$. For any $j \le s$ for which

$$A_j \ge N_j^{1/2+\delta}, \tag{2.13}$$

we apply the case $L = 1$ of Lemma 2. This yields integers r_j, b_j satisfying

$$1 \le r_j \le (N_j/A_j)^2 N_j^{\delta}, \tag{2.14}$$

$$|m\alpha_j r_j^2 - b_j| \le r_j \|m\alpha_j r_j\| \le (N_j/A_j)^4 N_j^{4\delta-2}. \tag{2.15}$$

If (2.13) fails, the last expression in (2.15) is at least 1, and we can trivially satisfy (2.14) and

$$|m\alpha_j r_j^2 - b_j| \le (N_j/A_j)^4 N^{4\delta-2}. \tag{2.16}$$

By (2.9), (2.12) and (1.4),

$$\begin{aligned} A_1^2 \ldots A_s^2 (N_1 \ldots N_s)^{-1-3\delta} (m/M)^{s/2} &\ge |\mathcal{B}|^{-2} N^{-\delta} (N_1 \ldots N_s)^{1-3\delta} (m/M)^{s/2} \\ &\ge N^{s-6s\delta} (MN^{-1})^{-2s/(s-2)} (m/M)^{s/2}. \end{aligned} \tag{2.17}$$

By the definition of M, the last expression in (2.17) is at least $m^{s\beta(s)+2s\delta}$. Thus

$$K_1 \ldots K_s \ge C_3(s,\delta)^s m^{s\beta(s)+s\delta},$$

where $K_j = A_j^2 N_j^{-1-3\delta}(m/M)^{1/2}$.

We apply (1.3). There are integers $x_1, \ldots, x_s$, not all zero, satisfying

$$\sum_{j=1}^{s} b_j x_j^2 \equiv 0 \pmod{m}, \tag{2.18}$$

$$0 \le x_j \le K_j \quad (j = 1, \ldots, s). \tag{2.19}$$

Taking $n_j = r_j x_j$ we have, by (2.14) and (2.19),

$$0 \le n_j \le (N_j/A_j)^2 N_j^{\delta} A_j^2 N_j^{-1-3\delta} (m/M)^{1/2} \le N_j.$$

Not all n_j are 0. Moreover,

$$\sum_{j=1}^{s} \alpha_j n_j^2 = \sum_{j=1}^{s} x_j^2 \alpha_j r_j^2 = m^{-1} \sum_{j=1}^{s} b_j x_j^2 + m^{-1} \sum_{j=1}^{s} x_j^2 (\alpha_j m r_j^2 - b_j).$$

By (2.18), (2.19) and (2.16),

$$\Big\|\sum_{j=1}^{s}\alpha_j n_j^2\Big\| \le m^{-1}\sum_{j=1}^{s} x_j^2|\alpha_j m r_j^2 - b_j|$$
$$\le m^{-1}\sum_{j=1}^{s} A_j^4 N_j^{-2-6\delta}(m/M)N_j^{2+4\delta}A_j^{-4} < M^{-1}\,,$$

contradicting our initial hypothesis. This proves the lemma.

Proof of Theorem 1. We proceed by induction on s. Clearly Heilbronn's theorem is equivalent to Theorem 1 when $s = 1$. Now suppose that $s > 1$ and the result has been proved for forms in $s-1$ variables. It is easily verified that, since $1/2 \le \beta(s) \le 1/2 + 1/(2s-4)$, we have

$$\sigma(s) \le 2 \quad \text{and} \quad \frac{s}{s-1}\sigma(s-1) \ge \sigma(s)\,. \tag{2.20}$$

If $N_j > N^{\varepsilon/4}$ $(j = 1,\ldots,s)$, then the induction step follows from Lemma 3. Thus we may suppose $N_j \le N^{\varepsilon/4}$ for some index j, let us say $j = s$. Consequently,

$$N_1\ldots N_{s-1} \ge N^{s-\varepsilon/4} \ge (N^{s/(s-1)-\varepsilon/4})^{s-1}\,.$$

By the induction hypothesis there are integers $n_1,\ldots,n_{s-1}$, not all zero, satisfying

$$0 \le n_i \le N_i \quad (i = 1,\ldots,s-1)\,,$$
$$\|\alpha_1 n_1^2 + \ldots + \alpha_{s-1}n_{s-1}^2\| < N^{-(s/(s-1)-\varepsilon/4)(\sigma(s-1)-\varepsilon/4)} \le N^{-\sigma(s)+\varepsilon}\,.$$

The last inequality follows from (2.20). This completes the induction step and proves Theorem 1.

3. The lattice method. We write $\boldsymbol{ab}$ for inner product in $\mathbb{R}^2$ and $|\boldsymbol{a}| = (\boldsymbol{aa})^{1/2}$. The area of the parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\det(\boldsymbol{a},\boldsymbol{b})$. Let

$$K_0 = \{\boldsymbol{x}\in\mathbb{R}^2 : |\boldsymbol{x}| < 1\}\,.$$

If $S,T\subset\mathbb{R}^2$ and $c\in\mathbb{R}$ then cS denotes the set $\{c\boldsymbol{s} : \boldsymbol{s}\in S\}$, and $S+T = \{\boldsymbol{s}+\boldsymbol{t} : \boldsymbol{s}\in S, \boldsymbol{t}\in T\}$.

To facilitate comparison with [19] and [2] we prove the following result in place of Theorem 2.

PROPOSITION. *Let $\varepsilon > 0$, $s \ge 2$, $N > C_5(s,\varepsilon)$ and*

$$\lambda(s) = \begin{cases} 1/2 + 2/s & (s = 2,3,4),\\ 4/s + (1-4/s)/\sigma(s) & (s = 5,6,7),\\ 1/2 + 1/(2\sigma(s)) & (s \ge 8).\end{cases}$$

Let Δ be a positive number satisfying

$$1 < \Delta^{\lambda(s)+\varepsilon} \le N \tag{3.1}$$

and let $\Lambda = \Delta^{1/2}\mathbb{Z}^2$. Then for any $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_s \in \mathbb{R}^2$ there are integers $n_1, \ldots, n_s$ satisfying (1.2) *and*

$$n_1^2\boldsymbol{\alpha}_1 + \ldots + n_s^2\boldsymbol{\alpha}_s \in \Lambda + K_0\,. \tag{3.2}$$

To deduce Theorem 2, we first note that

$$\lambda(s) = 1/(2\tau(s))\,,$$

as the reader may easily verify. Let $\Delta = N^{2(\tau(s)-\varepsilon)}$ so that (3.1) holds. Let $\boldsymbol{\alpha}_j = N^{\tau(s)-\varepsilon}(\alpha_j, \beta_j)$. Then (3.2) implies

$$|N^{\tau(s)-\varepsilon}(n_1^2\alpha_1 + \ldots + n_s^2\alpha_s) - N^{\tau(s)-\varepsilon}m| < 1$$

for some integer m, and similarly for the β_j. Now Theorem 2 follows at once.

In the same vein we have the following corollary of Theorem 1.

Corollary. *Let $s \ge 1$, $\delta > 0$ and $N > C_4(s,\delta)$. Suppose that $N_1, \ldots, N_s$ satisfy* (1.4). *Let S be a one-dimensional subspace of $\mathbb{R}^2$ and Λ_1 a lattice in S generated by a point $\boldsymbol{z}$ satisfying*

$$|\boldsymbol{z}| < N^{\sigma(s)-\delta}\,. \tag{3.3}$$

Then for any $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_s$ in S there are non-negative integers $n_1, \ldots, n_s$, not all zero, satisfying $n_i \le N_i$ and

$$n_1^2\boldsymbol{\alpha}_1 + \ldots + n_s^2\boldsymbol{\alpha}_s \in \Lambda_1 + K_0\,. \tag{3.4}$$

In the remainder of the paper, Δ, Λ are as in the Proposition. Let Π be the polar lattice of Λ, $\Pi = \Delta^{-1/2}\mathbb{Z}^2$. Let Π^* be the set of primitive points of Π. Evidently

$$|\boldsymbol{p}| \ge \Delta^{-1/2} \quad (\boldsymbol{p} \in \Pi^*)\,. \tag{3.5}$$

(Usually the lattice method is applied to general lattices in $\mathbb{R}^h$. The right-hand side of (3.5) would then be, essentially, Δ^{-1}. The stronger bound (3.5) is crucial to our proof.)

Let $\boldsymbol{p} \in \Pi^*$ and let $\boldsymbol{p}^\perp = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x}\boldsymbol{p} = 0\}$. Clearly $2\Lambda \cap \boldsymbol{p}^\perp$ is a lattice in $\boldsymbol{p}^\perp$ generated by a point $\boldsymbol{z}$ having

$$|\boldsymbol{z}| = 2\Delta|\boldsymbol{p}|\,. \tag{3.6}$$

In our application of the Corollary, we shall have $S = \boldsymbol{p}^\perp$, $\Lambda_1 = 2\Lambda \cap \boldsymbol{p}^\perp$.

Lemma 4. (i) *Let $\boldsymbol{p} \in \Pi^*$. Any point $\boldsymbol{a} \in \mathbb{R}^2$ may be written in the form*

$$\boldsymbol{a} = \boldsymbol{l} + \boldsymbol{s} + \boldsymbol{b} \tag{3.7}$$

where $\boldsymbol{l} \in \Lambda$, $\boldsymbol{s} \in \boldsymbol{p}^{\perp}$ *and*

$$|\boldsymbol{b}| \ll |\boldsymbol{p}|^{-1} \|\boldsymbol{p}\boldsymbol{a}\| . \tag{3.8}$$

(ii) *Let* $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ *be linearly independent points of* Π^*. *There is a positive integer* c,

$$c \ll \det(\boldsymbol{p}_1, \boldsymbol{p}_2)\Delta , \tag{3.9}$$

such that any $\boldsymbol{a} \in \mathbb{R}^2$ *may be written in the form*

$$\boldsymbol{a} = c^{-1}(\boldsymbol{k} + \boldsymbol{d}) \tag{3.10}$$

where $\boldsymbol{k} \in \Lambda$ *and*

$$|\boldsymbol{d}| \ll \Delta \max(|\boldsymbol{p}_1|, |\boldsymbol{p}_2|) \max(\|\boldsymbol{p}_1\boldsymbol{a}\|, \|\boldsymbol{p}_2\boldsymbol{a}\|) . \tag{3.11}$$

Proof. These are two special cases of Lemma 7.9 of [2].

LEMMA 5. *Let* $\varepsilon > 0$, $N > C_8(\varepsilon)$. *Let* $\mathcal{A}$ *be a subset of* Π^* *with* $|\boldsymbol{p}| \le N$ *for all* $\boldsymbol{p}$ *in* $\mathcal{A}$. *Suppose that any two vectors in* $\mathcal{A}$ *have determinant* $\le Z$. *Let* $\boldsymbol{e} \in \mathbb{R}^2$. *Let* U, V *be positive numbers such that for each* $\boldsymbol{p}$ *in* $\mathcal{A}$ *there are coprime integers* $q(\boldsymbol{p})$, $w(\boldsymbol{p})$ *having*

$$1 \le q(\boldsymbol{p}) \le U \le N, \qquad |q(\boldsymbol{p})\boldsymbol{e}\boldsymbol{p} - w(\boldsymbol{p})| \le V .$$

Suppose further that $\Delta \le N^2$,

$$ZU^2 V \Delta N^{\delta} \le 1 . \tag{3.12}$$

Then there is an integer q *and a subset* $\mathcal{C}$ *of* $\mathcal{A}$ *such that*

$$|\mathcal{C}| \ge |\mathcal{A}| N^{-\delta}, \qquad q(\boldsymbol{p}) = q \quad \text{for all } \boldsymbol{p} \in \mathcal{C} .$$

Proof. See [2], Lemma 7.6.

The starting point for the proof of the Proposition is the following variant of Lemma 1.

LEMMA 6. *Let* $\delta > 0$, $N > C_9(\delta)$. *Let* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ *be a sequence in* $\mathbb{R}^2$ *with*

$$\boldsymbol{x}_n \notin \Lambda + K_0 \qquad (n = 1, \ldots, N) .$$

Then

$$\sum_{\boldsymbol{p} \in \Pi,\, 0 < |\boldsymbol{p}| < N^{\delta}} \Big| \sum_{n=1}^{N} e(\boldsymbol{p}\boldsymbol{x}_n) \Big| \gg N .$$

Proof. See [2], Lemma 7.4.

Proof of the Proposition. For brevity, write $\sigma = \sigma(s)$, $\lambda = \lambda(s)$. We may suppose that $0 < \varepsilon < 1/2$. Let $\delta = \varepsilon/40$, so that

$$\Delta^{\lambda} \le N^{1-20\delta} . \tag{3.13}$$

Suppose that no integers $n_1, \dots, n_s$ satisfy (1.2) and (3.2). By Lemma 6, we have

$$\sum_{0<|\boldsymbol{p}|<N^\delta,\, \boldsymbol{p}\in\Pi} T_1(\boldsymbol{p})\dots T_s(\boldsymbol{p}) \gg N^s \tag{3.14}$$

where

$$T_i(\boldsymbol{p}) = \Big|\sum_{n=1}^{N} e(n^2 \boldsymbol{p}\boldsymbol{\alpha}_i)\Big|.$$

Since every $\boldsymbol{p}\in\Pi$ is an integer multiple of a primitive point, it follows that

$$\sum_{|\boldsymbol{p}|<N^\delta,\, \boldsymbol{p}\in\Pi^*} S(\boldsymbol{p}) \gg N^s$$

where

$$S(\boldsymbol{p}) = \sum_{t=1}^{[N^\delta/|\boldsymbol{p}|]} T_1(t\boldsymbol{p})\dots T_s(t\boldsymbol{p})\,.$$

We cover the interval $[\Delta^{-1/2}, N^\delta)$ with $O(\log N)$ intervals $[a, 2a)$. In view of (3.5), there is an a satisfying

$$\Delta^{-1/2} \le a < N^\delta\,, \tag{3.15}$$

$$\sum_{\boldsymbol{p}\in\Pi^*,\, a\le|\boldsymbol{p}|<2a} S(\boldsymbol{p}) \gg N^s/\log N\,.$$

There are $\ll \Delta a^2$ summands here, so that the contribution from $\boldsymbol{p}$ with

$$S(\boldsymbol{p}) < N^s(\log N)^{-2}\Delta^{-1}a^{-2}$$

is negligible. Covering $[N^s(\log N)^{-2}\Delta^{-1}a^{-2},\ a^{-1}N^{s+\delta}]$ with $O(\log N)$ intervals $[B, 2B)$, we see that there is a B with

$$N^s(\log N)^{-2}\Delta^{-1}a^{-2} \le B < a^{-1}N^{s+\delta}$$

and a subset $\mathcal{B}$ of Π^* with

$$a \le |\boldsymbol{p}| < 2a, \quad B \le S(\boldsymbol{p}) < 2B \quad \text{for } \boldsymbol{p}\in\mathcal{B}\,,$$

$$\sum_{\boldsymbol{p}\in\mathcal{B}} S(\boldsymbol{p}) \gg N^s(\log N)^{-2}\,.$$

It is convenient to write $X = N^sB^{-1}$, so that

$$X \le \Delta a^2 N^\delta\,, \tag{3.16}$$

$$S(\boldsymbol{p}) \ge N^s X^{-1} \quad (\boldsymbol{p}\in\mathcal{B}) \tag{3.17}$$

and clearly

$$|\mathcal{B}| \gg XN^{-\delta}\,. \tag{3.18}$$

Consider the following conditions:

(A) $X \ge N^{5\delta}$ and $\Delta a X^{2/s-1/2} \le N^{1-10\delta}$;

(B) $\Delta^{1/\sigma} a^{-1+1/\sigma} X^{2/s} \le N^{1-10\delta}$ if $s \ge 3$; $X < N^{5\delta}$ if $s = 2$.

We will first derive a contradiction provided that (A) or (B) holds. In conclusion we show that one of (A), (B) must be satisfied.

Suppose first that (A) is satisfied. Using $T_1 \ldots T_s \le T_1^s + \ldots + T_s^s$, (3.17) gives

$$\sum_{t \le N^\delta a^{-1}} T_i(t\boldsymbol{p})^s \gg N^s X^{-1} \tag{3.19}$$

for some index i depending on $\boldsymbol{p}$. There are only s possible i, and we may assume without loss of generality that (3.19) holds for $i = 1$.

We use the inequality

$$(b_1^l + \ldots + b_m^l)^{1/l} \le (b_1^r + \ldots + b_m^r)^{1/r}, \quad 0 < r \le l,$$

for positive numbers b_j ([12], p. 28). Thus

$$\sum_{t \le N^\delta a^{-1}} T_1(t\boldsymbol{p})^2 \gg N^2 X^{-2/s} \quad (\boldsymbol{p} \in \mathcal{B}). \tag{3.20}$$

We may apply Lemma 2 to the sum in (3.20). To see this,

$$a^{-1} X^{2/s} \le \Delta^{2/s} a^{2/s-1} N^{2\delta} \le \Delta^{1/s+1/2} N^{2\delta} \le \Delta^\lambda N^{2\delta} \le N^{1-3\delta}$$

from (3.16), (3.15), (3.13). Hence

$$N^2 X^{-2/s} \ge N^{1+\delta}[a^{-1} N^\delta].$$

The lemma yields natural numbers $q(\boldsymbol{p})$ for each $\boldsymbol{p}$ in $\mathcal{B}$ satisfying

$$q(\boldsymbol{p}) \ll a^{-1} X^{2/s} N^{2\delta}, \tag{3.21}$$

$$\|q(\boldsymbol{p})\boldsymbol{p}\boldsymbol{\alpha}_1\| \ll X^{2/s} N^{-2+2\delta}. \tag{3.22}$$

The next step is to apply Lemma 5 to a suitable subset of $\mathcal{B}$. The (x, y) plane may be covered by $\ll |\mathcal{B}| N^{-2\delta}$ angular sections centred at $\mathbf{0}$, of angle $|\mathcal{B}|^{-1} N^{2\delta} \le X^{-1} N^{4\delta}$. Here we have used (3.18) and the hypothesis $X \ge N^{5\delta}$. One of these sections must contain $\gg N^{2\delta}$ points of $\mathcal{B}$. Let $\mathcal{A}$ be the set of points of $\mathcal{B}$ lying in this section. In the notation of Lemma 5, we may take

$$\boldsymbol{e} = \boldsymbol{\alpha}_1, \quad Z = a^2 X^{-1} N^{4\delta}, \quad U \ll a^{-1} X^{2/s} N^{2\delta}, \quad V \ll X^{2/s} N^{-2+2\delta}$$

in view of the definition of $\mathcal{A}$, (3.21) and (3.22). Moreover, by (3.16), (3.15) and condition (A),

$$\begin{aligned} ZU^2 V \Delta N^{2\delta} &\ll a^2 X^{-1} N^{4\delta} (a^{-1} X^{2/s} N^{2\delta})^2 (N^{-2+2\delta} X^{2/s}) \Delta N^{2\delta} \\ &\ll X^{6/s-1} \Delta N^{-2+12\delta} \ll \begin{cases} \Delta^{6/s} N^{-2+30\delta} & (s \le 6) \\ \Delta N^{-2+12\delta} & (s > 6) \end{cases} \\ &\ll \Delta^{2\lambda} N^{-2+30\delta} \ll 1. \end{aligned}$$

We deduce from Lemma 5 that there is a subset $\mathcal{C}$ of $\mathcal{A}$ of cardinality $\gg N^{\delta}$, and a natural number q such that $q(\boldsymbol{p}) = q$ for every $\boldsymbol{p}$ in $\mathcal{C}$.

Let $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ be two linearly independent points in $\mathcal{C}$. We apply Lemma 4(ii) with $\boldsymbol{a}$ replaced by $q\boldsymbol{\alpha}_1$. There is a natural number c such that

$$c \ll \det(\boldsymbol{p}_1, \boldsymbol{p}_2)\Delta \le a^2 X^{-1} N^{4\delta} \Delta\,, \tag{3.23}$$

$$cq\boldsymbol{\alpha}_1 = \boldsymbol{k} + \boldsymbol{d}, \qquad \boldsymbol{k} \in \Lambda\,,$$

$$|\boldsymbol{d}| \ll \Delta a X^{2/s} N^{-2+2\delta}\,. \tag{3.24}$$

Here we use once again the definition of $\mathcal{A}$ and (3.22).

Now let $n = cq$. By (3.23), (3.21) and condition (A),

$$1 \le n \ll a^2 X^{-1} N^{4\delta} \Delta a^{-1} X^{2/s} N^{2\delta} = a X^{2/s-1} \Delta N^{6\delta} \ll N^{1-\delta}\,.$$

Moreover,

$$n^2 \boldsymbol{\alpha}_1 = cq\boldsymbol{k} + cq\boldsymbol{d}\,,$$

$$|cq\boldsymbol{d}| = n|\boldsymbol{d}| \ll a X^{2/s-1} \Delta N^{6\delta} \Delta a X^{2/s} N^{-2+2\delta}$$

$$\ll a^2 X^{4/s-1} \Delta^2 N^{-2+8\delta} \ll N^{-\delta}$$

by (3.24) and condition (A). Thus $1 \le n \le N$, $n^2\boldsymbol{\alpha}_1 \in \Lambda + K_0$. We have reached a contradiction when (A) holds.

Now suppose that condition (B) holds. We deal with the case $s \ge 3$ first. Fix any $\boldsymbol{p}$ in $\mathcal{B}$. From (3.17),

$$\sum_{t \le a^{-1} N^{\delta}} T_1(t\boldsymbol{p}) \ldots T_s(t\boldsymbol{p}) \ge N^s X^{-1}\,.$$

As in the proof of (2.8), there must be $A_1, \ldots, A_s$ in $[N^{-2}, N]$ and a set $\mathcal{A}$ of t having

$$1 \le t \le a^{-1} N^{\delta} \qquad (t \in \mathcal{A})\,, \tag{3.25}$$

$$A_i \le T_i(t\boldsymbol{p}) < 2A_i \qquad (i = 1, \ldots, s;\ t \in \mathcal{A}) \tag{3.26}$$

and

$$|\mathcal{A}|^2 A_1^2 \ldots A_s^2 \gg N^{2s-\delta} X^{-2}\,. \tag{3.27}$$

We may assume that

$$A_1 \ge \ldots \ge A_s\,. \tag{3.28}$$

The next step, in which we deduce a good rational approximation to $\boldsymbol{\alpha}_i \boldsymbol{p}$, is different according as $i \le 2$ or $i > 2$. By (3.27) and (3.28),

$$|\mathcal{A}|^2 A_2^{2(s-1)} \ge N^{2(s-1)-2\delta} X^{-2}\,,$$

$$|\mathcal{A}|^{2/(s-1)} A_2^2 \ge N^{2-2\delta} X^{-2/(s-1)}\,.$$

Since $s \geq 3$, this implies for $i = 1, 2$ that

$$\sum_{t \leq a^{-1}N^{\delta}} T_i(t\boldsymbol{p})^2 \geq |\mathcal{A}|A_i^2 \geq N^{2-2\delta}X^{-2/(s-1)} . \tag{3.29}$$

By (3.16), (3.15) and (3.13),

$$\begin{aligned} a^{-1}X^{2/(s-1)}N^{4\delta} &\leq \Delta^{2/(s-1)}a^{2/(s-1)-1}N^{6\delta} \leq \Delta^{1/2+1/(s-1)}N^{6\delta} \\ &\leq \Delta^{\lambda}N^{6\delta} \leq N . \end{aligned}$$

Consequently,

$$N^{1+\delta}[a^{-1}N^{\delta}] \leq N^{2-2\delta}X^{-2/(s-1)} . \tag{3.30}$$

In view of (3.30), we may apply Lemma 2 to (3.29), obtaining natural numbers q_1, q_2 having

$$q_i \leq a^{-1}N^{2+2\delta}A_i^{-2}|\mathcal{A}|^{-1} , \tag{3.31}$$

$$\|q_i^2\boldsymbol{\alpha}_i\boldsymbol{p}\| \leq q_i\|q_i\boldsymbol{\alpha}_i\boldsymbol{p}\| \leq a^{-1}N^{2+3\delta}A_i^{-4}|\mathcal{A}|^{-2} . \tag{3.32}$$

Now pick any $t \in \mathcal{A}$. If $A_i \geq N^{1/2+\delta}$ for an index $i \geq 3$, we may apply the case $L = 1$ of Lemma 2 to $T_i(t\boldsymbol{p})$. This gives a natural number r_i with

$$r_i \leq N^{2+\delta}A_i^{-2}, \quad \|r_i t\boldsymbol{\alpha}_i\boldsymbol{p}\| \leq A_i^{-2}N^{\delta} .$$

Writing $q_i = r_i t$, we then have, from (3.25),

$$q_i \leq a^{-1}N^{2+2\delta}A_i^{-2} , \tag{3.33}$$

$$\|q_i^2\boldsymbol{\alpha}_i\boldsymbol{p}\| \leq a^{-1}N^{2+5\delta}A_i^{-4} . \tag{3.34}$$

Just as in (2.16), we can in fact find q_i satisfying (3.33), (3.34) for every $i = 1, \ldots, s$.

Combining (3.31)–(3.34), we have

$$q_i \leq a^{-1}N^{2+2\delta}A_i^{-2}C(i)^{-1} , \tag{3.35}$$

$$\|q_i^2\boldsymbol{\alpha}_i\boldsymbol{p}\| \leq a^{-1}N^{2+5\delta}A_i^{-4}C(i)^{-2} , \tag{3.36}$$

where $C(i) = |\mathcal{A}|$ for $i \leq 2$, $C(i) = 1$ for $i > 2$. Applying Lemma 4(i) and (3.36), we obtain

$$q_i^2\boldsymbol{\alpha}_i = \boldsymbol{l}_i + \boldsymbol{t}_i + \boldsymbol{b}_i , \tag{3.37}$$

where $\boldsymbol{l}_i \in \Lambda$, $\boldsymbol{t}_i \in \boldsymbol{p}^{\perp}$ and

$$|\boldsymbol{b}_i| \ll a^{-1}a^{-1}N^{2+5\delta}A_i^{-4}C(i)^{-2} \ll a^{-2}N^{2+5\delta}A_i^{-4}C(i)^{-2} . \tag{3.38}$$

Recalling (3.6), we apply the Corollary, taking $S = \boldsymbol{p}^{\perp}$, $\Lambda_1 = 2\Lambda \cap \boldsymbol{p}^{\perp}$, and replacing $\boldsymbol{\alpha}_j$ by $2\boldsymbol{t}_j$ and N by $N^* = (a\Delta)^{1/\sigma}N^{2\delta}$. We replace N_i by

$$N_i^* = (a\Delta)^{1/\sigma}A_i^2N^{-2+3\delta}X^{2/s}C(i) .$$

We must verify (1.4). From (3.27),

$$N_1^* \ldots N_s^* = (a\Delta)^{s/\sigma}|\mathcal{A}|^2A_1^2 \ldots A_s^2N^{-2s+3s\delta}X^2 \geq (N^*)^s .$$

Thus there are non-negative integers $m_1, \ldots, m_s$, not all zero, satisfying

$$2m_1^2\boldsymbol{t}_1 + \ldots + 2m_s^2\boldsymbol{t}_s \in 2\Lambda + K_0\,, \tag{3.39}$$

$$m_i \le N_i^* \quad (i = 1, \ldots, s)\,. \tag{3.40}$$

Now let $n_i = q_i m_i$. Not all n_i are zero. Moreover, by (3.35), (3.40) and condition (B),

$$\begin{aligned} n_i &\le a^{-1}N^{2+2\delta}A_i^{-2}C(i)^{-1}(a\Delta)^{1/\sigma}A_i^2N^{-2+3\delta}X^{2/s}C(i) \\ &= \Delta^{1/\sigma}a^{1/\sigma-1}X^{2/s}N^{5\delta} \le N \end{aligned}$$

while

$$\begin{aligned} n_1^2\boldsymbol{\alpha}_1 + \ldots + n_s^2\boldsymbol{\alpha}_s = m_1^2\boldsymbol{l}_1 + \ldots + m_s^2\boldsymbol{l}_s + m_1^2\boldsymbol{t}_1 + \ldots + m_s^2\boldsymbol{t}_s \\ + m_1^2\boldsymbol{b}_1 + \ldots + m_s^2\boldsymbol{b}_s\,. \end{aligned}$$

Here $m_1^2\boldsymbol{l}_1 + \ldots + m_s^2\boldsymbol{t}_s \in \Lambda + \frac{1}{2}K_0$ by (3.39). By (3.40), (3.38) and condition (B),

$$\begin{aligned} |m_i^2\boldsymbol{b}_i| &\ll (a\Delta)^{2/\sigma}A_i^4N^{-4+6\delta}X^{4/s}C(i)^2a^{-2}N^{2+5\delta}A_i^{-4}C(i)^{-2} \\ &= \Delta^{2/\sigma}a^{2/\sigma-2}X^{4/s}N^{-2+11\delta} \ll N^{-\delta}\,. \end{aligned}$$

We conclude that

$$n_1^2\boldsymbol{\alpha}_1 + \ldots + n_s^2\boldsymbol{\alpha}_s \in \Lambda + K_0\,.$$

We have now reached a contradiction when condition (B) holds and $s \ge 3$.

Now let $s = 2$. Pick any $\boldsymbol{p} \in \mathcal{B}$. From (3.17) and condition (B) we have

$$\sum_{t \le a^{-1}N^\delta} T_1(t\boldsymbol{p})T_2(t\boldsymbol{p}) \gg N^{2-5\delta}\,,$$

$$\sum_{t \le a^{-1}N^\delta} T_i(t\boldsymbol{p}) \gg N^{1-5\delta} \quad (i = 1, 2)\,.$$

From Cauchy's inequality,

$$\sum_{t \le a^{-1}N^\delta} T_i(t\boldsymbol{p})^2 \gg (N^{1-5\delta})^2(a^{-1}N^\delta)^{-1} \gg N^{2-11\delta}a\,.$$

We may apply Lemma 2, since

$$N^{2-11\delta}a(N^\delta a^{-1})^{-1} = N^{2-12\delta}a^2 \gg N^{2-12\delta}\Delta^{-1} \gg N^{1+\delta}$$

from (3.15), (3.13). Thus there are natural numbers q_1, q_2 satisfying

$$q_i \ll a^{-2}N^{13\delta}\,, \tag{3.41}$$

$$\|q_i(\boldsymbol{\alpha}_i\boldsymbol{p})\| \ll a^{-1}N^{-2+12\delta}\,. \tag{3.42}$$

By Lemma 4(i),

$$q_i^2\boldsymbol{\alpha}_i = \boldsymbol{l}_i + \boldsymbol{s}_i + \boldsymbol{b}_i\,,$$

where $\boldsymbol{l}_i \in \Lambda$, $\boldsymbol{s}_i \in \boldsymbol{p}^{\perp}$ and

$$(3.43)\quad |\boldsymbol{b}_i| \ll a^{-1}q_i\|q_i(\boldsymbol{\alpha}_i\boldsymbol{p})\| \ll a^{-1}a^{-2}N^{13\delta}a^{-1}N^{-2+12\delta} = a^{-4}N^{-2+25\delta}\,.$$

Here we used (3.41) and (3.42).

We apply the Corollary as above, this time replacing N, N_1, N_2 by $\Delta a N^{2\delta}$. There are non-negative integers m_1, m_2, not both zero, with

$$(3.44)\qquad 2m_1^2\boldsymbol{s}_1 + 2m_2^2\boldsymbol{s}_2 \in 2\Lambda + K_0\,,$$

$$(3.45)\qquad m_i \le \Delta a N^{2\delta}\,.$$

Now let $n_i = m_i q_i$. Then

$$n_i \le \Delta a N^{2\delta}a^{-2}N^{14\delta} = a^{-1}\Delta N^{16\delta} \le \Delta^{3/2}N^{16\delta} \le N\,,$$

$$|m_i^2\boldsymbol{b}_i| \ll \Delta^2a^2N^{4\delta}a^{-4}N^{-2+25\delta} \ll a^{-2}\Delta^2N^{-2+29\delta} \ll \Delta^3N^{-2+29\delta} \ll N^{-\delta}$$

from (3.45), (3.43), (3.15) and (3.13). Just as above, we reach a contradiction when $s = 2$ and condition (B) holds.

It remains to show that one of (A), (B) is satisfied. If $X < N^{5\delta}$ we have, by (3.15) and (3.13),

$$\Delta^{1/\sigma}a^{-1+1/\sigma}X^{2/s}N^{10\delta} \le \Delta^{1/2+1/(2\sigma)}N^{15\delta} \le \Delta^{\lambda}N^{15\delta} \le N\,,$$

so that (B) holds. If $X \ge N^{5\delta}$ and $s = 2$ then

$$\Delta a X^{2/s-1/2} = \Delta a X^{1/2} \ll \Delta^{3/2}a^2N^{\delta} \ll \Delta^{3/2}N^{3\delta} \ll N^{1-11\delta}$$

from (3.16), (3.15), (3.13), so that (A) holds.

It remains to show that, for $s \ge 3$,

$$(3.46)\qquad \min(\Delta a X^{2/s-1/2}N^{10\delta}, \Delta^{1/\sigma}a^{-1+1/\sigma}X^{2/s}N^{10\delta}) \le N$$

whenever $\Delta^{-1/2} \le a \le N^{\delta}$, $N^{5\delta} \le X \le \Delta a^2 N^{\delta}$.

If $s = 3, 4$, the left-hand side of (3.46) is

$$\le \Delta a X^{2/s-1/2}N^{10\delta} \le \Delta^{1/2+2/s}a^{4/s}N^{11\delta} \le \Delta^{1/2+2/s}N^{13\delta} \le \Delta^{\lambda}N^{13\delta} \le N\,.$$

If $s = 5, 6, 7$ the left-hand side of (3.46) is

$$\le (\Delta^{1/\sigma}a^{-1+1/\sigma})^{1-4/s}(\Delta a)^{4/s}N^{10\delta}\,.$$

The exponent of a here is positive, so we obtain the bound

$$\le \Delta^{(1/\sigma)(1-4/s)+4/s}N^{12\delta} = \Delta^{\lambda}N^{12\delta} \le N\,.$$

Finally, if $s \ge 8$, the left-hand side of (3.46) is

$$\begin{aligned}&\le (\Delta a X^{2/s-1/2})^{1/2}(\Delta^{1/\sigma}a^{-1+1/\sigma}X^{2/s})^{1/2}N^{10\delta}\\ &\le \Delta^{1/2+1/(2\sigma)}a^{1/(2\sigma)}X^{2/s-1/4}N^{10\delta}\\ &\le \Delta^{1/2+1/(2\sigma)}N^{11\delta} = \Delta^{\lambda}N^{11\delta} \le N\,.\end{aligned}$$

This completes the proof of the Proposition.

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