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Bounded remainder sets

by

Sébastien Ferenczi (Marseille)

Definitions. Let L be a *lattice* in $\mathbb{R}^s$ (that is, a discrete subgroup of maximal order) and let α be an element of $\mathbb{R}^s$; (α, L) is said to be a *minimal couple* if for every nonzero linear form ϕ on $\mathbb{R}^s$ such that $\phi(L)$ is included in $\mathbb{Z}$, $\phi(\alpha)$ is not in $\mathbb{Z}$.

We define the *rotation* T on the set $X = \mathbb{R}^s/L$ by $Tx = x + \alpha \bmod L$; it preserves the Lebesgue measure λ on X, and (α, L) is minimal if and only if T is minimal, that is, has dense orbits; in particular, L and α must generate $\mathbb{R}^s$. If $\alpha = (\alpha_1, \ldots, \alpha_s)$ and L is $\mathbb{Z}^s$, this is equivalent to $(1, \alpha_1, \ldots, \alpha_s)$ being rationally independent.

A set A in $\mathbb{R}^s$ is L-*simple* if whenever $x \in A$, $y \in A$, $x - y \in L$, then $x = y$.

Let A be a subset of X; we say A is a *bounded remainder set* (BRS) if there exist real numbers a and C such that for every integer n and λ-almost every x in X,

$$\Big|\sum_{p=1}^{n} 1_A(T^p x) - na\Big| < C\,.$$

This definition also applies to L-simple subsets of $\mathbb{R}^s$, which we shall always identify with their projection on X.

It is a well-known result, which can for example be derived from the Markov–Kakutani fixed point theorem, that if A is measurable, then A is a BRS if and only if there exists a bounded function F such that

$$1_A - a = F - TF\,,$$

and in that case a can only be $\lambda(A)$.

For a set A of strictly positive measure and a point x in A, we denote by $\tau(x)$ the *return time* of x in A (that is, the least strictly positive integer n such that $T^n x$ is in A) and by $Sx = T^{\tau(x)}x$ the *induced map* of T on A, which exists by the Poincaré recurrence theorem.

Known results about BRS. If $s = 1$ and A is an interval, A is a BRS if and only if its length belongs to $\mathbb{Z}(\alpha)$ (Kesten [1]); a similar result holds when A is a finite union of intervals (Oren [3]).

If $s \geq 2$, there are no nontrivial rectangles which are BRS (Liardet [2]); it seems difficult to find nontrivial examples of BRS when $s \geq 2$; Szüsz ([6]) had one example of nontrivial parallelogram.

Rudolph [private communication] showed that whenever there exists a BRS of measure $a > 0$, the BRS are dense among the sets of measure a; this is true for every ergodic transformation.

Rauzy's sufficient condition

Let S be the induced map of T on A. If there exists a lattice M and an element β of $\mathbb{R}^s$ such that (β, M) is a minimal couple and $Sx = x + \beta \bmod M$, then A is a BRS (even if B is not measurable).

This criterion enabled Rauzy to find nonmeasurable examples of BRS in dimension $s = 1$ ([4]), and new nontrivial examples (parallelograms) in higher dimensions ([5]); however, this condition is not necessary, as can be seen in dimension 1 with the interval $[0, 2\alpha]$, though in this counter-example the set A breaks into a finite union of subsets which satisfy Rauzy's criterion. We can now give a

Necessary and sufficient condition generalizing Rauzy's criterion

Let A be a subset of $\mathbb{R}^s$, L-simple, measurable and with nonempty interior. Then A is a BRS if and only if there exist a lattice M' in $\mathbb{R}^{s+1}$ and a bounded function n from A to $\mathbb{N}$ such that, if ψ is the function from A to $\mathbb{R}^{s+1}$ defined by $\psi(x) = (x, n(x))$, and if Q is the translation of $\mathbb{R}^{s+1}/M'$ defined by $Q(z) = z + (0, \ldots, 0, 1)$, then $\psi(A)$ is a fundamental domain for Q, that is, for every z in $\psi(A)$, there exists a unique z' in $\psi(A)$ such that $z' \equiv Qz \bmod M'$. Thus we can define Q as a mapping from $\psi(A)$ to $\psi(A)$, and we have

$$S = \psi^{-1} Q \psi$$

(this last equality being defined λ-almost everywhere).

Proof of the condition. In all what follows, T, S and X will be as defined above and A will be a measurable L-simple set with nonempty interior.

Let W be a fundamental domain for the rotation T, containing the set A; for an element x in W, we denote by x' its projection on X. As a mapping from W to W, T can be viewed as a finite exchange of pieces (an exchange of two intervals if $s = 1$). The same is true for S, as a mapping from A to A, $A \subset W$:

LEMMA 1. *There exists a finite partition of A into sets A_i, and a finite number of elements e_i, $1 \le i \le r$, such that*,

$$Sx = x + e_i \quad \text{whenever } x \text{ is in } A_i .$$

Proof. A must contain an open set Ω. By Kronecker's theorem and compactness,

$$X = \bigcup_{n=1}^{+\infty} T^n \Omega = \bigcup_{n=1}^{N} T^n \Omega ,$$

for some finite N. Hence the return time $\tau(x)$ is bounded by N, and so takes only a finite number of values.

Now, for every x,

$$Sx = x + \tau(x)\alpha + g(x) ,$$

$g(x)$ being *the* element of L such that $x + \tau(x)\alpha + g(x)$ belongs to W. Then $g(x)$ must be bounded, and hence takes a finite number of values.

Now, if we partition A according to the values of $\tau(x)$ and $g(x)$, and if we define $e_i = \tau_i\alpha + g_i$, we get our lemma.

Proof that the condition is necessary. We suppose A is a BRS. Then

$$(1)\qquad 1_A(y) - \lambda(A) = F(y) - F(Ty) \quad \text{for almost every } y \text{ in } X .$$

This implies

$$e^{2\pi i TF}/e^{2\pi i F} = e^{2\pi i \lambda(A)} \quad \text{almost everywhere} .$$

Hence F and $\lambda(A)$ are an eigenvector and an eigenvalue for an ergodic rotation, and so there exist a linear form ϕ on $\mathbb{R}^s$ such that $\phi(L) \subset \mathbb{Z}$, an integer p and a measurable bounded integer function n such that

$$(2)\qquad \lambda(A) = \phi(\alpha) + p ,$$

$$(3)\qquad F(x') = \phi(x') + n(x') \quad \text{for almost all } x \text{ in } W .$$

The second equation lifts to W yielding

$$(4)\qquad F(x) = \phi(x) + n(x) ,$$

with some (bounded) modifications of the integer function n; and it would lift in the same way (with different functions n) to any other fundamental domain.

From ergodicity, we have

$$W = \bigcup_{i=1}^{r} \bigcup_{j=1}^{\tau_i - 1} T^j A_i .$$

Following Rauzy, we define a new fundamental domain by

$$Y = \bigcup_{i=1}^{r} \bigcup_{j=1}^{\tau_i - 1} (A_i + j\alpha)\,.$$

The sets $A_i + j\alpha$ can be seen as levels of a tower; on them, T is defined in the following manner: on the levels other than the top levels (that is, when $j < \tau_i$), $Tx = x + \alpha$; on the top levels, $Tx = x + \alpha + g_i$.

Now, if we write (4) for our new fundamental domain Y, and, together with (2) and the new expression for T, insert it into (1), we get

$$1_A(x) - \phi(\alpha) - p = \phi(x) - \phi(Tx) + n(x) - n(Tx)\,,$$

hence, as ϕ is linear, we get finally

$$1_A(x) - p = n(x) - n(x+\alpha) \quad \text{if } x \text{ is not in a top level}\,,$$
$$1_A(x) - p = n(x) - n(x+\alpha+g_i) - \phi(g_i) \quad \text{if } x \text{ is in a top level above } A_i\,.$$

Suppose we already know $n(x)$ on the basis A; this defines n on the whole tower, by $n(x+\alpha) = n(x)+p-1$ on the first floor, $n(x+2\alpha) = n(x)+2p-1$ on the second floor, and so on as long as we do not reach the top. We just have to write the compatibility relation at the top:

$$n(x) - n(x+\alpha) = 1 - p\,,$$
$$n(x+\alpha) - n(x+2\alpha) = -p\,,$$
$$n(x + (\tau_i - 1)\alpha) - n(x + \tau_i\alpha + g_i) = -p + \phi(g_i)\,,$$

hence

$$n(x) - n(Sx) = 1 - p\tau_i + \phi(g_i) \quad \text{whenever } x \in A_i\,.$$

Let $m_i, 1 \le i \le r$, be the integer $p\tau_i - \phi(g_i)$; these integers satisfy the following property: if $(q_i, 1 \le i \le r)$ is an r-uple of integers such that $\sum q_i e_i = 0$, then

$$\sum q_i m_i = 0\,. \tag{5}$$

This is easy to see, since if $\sum q_i e_i = 0$, then $\sum q_i \tau_i = 0$ and $\sum q_i g_i = 0$, hence also $\phi(\sum q_i g_i) = 0$ and so $\sum q_i m_i = 0$.

Also,

$$m_i = 1 + n(Sx) - n(x) \quad \text{for almost all } x \text{ in } A_i\,. \tag{6}$$

Let now M be the set ($\sum q_i e_i$, for all r-uples of integers q_i such that $\sum q_i m_i = 0$).

M is a lattice: it is clear that M is a discrete subgroup of $\mathbb{R}^s$, so it suffices to show that its dimension as a $\mathbb{Q}$-vector space is exactly s.

Consider the mapping Φ from $\mathbb{Q}^r$ to $\mathbb{R}^s$ given by $\Phi(q_1, \ldots, q_r) = \sum q_i e_i$; its image is contained in $\mathbb{Q}(\alpha) + \mathbb{Q}(L)$, so must be of dimension at most $s+1$; but since S, being the induced map of a minimal map on a set with nonempty interior, has dense orbits in an open set, $\dim \operatorname{Im} \Phi$ must be exactly $s+1$; hence $\operatorname{Ker} \Phi$ is of dimension $r-s-1$.

Consider now the set $B = (\sum q_i m_i = 0)$; as the m_i are not all zero (they have average one), B is of dimension 1, and contained in $\operatorname{Ker} \Phi$ by (5); hence $\Phi(B)$ is of dimension s.

Now choose k such that m_k is not zero, and put $\beta = e_k/m_k$; we have

$$e_i \equiv m_i \beta \bmod M \quad \text{for all } i\,. \tag{7}$$

As we have $Sx \equiv x + m_i\beta \bmod M$, and as S has dense orbits in an open set, (β, M) must be a minimal couple.

So we have already an *intermediate form of the necessary condition*: there exist a lattice M in $\mathbb{R}^s$, an element β of $\mathbb{R}^s$, a bounded function n from A to $\mathbb{Z}$, and a partition A_i of A, such that

$$(\beta, M) \text{ is minimal}\,,$$
$$m_i = 1 + n(Sx) - n(x) \quad \text{when } x \in A_i\,,$$
$$Sx \equiv x + m_i\beta \bmod M \quad \text{when } x \in A_i\,.$$

Note that A is not necessarily M-simple; it suffices that some m_j is zero, to have $x \in A$, $Sx \in A$, $Sx \equiv x \bmod M$ but $x \neq Sx$.

We now define $M' \subset \mathbb{R}^{s+1}$ (viewed naturally as $\mathbb{R}^s \times \mathbb{R}$) as the set $\Phi'(\mathbb{Z}^r)$, where

$$\Phi'(q_1, \ldots, q_r) = \Big(\sum q_i e_i, -\sum q_i m_i\Big)\,.$$

In $\mathbb{Q}^r$, $\operatorname{Ker} \Phi' = \operatorname{Ker} \Phi$ (by (5)), so $\dim \mathbb{Q}(M') = s+1$ and M' is a lattice.

For all i, $(e_i, -m_i)$ is in M', hence $(x+e_i, 0) \equiv (x, m_i) \bmod M'$, hence for almost all x

$$(x+e_i, 0) \equiv (x, n(x) - n(Sx) + 1) \bmod M'\,,$$

thus

$$(Sx, 0) \equiv (x, n(x) - n(Sx) + 1) \bmod M'\,,$$

therefore

$$(Sx, n(Sx)) \equiv (x, n(x)+1) \bmod M'\,,$$

or in other terms $\psi S = Q\psi$.

$\psi(A)$ is M'-simple: if $(x, n(x)) \equiv (x', n(x')) \bmod M'$, then $x' = x + \sum q_i e_i = x + c\alpha + d$, c being an integer and d an element of L; so x' is some $T^c x$, and, as x and x' are in A, x' is some $S^b x$, hence $(x, n(x)) \equiv (S^b x, n(S^b x)) \equiv (x, n(x)+b) \bmod M'$; hence $(0, b)$ is in M', thus $0 = \sum q_i e_i$ and $b = \sum q_i m_i$, and so $b = 0$ by (5), and $x = x'$.

Hence $Q(x, n(x)) = (Sx, n(Sx))$ is a representation of the rotation Q as a mapping from $\psi(A)$ to $\psi(A)$, and we can write $S = \psi^{-1} Q \psi$. This yields the necessity of our condition (since n is bounded and is a coboundary, we can make it positive by adding some constant).

Note that $((0,\ldots,0,1),M')$ is *not* a minimal couple.

Proof that the condition is sufficient. *For this direction, we do not need the assumption of measurability of A.* We suppose A satisfies the assumptions of our condition. By Lemma 1, A is partitioned into r sets by the different forms of S. We partition it further according to the finite set of values taken by the function $m(x)=n(x)-n(Sx)+1$. This gives us t different couples (e_j,m_j). We define a mapping Φ'' from $\mathbb{Q}^t$ to $\mathbb{R}^{s+1}$ by

$$\Phi''(q_1,\ldots,q_t)=\Big(\sum q_ie_i,-\sum q_im_i\Big)\,.$$

From $\psi S=Q\psi$, we deduce that M' must contain all the $(e_i,-m_i)$, and so must contain $\Phi''(\mathbb{Q}^t)$. As $\operatorname{Ker}\Phi''=((q_i)$ such that $\sum q_ie_i=0$ and $\sum q_im_i=0)$, we have $\dim\Phi''(\mathbb{Q}^t)\ge s+1$, with equality if and only if (5) is satisfied.

But, since we know M' is a lattice, we conclude simultaneously that $M'=\Phi''(\mathbb{Q}^t)$ and that (5) is satisfied (with t-uples instead of r-uples of integers). In particular, $e_i=e_j$ must imply $m_i=m_j$ and in fact $t=r$.

Now, the τ_i and g_i being defined as in the proof of Lemma 1, we shall construct a linear map ϕ from $\mathbb{R}^s$ to $\mathbb{R}$, and a rational number p, such that

$$\phi(g_i)=p\tau_i-m_i \quad \text{for all } i\,.$$

We know from minimality that the vector space $\mathbb{Q}(e_i)$, $1\le i\le r$, is of dimension $s+1$. We choose a basis for it, for example $e_1,\ldots,e_{s+1}$. The remaining e_j satisfy rational relations of the form

$$e_j=a_{j,1}e_1+\ldots+a_{j,s+1}e_{s+1}\,,\quad s+2\le j\le r\,.$$

By minimality of (α,L), these imply also

$$\tau_j=a_{j,1}\tau_1+\ldots+a_{j,s+1}\tau_{s+1}\,,\quad s+2\le j\le r\,,$$
$$g_j=a_{j,1}g_1+\ldots+a_{j,s+1}g_{s+1}\,,\quad s+2\le j\le r\,,$$

and so

$$m_j=a_{j,1}m_1+\ldots+a_{j,s+1}m_{s+1}\,,\quad s+2\le j\le r\,.$$

So the g_i, $1\le i\le s+1$, must generate $\mathbb{Q}(L)$; thus we can choose s of them to form a basis of $\mathbb{Q}(L)$, for example the first s. This means we have

$$g_{s+1}=b_1g_1+\ldots+b_sg_s\,,$$

while

$$\tau_{s+1}\ne b_1\tau_1+\ldots+b_s\tau_s\,,$$

since the e_i generate a space of dimension $s+1$.

We define

$$p=(m_{s+1}-(b_1m_1+\ldots+b_sm_s))/(\tau_{s+1}-(b_1\tau_1+\ldots+b_s\tau_s))\,.$$

Then we define ϕ by

$$\phi(g_i) = p\tau_i - m_i \quad \text{for } 1 \le i \le s\,.$$

This relation will remain true also for $i = s+1$, and for $s+2 \le i \le r$. This defines ϕ on the $\mathbb{R}$-vector space generated by the g_i, which is $\mathbb{R}^s$.

Then we can define a function F from the new fundamental domain Y (defined as in the first part of the proof) to $\mathbb{R}$ by

$$F(y) = \begin{cases} \phi(y) + n(y) & \text{if } y \text{ is in } A, \\ \phi(y) + n(y) + jp - 1 & \text{if } y \text{ is in some } A_i + j\alpha,\ j \ge 1. \end{cases}$$

It is easy to check that F is bounded and that

$$1_A - \lambda(A) = \phi(y) - \phi(Ty) \qquad \text{for } \lambda\text{-almost all } y \text{ in } Y\,,$$

which implies

$$\Big|\sum_{p=1}^{n} 1_A(T^p y) - na\Big| < C \qquad \text{for almost all } y \text{ in } Y\,,$$

and so

$$\Big|\sum_{p=1}^{n} 1_A(T^p x) - na\Big| < C \qquad \text{for almost every } x \text{ in } X\,;$$

which means A is a BRS, and also (which was not in any way implied by the computations) that p is an integer and F factorizes to X. (These last assertions are also consequences of a deep result of Rauzy, which is true even if A is not a BRS: minimality implies not only $\mathbb{Q}(e_i) = \mathbb{Q}(\alpha) + L$, but also $\mathbb{Z}(e_i) = \mathbb{Z}(\alpha) + L$.)

Another form of the necessary and sufficient condition

A measurable set A with nonempty interior is a BRS iff there exist a lattice M in $\mathbb{R}^s$, an element β of $\mathbb{R}^s$, a partition of A into sets B_i, $1 \le i \le u$, such that, if we denote by S_i the map induced by T (or S) on B_i, then

$$(\beta, M) \text{ is minimal}\,,$$

$$Sx - x \in \mathbb{Z}\beta + M \qquad \text{for almost all } x\,,$$

$$S_i x \equiv x + k\beta \bmod M \qquad \text{whenever } S_i = S^k\,.$$

P r o o f. This is easily deduced from what we called the intermediate form of the condition by partitioning A according to the values of $n(x)$.

In the other direction, if we are given the sets B_i, it is easy to build a function n. This is done step by step, for example taking $n = 0$ in B_1, then extending it to SB_1 by the relation $n(x) - n(Sx) = m_1 - 1$, and so on, the relations above guaranteeing there is no compatibility problem.

Note that, in contrast to A, the B_i are M-simple: if $x \equiv y \bmod M$, with x and y in the same B_i, then y must be some $T^c x$, hence some $S_i^k x$, and hence $y \equiv x + l\beta \bmod M$, with l a sum of k strictly positive terms; hence $l = 0$, $k = 0$ and $x = y$.

A by-product of the proof

If A and B are subsets of $\mathbb{R}$, if $C = A \times B \subset \mathbb{R}^2$ is a BRS for the rotation by $\alpha = (\alpha_1, \alpha_2)$ modulo $\mathbb{Z}$, with $\lambda(A) \neq 1$ and $\lambda(B) \neq 1$, then there exists a relation

$$p\alpha_1\alpha_2 + q\alpha_1 + r\alpha_2 + s = 0\,, \quad p, q, r, s \in \mathbb{Z}\,.$$

In particular, when α_1 is fixed, there exists only a denumerable set of α_2 such that there can exist non-trivial product BRS; this set is empty if α_1 is algebraic of degree 2.

Proof. Note simply that if C is a BRS, A and B must also be BRS. The first part of the proof shows that we must have

$$\lambda(A) = e\alpha_1 + f\,, \quad \lambda(B) = g\alpha_2 + h\,, \quad \lambda(A)\lambda(B) = \phi(\alpha_1, \alpha_2) + l\,,$$

e, f, g, h, l being integers and ϕ a linear form with integer coefficients; hence the relation follows (algebraicity of degree 2 is excluded because of the minimality of the rotation).

Thus we can exclude "most" of the rectangles.

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CNRS, URA 225
163 AVENUE DE LUMINY
F-13288 MARSEILLE CEDEX 9, FRANCE
E-mail: FERENCZI@LUMIMATH.UNIV-MRS.FR

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