Bounded remainder sets

by

Sébastien Ferenczi (Marseille)

Definitions. Let $L$ be a lattice in $\mathbb{R}^s$ (that is, a discrete subgroup of maximal order) and let $\alpha$ be an element of $\mathbb{R}^s$; $(\alpha, L)$ is said to be a minimal couple if for every nonzero linear form $\phi$ on $\mathbb{R}^s$ such that $\phi(L)$ is included in $\mathbb{Z}$, $\phi(\alpha)$ is not in $\mathbb{Z}$.

We define the rotation $T$ on the set $X = \mathbb{R}^s/L$ by $Tx = x + \alpha \mod L$; it preserves the Lebesgue measure $\lambda$ on $X$, and $(\alpha, L)$ is minimal if and only if $T$ is minimal, that is, has dense orbits; in particular, $L$ and $\alpha$ must generate $\mathbb{R}^s$. If $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $L$ is $\mathbb{Z}^s$, this is equivalent to $(1, \alpha_1, \ldots, \alpha_s)$ being rationally independent.

A set $A$ in $\mathbb{R}^s$ is $L$-simple if whenever $x \in A$, $y \in A$, $x - y \in L$, then $x = y$.

Let $A$ be a subset of $X$; we say $A$ is a bounded remainder set (BRS) if there exist real numbers $a$ and $C$ such that for every integer $n$ and $\lambda$-almost every $x$ in $X$,

$$\left| \sum_{p=1}^{n} 1_A(T^p x) - na \right| < C.$$

This definition also applies to $L$-simple subsets of $\mathbb{R}^s$, which we shall always identify with their projection on $X$.

It is a well-known result, which can for example be derived from the Markov–Kakutani fixed point theorem, that if $A$ is measurable, then $A$ is a BRS if and only if there exists a bounded function $F$ such that

$$1_A - a = F - TF,$$

and in that case $a$ can only be $\lambda(A)$.

For a set $A$ of strictly positive measure and a point $x$ in $A$, we denote by $\tau(x)$ the return time of $x$ in $A$ (that is, the least strictly positive integer $n$ such that $T^n x$ is in $A$) and by $Sx = T^{\tau(x)} x$ the induced map of $T$ on $A$, which exists by the Poincaré recurrence theorem.
**Known results about BRS.** If \( s = 1 \) and \( A \) is an interval, \( A \) is a BRS if and only if its length belongs to \( \mathbb{Z}(\alpha) \) (Kesten [1]); a similar result holds when \( A \) is a finite union of intervals (Oren [3]).

If \( s \geq 2 \), there are no nontrivial rectangles which are BRS (Liardet [2]); it seems difficult to find nontrivial examples of BRS when \( s \geq 2 \); Szusz ([6]) had one example of nontrivial parallelogram.

Rudolph [private communication] showed that whenever there exists a BRS of measure \( a > 0 \), the BRS are dense among the sets of measure \( a \); this is true for every ergodic transformation.

**Rauzy’s sufficient condition**

Let \( S \) be the induced map of \( T \) on \( A \). If there exists a lattice \( M \) and an element \( \beta \) of \( \mathbb{R}^s \) such that \((\beta, M)\) is a minimal couple and \( Sx = x + \beta \mod M \), then \( A \) is a BRS (even if \( B \) is not measurable).

This criterion enabled Rauzy to find nonmeasurable examples of BRS in dimension \( s = 1 \) ([4]), and new nontrivial examples (parallelograms) in higher dimensions ([5]); however, this condition is not necessary, as can be seen in dimension 1 with the interval \([0, 2\alpha] \), though in this counter-example the set \( A \) breaks into a finite union of subsets which satisfy Rauzy’s criterion. We can now give a

**Necessary and sufficient condition generalizing Rauzy’s criterion**

Let \( A \) be a subset of \( \mathbb{R}^s \), \( L \)-simple, measurable and with nonempty interior. Then \( A \) is a BRS if and only if there exist a lattice \( M' \) in \( \mathbb{R}^{s+1} \) and a bounded function \( n \) from \( A \) to \( \mathbb{N} \) such that, if \( \psi \) is the function from \( A \) to \( \mathbb{R}^{s+1} \) defined by \( \psi(x) = (x, n(x)) \), and if \( Q \) is the translation of \( \mathbb{R}^{s+1}/M' \) defined by \( Q(z) = z + (0, \ldots, 0, 1) \), then \( \psi(A) \) is a fundamental domain for \( Q \), that is, for every \( z \) in \( \psi(A) \), there exists a unique \( z' \) in \( \psi(A) \) such that \( z' \equiv Qz \mod M' \). Thus we can define \( Q \) as a mapping from \( \psi(A) \) to \( \psi(A) \), and we have

\[ S = \psi^{-1}Q\psi \]

(this last equality being defined \( \lambda \)-almost everywhere).

**Proof of the condition.** In all what follows, \( T, S \) and \( X \) will be as defined above and \( A \) will be a measurable \( L \)-simple set with nonempty interior.

Let \( W \) be a fundamental domain for the rotation \( T \), containing the set \( A \); for an element \( x \) in \( W \), we denote by \( x' \) its projection on \( X \). As a mapping from \( W \) to \( W \), \( T \) can be viewed as a finite exchange of pieces (an exchange of two intervals if \( s = 1 \)). The same is true for \( S \), as a mapping from \( A \) to \( A \), \( A \subset W \).
Lemma 1. There exists a finite partition of \( A \) into sets \( A_i \), and a finite number of elements \( e_i \), \( 1 \leq i \leq r \), such that,

\[
Sx = x + e_i \quad \text{whenever } x \text{ is in } A_i.
\]

Proof. A must contain an open set \( \Omega \). By Kronecker’s theorem and compactness,

\[
X = \bigcup_{n=1}^{+\infty} T^n \Omega = \bigcup_{n=1}^N T^n \Omega,
\]

for some finite \( N \). Hence the return time \( \tau(x) \) is bounded by \( N \), and so takes only a finite number of values.

Now, for every \( x \),

\[
Sx = x + \tau(x) \alpha + g(x),
\]

\( g(x) \) being the element of \( L \) such that \( x + \tau(x) \alpha + g(x) \) belongs to \( W \). Then \( g(x) \) must be bounded, and hence takes a finite number of values.

Now, if we partition \( A \) according to the values of \( \tau(x) \) and \( g(x) \), and if we define \( e_i = \tau_i \alpha + g_i \), we get our lemma.

Proof that the condition is necessary. We suppose \( A \) is a BRS. Then

1. \( 1_A(y) - \lambda(A) = F(y) - F(Ty) \) for almost every \( y \) in \( X \).

This implies

\[
e^{2\pi i T F} e^{2\pi i F} = e^{2\pi i \lambda(A)} \quad \text{almost everywhere}.
\]

Hence \( F \) and \( \lambda(A) \) are an eigenvector and an eigenvalue for an ergodic rotation, and so there exist a linear form \( \phi \) on \( \mathbb{R}^s \) such that \( \phi(L) \subset \mathbb{Z} \), an integer \( p \) and a measurable bounded integer function \( n \) such that

2. \( \lambda(A) = \phi(\alpha) + p \),

3. \( F(x') = \phi(x') + n(x') \) for almost all \( x \) in \( W \).

The second equation lifts to \( W \) yielding

4. \( F(x) = \phi(x) + n(x) \),

with some (bounded) modifications of the integer function \( n \); and it would lift in the same way (with different functions \( n \)) to any other fundamental domain.

From ergodicity, we have

\[
W = \bigcup_{i=1}^r \bigcup_{j=1}^{\tau_i - 1} T^j A_i.
\]
Following Rauzy, we define a new fundamental domain by

\[ Y = \bigcup_{i=1}^{r} \bigcup_{j=1}^{\tau_i-1} (A_i + j\alpha). \]

The sets \( A_i + j\alpha \) can be seen as levels of a tower; on them, \( T \) is defined in the following manner: on the levels other than the top levels (that is, when \( j < \tau_i \)), \( Tx = x + \alpha \); on the top levels, \( Tx = x + \alpha + g_i \).

Now, if we write (4) for our new fundamental domain \( Y \), and, together with (2) and the new expression for \( T \), insert it into (1), we get

\[ 1_A (x) - \phi(\alpha) - p = \phi(x) - \phi(Tx) + n(x) - n(Tx), \]

hence, as \( \phi \) is linear, we get finally

\[ 1_A (x) - p = n(x) - n(x + \alpha) \quad \text{if} \quad x \text{ is not in a top level}, \]
\[ 1_A (x) - p = n(x) - n(x + \alpha + g_i) - \phi(g_i) \quad \text{if} \quad x \text{ is in a top level above} \ A_i. \]

Suppose we already know \( n(x) \) on the basis \( A \); this defines \( n \) on the whole tower, by \( n(x + \alpha) = n(x) + p - 1 \) on the first floor, \( n(x + 2\alpha) = n(x) + 2p - 1 \) on the second floor, and so on as long as we do not reach the top. We just have to write the compatibility relation at the top:

\[ n(x) - n(x + \alpha) = 1 - p, \]
\[ n(x + \alpha) - n(x + 2\alpha) = -p, \]
\[ n(x + (\tau_i - 1)\alpha) - n(x + \tau_i\alpha + g_i) = -p + \phi(g_i), \]

hence

\[ n(x) - n(Sx) = 1 - p\tau_i + \phi(g_i) \quad \text{whenever} \quad x \in A_i. \]

Let \( m_i, 1 \leq i \leq r \), be the integer \( p\tau_i - \phi(g_i) \); these integers satisfy the following property: if \( (q_i, 1 \leq i \leq r) \) is an \( r \)-uple of integers such that \( \sum q_ie_i = 0 \), then

\[ \sum q_im_i = 0. \]  

This is easy to see, since if \( \sum q_ie_i = 0 \), then \( \sum q_i\tau_i = 0 \) and \( \sum q_i\alpha = 0 \), hence also \( \phi(\sum q_ig_i) = 0 \) and so \( \sum q_im_i = 0 \).

Also,

\[ m_i = 1 + n(Sx) - n(x) \quad \text{for almost all} \quad x \in A_i. \]

Let now \( M \) be the set \( (\sum q_ie_i, \text{ for all} \ r \text{-uples of integers} \ q_i \text{ such that} \ \sum q_i = 0) \).

\( M \) is a lattice: it is clear that \( M \) is a discrete subgroup of \( \mathbb{R}^s \), so it suffices to show that its dimension as a \( \mathbb{Q} \)-vector space is exactly \( s \).

Consider the mapping \( \Phi \) from \( \mathbb{Q}^r \) to \( \mathbb{R}^s \) given by \( \Phi(q_1, \ldots, q_r) = \sum q_ie_i \); its image is contained in \( \mathbb{Q}(\alpha) + \mathbb{Q}(L) \), so must be of dimension at most \( s + 1 \); but since \( S \), being the induced map of a minimal map on a set with
nonempty interior, has dense orbits in an open set, \( \dim \text{Im} \Phi \) must be exactly \( s + 1 \); hence \( \ker \Phi \) is of dimension \( r - s - 1 \).

Consider now the set \( B = (\sum q_i m_i = 0) \); as the \( m_i \) are not all zero (they have average one), \( B \) is of dimension 1, and contained in \( \ker \Phi \) by (5); hence \( \Phi(B) \) is of dimension \( s \).

Now choose \( k \) such that \( m_k \) is not zero, and put \( \beta = e_k/m_k \); we have

\[
e_i \equiv m_i \beta \mod M \quad \text{for all } i.
\]

As we have \( Sx \equiv x + m_i \beta \mod M \), and as \( S \) has dense orbits in an open set, \( (\beta, M) \) must be a minimal couple.

So we have already an intermediate form of the necessary condition: there exist a lattice \( M \) in \( \mathbb{R}^s \), an element \( \beta \) of \( \mathbb{R}^s \), a bounded function \( n \) from \( A \) to \( \mathbb{Z} \), and a partition \( A_i \) of \( A \), such that

\[
(\beta, M) \text{ is minimal }, \quad m_i = 1 + n(Sx) - n(x) \quad \text{when } x \in A_i, \quad Sx \equiv x + m_i \beta \mod M \quad \text{when } x \in A_i.
\]

Note that \( A \) is not necessarily \( M \)-simple; it suffices that some \( m_j \) is zero, to have \( x \in A, Sx \in A, Sx \equiv x \mod M \) but \( x \neq Sx \).

We now define \( M' \subset \mathbb{R}^{s+1} \) (viewed naturally as \( \mathbb{R}^s \times \mathbb{R} \)) as the set \( \Phi'(\mathbb{Z}') \), where

\[
\Phi'(q_i, \ldots, q_r) = \left( \sum q_i e_i, -\sum q_i m_i \right).
\]

In \( \mathbb{Q}^s \), \( \ker \Phi' = \ker \Phi \) by (5), so \( \dim \mathbb{Q}(M') = s + 1 \) and \( M' \) is a lattice.

For all \( i \), \( (e_i, -m_i) \) is in \( M' \), hence \( (x + e_i, 0) \equiv (x, m_i) \mod M' \), hence for almost all \( x \)

\[
(x + e_i, 0) \equiv (x, n(x) - n(Sx) + 1) \mod M',
\]

thus

\[
(Sx, 0) \equiv (x, n(x) - n(Sx) + 1) \mod M',
\]

therefore

\[
(Sx, n(Sx)) \equiv (x, n(x) + 1) \mod M',
\]
or in other terms \( \psi S = Q \psi \).

\( \psi(A) \) is \( M' \)-simple: if \( (x, n(x)) \equiv (x', n(x')) \mod M' \), then \( x' = x + \sum q_i e_i = x + ce \) \( d \) an integer and \( d \) an element of \( L \); so \( x' \) is some \( T x \), and, as \( x \) and \( x' \) are in \( A \), \( x' \) is some \( S^k x \), hence \( (x, n(x)) \equiv (S^k x, n(S^k x)) \equiv (x, n(x) + b) \mod M' \); hence \( (0, b) \) is in \( M' \), thus \( 0 = \sum q_i e_i = c \approx q_i m_i \), and so \( b = 0 \) by (5), and \( x = x' \).

Hence \( Q(x, n(x)) = (Sx, n(Sx)) \) is a representation of the rotation \( Q \) as a mapping from \( \psi(A) \) to \( \psi(A) \), and we can write \( S = \psi^{-1} Q \psi \). This yields the necessity of our condition (since \( n \) is bounded and is a coboundary, we can make it positive by adding some constant).
Note that \(((0, \ldots, 0, 1), M')\) is not a minimal couple.

Proof that the condition is sufficient. For this direction, we do not need the assumption of measurability of \(A\). We suppose \(A\) satisfies the assumptions of our condition. By Lemma 1, \(A\) is partitioned into \(r\) sets by the different forms of \(S\). We partition it further according to the finite set of values taken by the function \(m(x) = n(x) - n(Sx) + 1\). This gives us \(t\) different couples \((e_j, m_j)\). We define a mapping \(\Phi''\) from \(Q^t\) to \(R^{s+1}\) by

\[
\Phi''(q_1, \ldots, q_t) = \left( \sum q_i e_i, -\sum q_i m_i \right).
\]

From \(\psi S = Q\psi\), we deduce that \(M'\) must contain all the \((e_i, -m_i)\), and so must contain \(\Phi''(Q^t)\). As \(\text{Ker} \Phi'' = ((q_i)\text{ such that } \sum q_i e_i = 0\) and \(\sum q_i m_i = 0)\), we have \(\text{dim} \Phi''(Q^t) \geq s + 1\), with equality if and only if (5) is satisfied.

But, since we know \(M'\) is a lattice, we conclude simultaneously that \(M' = \Phi''(Q^t)\) and that (5) is satisfied (with \(t\)-uples instead of \(r\)-uples of integers). In particular, \(e_i = e_j\) must imply \(m_i = m_j\) and in fact \(t = r\).

Now, the \(\tau_i\) and \(g_i\) being defined as in the proof of Lemma 1, we shall construct a linear map \(\phi\) from \(R^s\) to \(R\), and a rational number \(p\), such that

\[
\phi(g_i) = p\tau_i - m_i \text{ for all } i.
\]

We know from minimality that the vector space \(Q(e_i), 1 \leq i \leq r\), is of dimension \(s + 1\). We choose a basis for it, for example \(e_1, \ldots, e_{s+1}\). The remaining \(e_j\) satisfy rational relations of the form

\[
e_j = a_{j,1} e_1 + \ldots + a_{j,s+1} e_{s+1}, \quad s + 2 \leq j \leq r.
\]

By minimality of \((\alpha, L)\), these imply also

\[
\tau_j = a_{j,1} \tau_1 + \ldots + a_{j,s+1} \tau_{s+1}, \quad s + 2 \leq j \leq r,
\]

\[
g_j = a_{j,1} g_1 + \ldots + a_{j,s+1} g_{s+1}, \quad s + 2 \leq j \leq r,
\]

and so

\[
m_j = a_{j,1} m_1 + \ldots + a_{j,s+1} m_{s+1}, \quad s + 2 \leq j \leq r.
\]

So the \(g_i, 1 \leq i \leq s + 1\), must generate \(Q(L)\); thus we can choose \(s\) of them to form a basis of \(Q(L)\), for example the first \(s\). This means we have

\[
g_{s+1} = b_1 g_1 + \ldots + b_s g_s,
\]

while

\[
\tau_{s+1} \neq b_1 \tau_1 + \ldots + b_s \tau_s,
\]

since the \(e_i\) generate a space of dimension \(s + 1\).

We define

\[
p = (m_{s+1} - (b_1 m_1 + \ldots + b_s m_s))/((\tau_{s+1} - (b_1 \tau_1 + \ldots + b_s \tau_s))
\]
Then we define \( \phi \) by
\[
\phi(g_i) = p\tau_i - m_i \quad \text{for} \quad 1 \leq i \leq s.
\]
This relation will remain true also for \( i = s + 1 \), and for \( s + 2 \leq i \leq r \). This defines \( \phi \) on the \( \mathbb{R} \)-vector space generated by the \( g_i \), which is \( \mathbb{R}^s \).

Then we can define a function \( F \) from the new fundamental domain \( Y \) (defined as in the first part of the proof) to \( \mathbb{R} \) by
\[
F(y) = \begin{cases} 
\phi(y) + n(y) & \text{if } y \text{ is in } A, \\
\phi(y) + n(y) + jp - 1 & \text{if } y \text{ is in some } A_i + j\alpha, \, j \geq 1.
\end{cases}
\]
It is easy to check that \( F \) is bounded and that
\[
1_A - \lambda(A) = \phi(y) - \phi(Ty) \quad \text{for } \lambda \text{-almost all } y \in Y,
\]
which implies
\[
\left| \sum_{p=1}^{n} 1_A(T^p y) - na \right| < C \quad \text{for almost all } y \in Y,
\]
and so
\[
\left| \sum_{p=1}^{n} 1_A(T^p x) - na \right| < C \quad \text{for almost every } x \in X;
\]
which means \( A \) is a BRS, and also (which was not in any way implied by the computations) that \( p \) is an integer and \( F \) factorizes to \( X \). (These last assertions are also consequences of a deep result of Rauzy, which is true even if \( A \) is not a BRS: minimality implies not only \( \mathbb{Q}(e_i) = \mathbb{Q}(\alpha) + L \), but also \( \mathbb{Z}(e_i) = \mathbb{Z}(\alpha) + L \).

Another form of the necessary and sufficient condition

A measurable set \( A \) with nonempty interior is a BRS iff there exist a lattice \( M \) in \( \mathbb{R}^s \), an element \( \beta \) of \( \mathbb{R}^s \), a partition of \( A \) into sets \( B_i \), \( 1 \leq i \leq u \), such that, if we denote by \( S_i \) the map induced by \( T \) (or \( S \)) on \( B_i \), then
\[
(\beta, M) \text{ is minimal,} \quad S_i x \in \mathbb{Z}\beta + M \quad \text{for almost all } x, \\
S_i x \equiv x + k\beta \mod M \quad \text{whenever } S_i = S^{k}.
\]

Proof. This is easily deduced from what we called the intermediate form of the condition by partitioning \( A \) according to the values of \( n(x) \).

In the other direction, if we are given the sets \( B_i \), it is easy to build a function \( n \). This is done step by step, for example taking \( n = 0 \) in \( B_1 \), then extending it to \( SB_1 \) by the relation \( n(x) - n(Sx) = m_1 - 1 \), and so on, the relations above guaranteeing there is no compatibility problem.

Note that, in contrast to \( A \), the \( B_i \) are \( M \)-simple: if \( x \equiv y \mod M \), with \( x \) and \( y \) in the same \( B_i \), then \( y \) must be some \( T^k x \), hence some \( S_i^k x \), and
hence \( y \equiv x + l\beta \mod M \), with \( l \) a sum of \( k \) strictly positive terms; hence \( l = 0, k = 0 \) and \( x = y \).

**A by-product of the proof**

If \( A \) and \( B \) are subsets of \( \mathbb{R} \), if \( C = A \times B \subset \mathbb{R}^2 \) is a BRS for the rotation by \( \alpha = (\alpha_1, \alpha_2) \) modulo \( \mathbb{Z} \), with \( \lambda(A) \neq 1 \) and \( \lambda(B) \neq 1 \), then there exists a relation

\[
p\alpha_1 \alpha_2 + q\alpha_1 + r\alpha_2 + s = 0, \quad p, q, r, s \in \mathbb{Z}.
\]

In particular, when \( \alpha_1 \) is fixed, there exists only a denumerable set of \( \alpha_2 \) such that there can exist non-trivial product BRS; this set is empty if \( \alpha_1 \) is algebraic of degree 2.

**Proof.** Note simply that if \( C \) is a BRS, \( A \) and \( B \) must also be BRS. The first part of the proof shows that we must have

\[
\lambda(A) = e\alpha_1 + f, \quad \lambda(B) = g\alpha_2 + h, \quad \lambda(A)\lambda(B) = \phi(\alpha_1, \alpha_2) + l,
\]

\( e, f, g, h, l \) being integers and \( \phi \) a linear form with integer coefficients; hence the relation follows (algebraicity of degree 2 is excluded because of the minimality of the rotation).

Thus we can exclude "most" of the rectangles.

**References**


