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On a diophantine inequality involving prime numbers

by

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In 1952 Piatetski-Shapiro [4] considered the following analogue of the Goldbach–Waring problem.

Assume that $c > 1$ is not integer and let ε be a positive number. If r is a sufficiently large integer (depending only on c) then the inequality

$$|p_1^c + p_2^c + \ldots + p_r^c - N| < \varepsilon \tag{1}$$

has a solution in prime numbers $p_1, \ldots, p_r$ for sufficiently large N. More precisely, if the least r such that (1) has a solution in prime numbers for every $\varepsilon > 0$ and $N > N_0(c, \varepsilon)$ is denoted by $H(c)$ then it is proved in [4] that

$$\limsup_{c\to\infty} \frac{H(c)}{c \log c} \le 4\,.$$

Piatetski-Shapiro also proved that if $1 < c < 3/2$ then $H(c) \le 5$. One can conjecture that if c is near to unity then $H(c) \le 3$. On the other hand, instead of fixed ε, we may consider ε depending on N and tending to zero as N tends to infinity.

This conjecture is proved in [7] for $1 < c < 27/26$ and

$$\varepsilon = N^{-(1/c)(27/26-c)} \log^{13} N\,.$$

In this paper we sharpen the last result and prove the following theorem:

Theorem. *Let $1 < c < 15/14$. There exists a number $N_0(c) > 0$ such that for each real number $N > N_0(c)$ the inequality*

$$|p_1^c + p_2^c + p_3^c - N| < N^{-(1/c)(15/14-c)} \log^9 N$$

has a solution in prime numbers p_1, p_2, p_3.

Notation

c — fixed real number such that $1 < c < 15/14$,
N — sufficiently large number,

(2) $$X=(N/2)^{1/c}\,,$$

(3) $$\tau=X^{13/28-c/2}\,,$$

(4) $$T=X^{41/56+c/4}\,,$$

(5) $$\varepsilon=X^{-(15/14-c)}\log^8 X\,,$$

(6) $$\Delta=\frac{1}{10}\varepsilon\,,$$

(7) $$r=[\log X]\,,$$

where $[\alpha]$ denotes the integer part of the real number α;

(8) $$K=X^{15/14-c}(\log X)^{-6}\,,$$

m, n, k, l, d, r (with or without subscripts) — integers,
p, p_1, $p_2,\ldots$ — prime numbers,
$\tau(n)$ — the number of positive divisors of n,
$\Lambda(n)$ — von Mangoldt's function,
$\Psi(x)=\sum_{n\le x}\Lambda(n)$ — Chebyshev's function,
$\varrho=\beta+i\gamma$ — non-trivial zero of the Riemann zeta function $\zeta(s)$,
$\sum_{a<\gamma<b}$ — sum over the non-trivial zeroes of $\zeta(s)$ such that $a<\gamma<b$,
$e(x)=e^{2\pi ix}$,
$A\asymp B$ means $A\ll B\ll A$,

(9) $$S(x)=\sum_{X/2<p\le X}\log p\cdot e(p^c x)\,,$$

(10) $$I(x)=\int\limits_{X/2}^{X} e(t^c x)\,dt\,,$$

(11) $$I_\varrho(x)=\int\limits_{X/2}^{X} e(t^c x)t^{\varrho-1}\,dt\,,$$

(12) $$J(x)=\sum_{|\gamma|\le T} I_\varrho(x)\,.$$

The constants in O-terms and $\ll$-symbols are absolute or depend only on c.

To prove the theorem we need some lemmas.

Lemma 1. *Let a, δ be real numbers, $0<\delta<a/4$, and let k be an integer. There exists a function $\varphi(y)$ which is k times continuously differentiable and such that*

$$\begin{aligned}\varphi(y)&=1 &&\text{for } |y|\le a-\delta\,,\\ 0<\varphi(y)&<1 &&\text{for } a-\delta<|y|<a+\delta\,,\\ \varphi(y)&=0 &&\text{for } |y|\ge a+\delta\,,\end{aligned}$$

and its Fourier transform

$$\Phi(x) = \int_{-\infty}^{\infty} e(-xy)\varphi(y)\,dy$$

satisfies the inequality

$$|\Phi(x)| \le \min\left(2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|}\left(\frac{k}{2\pi|x|\delta}\right)^k\right).$$

Proof. See [4] or [5].

Throughout this paper we denote by $\varphi(y)$ the function from Lemma 1 with parameters $a = \frac{9}{10}\varepsilon$, $\delta = \Delta$, $k = r$, and by $\Phi(x)$ the Fourier transform of $\varphi(y)$.

LEMMA 2. *Assume that* $G(x)$, $F(x)$ *are real functions defined in* $[a, b]$, $|G(x)| \le H$ *for* $a \le x \le b$ *and* $G(x)/F'(x)$ *is a monotonous function. Set*

$$I = \int_a^b G(x)e(F(x))\,dx.$$

If $F'(x) \ge h > 0$ *for all* $x \in [a, b]$ *or if* $F'(x) \le -h < 0$ *for all* $x \in [a, b]$ *then*

$$|I| \ll H/h.$$

If $F''(x) \ge h > 0$ *for all* $x \in [a, b]$ *then*

$$|I| \ll H/\sqrt{h}.$$

Proof. See [6, p. 71].

LEMMA 3. *Suppose that* $f''(t)$ *exists, is continuous on* $[a, b]$ *and satisfies*

$$f''(t) \asymp \lambda \quad (\lambda > 0) \qquad \text{for } t \in [a, b].$$

Then

$$\Big|\sum_{a<n\le b} e(f(n))\Big| \ll (b-a)\lambda^{1/2} + \lambda^{-1/2}.$$

Proof. See [6, p. 104].

LEMMA 4. *If* $2 \le t \le x$ *then*

$$\Psi(x) = x - \sum_{|\gamma|\le t} \frac{x^{\varrho}}{\varrho} + O\left(\frac{x\log^2 x}{t}\right).$$

Proof. See [3, p. 80].

LEMMA 5. *Assume that* $1 \le R \le Y$. *There exists a number* $\gamma_1 > 0$ *such that if*

$$\theta(R) = \frac{\gamma_1}{\log^{2/3}(R+10)\log\log(R+10)}$$

then the following estimate holds:

$$\sum_{0<\gamma\le R} Y^{\beta} \ll \begin{cases} Y^{1/2} R \log^6 Y, & Y^{3/4} \le R \le Y, \\ e^{2\log Y + 3\log R - 2\sqrt{3\log Y \log R}} \log^6 Y, & Y^{1/3} \le R \le Y^{3/4}, \\ Y^{1-\theta(R)} R^{2.4\theta(R)} \log^{45} Y, & 1 \le R \le Y^{1/3}. \end{cases}$$

Proof. This lemma can be deduced from the density theorems of Ingham (see [6, p. 236]) and Huxley (see [2]) combined with the estimate of I. M. Vinogradov for the zero-free region of $\zeta(s)$ (see [3, p. 100]). Detailed computations may be found in [1].

LEMMA 6. *We have*

$$\int_{-\infty}^{\infty} I^3(x) e(-Nx) \Phi(x)\, dx \gg \varepsilon X^{3-c}.$$

Proof. Denote the above integral by H. We have

$$H = \int_{X/2}^{X}\int_{X/2}^{X}\int_{X/2}^{X}\int_{-\infty}^{\infty} e((t_1^c + t_2^c + t_3^c - N)x)\Phi(x)\, dx\, dt_1\, dt_2\, dt_3$$

(the change of the order of integration is legitimate because of the absolute convergence of the integral). Using the Fourier inversion formula we get

$$H = \int_{X/2}^{X}\int_{X/2}^{X}\int_{X/2}^{X} \varphi(t_1^c + t_2^c + t_3^c - N)\, dt_1\, dt_2\, dt_3$$

and by the definition of $\varphi(y)$ we get

$$H \ge \underset{|t_1^c+t_2^c+t_3^c-N|<4\varepsilon/5}{\int_{X/2}^{X}\int_{X/2}^{X}\int_{X/2}^{X}} dt_1\, dt_2\, dt_3 \ge \int_{\lambda X}^{\mu X}\int_{\lambda X}^{\mu X} \Big(\int_{\mathfrak{M}} dt_3\Big)\, dt_1\, dt_2,$$

where λ and μ are real numbers such that

$$\frac{1}{2} < \left(\frac{1}{2}\right)^{1/c} < \lambda < \mu < \left(\frac{1}{2}\left(2 - \frac{1}{2^c}\right)\right)^{1/c} < 1 \tag{13}$$

and

$$\mathfrak{M} = [X/2, X] \cap [(N - \tfrac{4}{5}\varepsilon - t_1^c - t_2^c)^{1/c}, (N + \tfrac{4}{5}\varepsilon - t_1^c - t_2^c)^{1/c}].$$

Because of (13) and the choice of N, X, ε it is easy to show that in fact $\mathfrak{M}$ is the right interval of the above intersection. Thus by the mean-value theorem

$$H \gg \varepsilon \int\limits_{\lambda X}^{\mu X} \int\limits_{\lambda X}^{\mu X} (\xi_{t_1,t_2})^{1/c-1} dt_1\, dt_2\,,$$

where $\xi_{t_1,t_2} \asymp X^c$. Therefore, $H \gg \varepsilon X^{3-c}$, which proves the lemma.

LEMMA 7. *We have*

(i) $$\int\limits_{-\tau}^{\tau} |S^2(x)|\, dx \ll X^{2-c} \log^3 X\,,$$

(ii) $$\int\limits_{-\tau}^{\tau} |I^2(x)|\, dx \ll X^{2-c} \log X\,,$$

(iii) $$\int\limits_{n}^{n+1} |S^2(x)|\, dx \ll X \log^3 X$$

uniformly with respect to n.

Proof. We only prove (i). Inequalities (ii) and (iii) can be proved likewise.

We have

(14) $$\begin{aligned}\int\limits_{-\tau}^{\tau} |S^2(x)|\, dx &= \sum_{X/2<p_1,p_2\le X} \log p_1\, \log p_2 \int\limits_{-\tau}^{\tau} e((p_1^c - p_2^c)x)\, dx \\ &\ll \sum_{X/2<p_1,p_2\le X} \log p_1 \log p_2 \min\left(\tau, \frac{1}{|p_1^c - p_2^c|}\right) \\ &\ll U\tau \log^2 X + V \log^2 X\,,\end{aligned}$$

where

$$U = \sum_{\substack{X/2<n_1,n_2\le X\\ |n_1^c - n_2^c|\le 1/\tau}} 1\,, \qquad V = \sum_{\substack{X/2<n_1,n_2\le X\\ 1/\tau<|n_1^c - n_2^c|}} \frac{1}{|n_1^c - n_2^c|}\,.$$

We have

$$U \ll \sum_{X/2<n_1\le X} \sum_{\substack{X/2<n_2\le X\\ (n_1^c-1/\tau)^{1/c}\le n_2\le (n_1^c+1/\tau)^{1/c}}} 1 \ll \sum_{X/2<n_1\le X} (1 + (n_1^c + 1/\tau)^{1/c} - (n_1^c - 1/\tau)^{1/c})$$

and by the mean-value theorem

(15) $$U \ll X + \frac{1}{\tau} X^{2-c}\,.$$

Obviously $V \le \sum_l V_l$ where

$$V_l = \sum_{\substack{X/2<n_1,n_2\le X\\ l<|n_1^c-n_2^c|\le 2l}} \frac{1}{|n_1^c-n_2^c|} \tag{16}$$

and l takes the values $2^k/\tau$, $k=0,1,2,\ldots$, with $l \le X^c$. We have

$$V_l \ll \frac{1}{l} \sum_{X/2<n_1\le X} \sum_{\substack{X/2<n_2\le X\\ (n_1^c+l)^{1/c}\le n_2\le (n_1^c+2l)^{1/c}}} 1\,.$$

For $l \ge 1/\tau$ and $X/2 < n_1 \le X$ it is easy to see that

$$(n_1^c+2l)^{1/c} - (n_1^c+l)^{1/c} > 1\,.$$

Hence

$$V_l \ll \frac{1}{l} \sum_{X/2<n_1\le X} ((n_1^c+2l)^{1/c} - (n_1^c+l)^{1/c}) \ll X^{2-c} \tag{17}$$

by the mean-value theorem.

The conclusion follows from formulas (3) and (14)–(17).

LEMMA 8. *Let a_m, b_n be arbitrary complex numbers and let*

$$\tau \le |x| \le K\,, \qquad X^{1/4} < R \le X^{1/2}\,,$$
$$X^{1/4} < L < L_1 \le 2L\,, \qquad LR \le X\,.$$

Define

$$W = \sum_{L<n\le L_1} \sum_{X^{1/4}<m\le R} a_m b_n e((mn)^c x)\,.$$

Then

$$|W| \ll (\mathcal{A}\mathcal{B})^{1/2} X^{3/7} (\log X)^{-1}\,,$$

where

$$\mathcal{A} = \sum_{X^{1/4}<m\le R} |a_m|^2\,, \qquad \mathcal{B} = \sum_{L<n\le L_1} |b_n|^2\,.$$

Proof. Obviously, we may suppose that $\tau \le x \le K$. Take $s \in [1,R]$ whose exact value will be determined later. We define the numbers R_i, $0 \le i \le Q$, in the following way:

$$R_0 = X^{1/4}\,, \qquad R_{i+1} = \min(R_i + s, R)\,, \qquad R_Q = R\,.$$

Obviously $Q \ll R/s$. We have

$$W = \sum_{L<n\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1}<m\le R_i} a_m b_n e((mn)^c x)\,,$$

hence

$$|W| \le \sum_{L<n\le L_1} |b_n| \Big| \sum_{1\le i\le Q} \sum_{R_{i-1}<m\le R_i} a_m e((mn)^c x) \Big|$$

and Cauchy's inequality gives

$$\begin{aligned} |W|^2 &\le \mathcal{B} \sum_{L<n\le L_1} \Big| \sum_{1\le i\le Q} \sum_{R_{i-1}<m\le R_i} a_m e((mn)^c x) \Big|^2 \\ &\le \mathcal{B}Q \sum_{L<n\le L_1} \sum_{1\le i\le Q} \Big| \sum_{R_{i-1}<m\le R_i} a_m e((mn)^c x) \Big|^2 \\ &= \mathcal{B}Q \sum_{L<n\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1}<m_1,m_2\le R_i} a_{m_1} \overline{a}_{m_2} e((m_1^c - m_2^c) n^c x) . \end{aligned}$$

After some rearrangements we obtain

$$\begin{aligned} (18) \quad |W|^2 &\ll \mathcal{B}Q \Big(\sum_{L<n\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1}<m\le R_i} |a_m|^2 \\ &\quad + \sum_{1\le i\le Q} \sum_{\substack{R_{i-1}<m_1,m_2\le R_i \\ m_1\ne m_2}} |a_{m_1}|\, |a_{m_2}| \\ &\quad \times \Big| \sum_{L<n\le L_1} e((m_1^c - m_2^c) n^c x) \Big| \Big) \\ &\ll \mathcal{B}Q \Big(\mathcal{A}L + \sum_{1\le h\le s} \sum_{1\le i\le Q} \sum_{R_{i-1}<m\le R_i - h} |a_m|\, |a_{m+h}| \\ &\quad \times \Big| \sum_{L<n\le L_1} e(f(n)) \Big| \Big) , \end{aligned}$$

where

$$f(n) = ((m+h)^c - m^c) n^c x .$$

It is easy to see that if $L < n \le L_1$ then

$$f''(n) \asymp ((m+h)^c - m^c) L^{c-2} x$$

and by Lemma 3 we obtain

$$\begin{aligned} \Big| \sum_{L<n\le L_1} e(f(n)) \Big| &\ll ((m+h)^c - m^c)^{1/2} L^{c/2} x^{1/2} \\ &\quad + ((m+h)^c - m^c)^{-1/2} L^{1-c/2} x^{-1/2} \\ &\ll ((m+h)^c - m^c)^{1/2} L^{c/2} K^{1/2} \\ &\quad + ((m+h)^c - m^c)^{-1/2} L^{1-c/2} \tau^{-1/2} \\ &\ll ((m+h)^c - m^c)^{1/2} L^{c/2} K^{1/2} \\ &\ll s^{1/2} R^{(c-1)/2} L^{c/2} K^{1/2} . \end{aligned}$$

We substitute this estimate in (18) to get

$$|W|^2 \ll \mathcal{B}Q(\mathcal{A}L + s^{1/2}R^{(c-1)/2}L^{c/2}K^{1/2}\Sigma_0)\,, \tag{19}$$

where

$$\Sigma_0 = \sum_{1\le h\le s}\sum_{1\le i\le Q}\sum_{R_{i-1}<m\le R_i-h}|a_m|\,|a_{m+h}|\,.$$

Clearly

$$\begin{aligned}\Sigma_0 &\le \sum_{1\le h\le s}\sum_{X^{1/4}<m\le R-h}|a_m|\,|a_{m+h}|\\ &\le \sum_{1\le h\le s}\Big(\sum_{X^{1/4}<m\le R-h}|a_m|^2\Big)^{1/2}\Big(\sum_{X^{1/4}<m\le R-h}|a_{m+h}|^2\Big)^{1/2}\le s\mathcal{A}\,.\end{aligned}$$

The last estimate, (19) and the inequality $Q \ll R/s$ give us

$$|W|^2 \ll \mathcal{A}\mathcal{B}LR(1/s + s^{1/2}R^{(c-1)/2}L^{(c-2)/2}K^{1/2})\,. \tag{20}$$

We now determine s in such a way that the two summands in brackets are equal. This gives the value

$$s = K^{-1/3}R^{(1-c)/3}L^{(2-c)/3}\,.$$

The assumption of our lemma and (8) imply easily $1 \le s \le R$. Substituting s in (20) we get after some calculations

$$|W|^2 \ll \mathcal{A}\mathcal{B}X^{6/7}(\log X)^{-2}\,,$$

i.e.

$$|W| \ll (\mathcal{A}\mathcal{B})^{1/2}X^{3/7}(\log X)^{-1}\,,$$

which is the desired result.

Lemma 9. *Suppose that a_m, b_n are arbitrary complex numbers and that $L < L_1 \le 2L$, $L \le X$. Set*

$$V = \sum_{L<n\le L_1}\sum_{X^{1/4}<m\le X/n} a_m b_n e((mn)^c x)\,.$$

Then

$$|V| \ll \log X \cdot \Big|\sum_{L<n\le L_1}\sum_{X^{1/4}<m\le X/L} a'_m b'_n e((mn)^c x)\Big|$$

for some complex numbers a'_m, b'_n such that $|a'_m| \le |a_m|$, $|b'_n| \le |b_n|$.

Proof. Let q be an odd integer such that $2X/L \le q \le 4X/L$. We have

$$\begin{aligned}V = \sum_{L<n\le L_1}\sum_{X^{1/4}<m\le X/L} & a_m b_n e((mn)^c x)\\ &\times \sum_{X^{1/4}<m_1\le X/n}\frac{1}{q}\sum_{|k|\le(q-1)/2} e\Big(\frac{k(m-m_1)}{q}\Big)\end{aligned} \tag{21}$$

$$= \sum_{|k|\le (q-1)/2} \frac{1}{\max(1,|k|)} \sum_{L<n\le L_1} \sum_{X^{1/4}<m\le X/L} a_m^{(k)} b_n^{(k)} e((mn)^c x)\,,$$

where

$$a_m^{(k)} = a_m e\Big(\frac{km}{q}\Big), \qquad b_n^{(k)} = b_n \frac{\max(1,|k|)}{q} \sum_{X^{1/4}<m_1\le X/n} e\Big(-\frac{km_1}{q}\Big).$$

Obviously $|a_m^{(k)}| = |a_m|$ and because of the well-known estimate

$$\Big|\sum_{a<k\le b} e(\alpha k)\Big| \ll \min(b-a, 1/||\alpha||)$$

we get

$$|b_n^{(k)}| \ll |b_n|\,.$$

The absolute value of the double sum over m and n on the right-hand side of (21) takes its maximum for some value $k = k_0$.

Denote by a'_m, b'_n the numbers $a_m^{(k_0)}$, $b_n^{(k_0)}$, multiplied by a sufficiently small positive constant.

Obviously this proves the lemma.

LEMMA 10. *Assume that* $\tau \le |x| \le K$. *Then*

$$|S(x)| \ll X^{13/14} \log^3 X\,.$$

P r o o f. Without loss of generality we may assume that $\tau \le x \le K$. Clearly

$$S(x) = V_0(x) - V_1(x) + O(X^{1/2})\,,$$

where

$$V_i(x) = \sum_{X^{1/4}<n\le 2^{-i}X} \Lambda(n) e(n^c x)\,, \qquad i = 0,1\,.$$

Hence it is sufficient to prove that

$$|V_0(x)|\,,\ |V_1(x)| \ll X^{13/14} \log^3 X\,. \tag{22}$$

Consider, for instance, $V_0(x)$. We use Vaughan's identity (see [8]) to get

$$V_0(x) = S_1 - S_2 - S_3\,, \tag{23}$$

where

$$S_1 = \sum_{d\le X^{1/4}} \mu(d) \sum_{l\le X/d} \log l \cdot e((ld)^c x)\,,$$

$$S_2 = \sum_{k\le X^{1/2}} \sum_{r\le X/k} c_k e((kr)^c x)\,,$$

$$S_3 = \sum_{X^{1/4}<m\le X^{3/4}} \sum_{X^{1/4}<n\le X/m} a_m \Lambda(n) e((mn)^c x)\,,$$

where $|c_k| \le \log k$, $|a_m| \le \tau(m)$.

We split the sum S_2 in the following way:

$$S_2 = S_2^{(1)} + S_2^{(2)} + O(X^{3/4}\log X)\,, \tag{24}$$

where

$$S_2^{(1)} = \sum_{k\le X^{1/4}} \sum_{r\le X/k} c_k e((kr)^c x)\,,$$
$$S_2^{(2)} = \sum_{X^{1/4}<k\le X^{1/2}} \sum_{X^{1/4}<r\le X/k} c_k e((kr)^c x)\,.$$

Let us treat S_1. We have

$$|S_1| \le \sum_{d\le X^{1/4}} \Big| \sum_{l\le X/d} \log l\cdot e((ld)^c x)\Big|\,. \tag{25}$$

We break the sum over l into sums of the type

$$T_L = \sum_{L<l\le L_1} \log l\cdot e((ld)^c x)$$

whose number is $O(\log X)$ and $L < L_1 \le 2L$, $L_1 \le X/d$. Abel's transformation formula gives us

$$|T_L| \ll \log X \max_{L_2\in[L,L_1]} \Big| \sum_{L<l\le L_2} e(g(l))\Big|\,, \tag{26}$$

where $g(l) = (ld)^c x$. Obviously $g''(l) \asymp L^{c-2}d^c x$ for $L \le l \le L_2$ and by Lemma 3

$$\begin{aligned}\Big| \sum_{L<l\le L_2} e(g(l))\Big| &\ll L^{c/2}d^{c/2}x^{1/2} + L^{1-c/2}d^{-c/2}x^{-1/2}\\ &\le L^{c/2}d^{c/2}K^{1/2} + L^{1-c/2}d^{-c/2}\tau^{-1/2}\\ &\le X^{c/2}K^{1/2} + X^{1-c/2}d^{-1}\tau^{-1/2} \ll X^{c/2}K^{1/2}\,.\end{aligned}$$

The last estimate, together with (25) and (26), implies

$$|S_1| \ll \log^2 X\cdot X^{1/4+c/2}K^{1/2} \ll X^{11/14}\,. \tag{27}$$

We estimate $S_2^{(1)}$ in the same way to get

$$|S_2^{(1)}| \ll X^{11/14}\,. \tag{28}$$

To estimate $S_2^{(2)}$ and S_3 we need Lemmas 8 and 9. Consider S_3. We split it in the following way:

$$S_3 = W_1 + W_2 + W_3\,, \tag{29}$$

where

$$W_1 = \sum_{X^{1/2}<n\le X^{3/4}} \sum_{X^{1/4}<m\le X/n} a_m \Lambda(n) e((mn)^c x)\,,$$

$$W_2 = \sum_{X^{1/2}<n\le X^{3/4}} \sum_{X^{1/4}<m\le X/n} \Lambda(m) a_n e((mn)^c x)\,,$$

$$W_3 = \sum_{X^{1/4}<n\le X^{1/2}} \sum_{X^{1/4}<m\le X^{1/2}} a_m \Lambda(n) e((mn)^c x)\,.$$

We break W_1 into sums of the type

$$W_1(L) = \sum_{L<n\le L_1} \sum_{X^{1/4}<m\le X/n} a_m \Lambda(n) e((mn)^c x)$$

whose number is $O(\log X)$ and $X^{1/2} \le L < L_1 \le X^{3/4}$, $L_1 \le 2L$. According to Lemma 9 there exist complex numbers a'_m, b'_n such that $|a'_m| \le |a_m| \le \tau(m)$, $|b'_n| \le \Lambda(n)$ and

$$|W_1(L)| \ll (\log X)\cdot |W'_1(L)|\,,$$

where

$$W'_1(L) = \sum_{L<n\le L_1} \sum_{X^{1/4}<m\le X/L} a'_m b'_n e((mn)^c x)\,.$$

Using the well-known inequalities

$$\sum_{m\le y} \tau^2(m) \ll y\log^3 y\,, \qquad \sum_{n\le y} \Lambda^2(n) \ll y\log y$$

and Lemma 8 we obtain

$$|W'_1(L)| \ll \left(\frac{X}{L}\log^3 X \cdot L\log X\right)^{1/2} X^{3/7}(\log X)^{-1} = X^{13/14}\log X\,,$$

hence

$$|W_1| \ll X^{13/14}\log^3 X\,. \tag{30}$$

We estimate W_2 and W_3 in the same way (for W_3, of course, we do not use Lemma 9) to get

$$|W_2|,\ |W_3| \ll X^{13/14}\log^3 X\,. \tag{31}$$

We treat $S_2^{(2)}$ analogously:

$$S_2^{(2)} = U_1 + U_2\,, \tag{32}$$

where

$$U_1 = \sum_{X^{1/4}<r\le X^{1/2}} \sum_{X^{1/4}<k\le X^{1/2}} c_k e((kr)^c x)\,,$$
$$U_2 = \sum_{X^{1/2}<r\le X^{3/4}} \sum_{X^{1/4}<k\le X/r} c_k e((kr)^c x)\,.$$

We estimate these sums just as W_1, W_2, W_3, with the help of Lemmas 8 and 9, using the inequality

$$\sum_{k\le y} \log^2 k \ll y \log^2 y\,.$$

Thus we get

$$|U_1|\,,\ |U_2| \ll X^{13/14}\log^3 X\,. \tag{33}$$

The inequality (22) follows from (23), (24), (27)–(33). The lemma is proved.

Lemma 11. *If* $1\le T_1\le T$ *then*

$$\frac{1}{\sqrt{T_1}} \sum_{0<\gamma\le T_1} X^\beta \ll X e^{-(\log X)^{1/4}}\,.$$

Proof. Assume that $T_1\le X^{1/3}$. Then using Lemma 5 we have

$$\frac{1}{\sqrt{T_1}} \sum_{0<\gamma\le T_1} X^\beta \le \sum_{0<\gamma\le T_1} X^\beta \le \sum_{0<\gamma\le X^{1/3}} X^\beta$$
$$\ll \log^{45} X\cdot X^{1-0.2\theta(X^{1/3})} \ll X e^{-(\log X)^{1/4}}\,.$$

If $X^{1/3}\le T_1\le X^{3/4}$ then using Lemma 5 we get

$$\frac{1}{\sqrt{T_1}} \sum_{0<\gamma\le T_1} X^\beta \ll \log^6 X\cdot e^\omega\,,$$

where

$$\omega = 2\log X + \tfrac{5}{2}\log T_1 - 2\sqrt{3\log X\log T_1}\,.$$

Obviously ω is a convex function of $\sqrt{\log T_1}$. Thus

$$\max_{X^{1/3}\le T_1\le X^{3/4}} \omega = \max(\omega|_{T_1=X^{1/3}},\, \omega|_{T_1=X^{3/4}}) = \tfrac{7}{8}\log X\,.$$

Hence

$$\log^6 X\cdot e^\omega \ll X^{7/8}\log^6 X \ll X e^{-(\log X)^{1/4}}\,.$$

If $X^{3/4}\le T_1\le T$ then Lemma 5 gives us

$$\frac{1}{\sqrt{T_1}} \sum_{0<\gamma\le T_1} X^\beta \ll \log^6 X\cdot \frac{1}{\sqrt{T_1}} X^{1/2} T_1$$
$$\le \log^6 X\cdot X^{1/2}T^{1/2} \ll X e^{-(\log X)^{1/4}}\,.$$

Lemma 11 is proved.

LEMMA 12. *If* $1 \le T_1 \le T$ *then*

$$\frac{1}{T_1}\sum_{0<\gamma\le T_1} X^{\beta} \ll Xe^{-(\log X)^{1/4}}.$$

Proof. This lemma follows immediately from Lemma 11.

LEMMA 13. *If* $|x| \le \tau$ *then*

$$|J(x)| \ll Xe^{-(\log X)^{1/5}},$$

where $J(x)$ *is defined in* (12).

Proof. Without loss of generality we may assume that $x \ge 0$. By the definition we have

$$I_{\varrho}(x) = \int\limits_{X/2}^{X} t^{\beta-1} e(F(t))\,dt$$

where

$$F(t) = t^c x + \frac{\gamma}{2\pi}\log t.$$

Define the following sets of non-trivial zeroes of $\zeta(s)$:

$$M_1 = \{\varrho \mid |\gamma| \le T,\ -\gamma/2\pi > \tfrac{3}{2}cX^c x\},$$
$$M_2 = \{\varrho \mid |\gamma| \le T,\ \tfrac{1}{2}c(X/2)^c x \le -\gamma/2\pi \le \tfrac{3}{2}cX^c x\},$$
$$M_3 = \{\varrho \mid |\gamma| \le T,\ -\gamma/2\pi < \tfrac{1}{2}c(X/2)^c x\}.$$

(The set M_2 may be empty.)

Let $X^{-c} \le x \le \tau$. If $\varrho \in M_3$ then

$$F'(t) \gg \frac{1}{X}\left(c\left(\frac{X}{2}\right)^c x - \frac{-\gamma}{2\pi}\right) > 0$$

and according to Lemma 2

$$|I_{\varrho}(x)| \ll \frac{X^{\beta}}{c(X/2)^c x + \gamma/2\pi}.$$

Hence

$$\begin{aligned}\sum_{\varrho\in M_3} |I_{\varrho}(x)| &\ll \sum_{-\pi c(X/2)^c x\le\gamma\le T} \frac{X^{\beta}}{c(X/2)^c x + \gamma/2\pi}\\ &\ll \sum_{-\pi c(X/2)^c x\le\gamma\le X^c x} \frac{X^{\beta}}{X^c x} + \sum_{X^c x<\gamma\le T}\frac{X^{\beta}}{\gamma}\\ &\ll \log X\cdot \max_{1\le T_1\le T}\left(\frac{1}{T_1}\sum_{0<\gamma\le T_1} X^{\beta}\right).\end{aligned}$$

Using Lemma 12 we obtain

$$\sum_{\varrho\in M_3} |I_\varrho(x)| \ll Xe^{-(\log X)^{1/5}}\,. \tag{34}$$

If $\varrho \in M_1$ then

$$F'(t) \le \frac{1}{t}\left(cX^c x + \frac{\gamma}{2\pi}\right) \le \frac{1}{X}\left(cX^c x + \frac{\gamma}{2\pi}\right) < 0$$

and by Lemma 2 we get

$$|I_\varrho(x)| \ll \frac{X^\beta}{-\gamma/2\pi - cX^c x} \ll \frac{X^\beta}{|\gamma|}\,.$$

Therefore

$$\sum_{\varrho\in M_1} |I_\varrho(x)| \ll \sum_{0<\gamma\le T_1} \frac{X^\beta}{\gamma} \ll \log X \cdot \max_{1\le T_1\le T}\left(\frac{1}{T_1}\sum_{0<\gamma\le T_1} X^\beta\right)$$

and Lemma 12 gives

$$\sum_{\varrho\in M_1} |I_\varrho(x)| \ll Xe^{-(\log X)^{1/5}}\,. \tag{35}$$

If $\varrho \in M_2$ then $-\gamma/2\pi \asymp X^c x$ and $F''(t) \asymp X^{c-2}x$. According to Lemma 2

$$|I_\varrho(x)| \ll \frac{X^\beta}{\sqrt{X^c x}}\,.$$

Hence, using Lemma 11 we get

$$\sum_{\varrho\in M_2} |I_\varrho(x)| \ll \frac{1}{\sqrt{X^c x}} \sum_{0<\gamma\le X^c x} X^\beta \ll Xe^{-(\log X)^{1/4}}\,. \tag{36}$$

(If the set M_2 is empty then the last estimate is also true.)

Inequalities (34)–(36) give us

$$|J(x)| \ll Xe^{-(\log X)^{1/5}}\,. \tag{37}$$

If $0 \le x \le X^{-c}$ then estimating $|I_\varrho(x)|$ trivially we have

$$\sum_{\varrho\in M_2} |I_\varrho(x)| \ll \sum_{\varrho\in M_2} X^\beta \ll X^{\beta_0}\,,$$

where $\beta_0 = \max_{\varrho\in M_2} \beta < 1$. The terms $\sum_{\varrho\in M_1} |I_\varrho(x)|$ and $\sum_{\varrho\in M_3} |I_\varrho(x)|$ can be estimated as in the previous case. Thus (37) is always true.

Lemma 14. *If* $|x| \le \tau$ *then*

$$S(x) = I(x) + O(Xe^{-(\log X)^{1/5}})\,.$$

P r o o f. Obviously

$$S(x) = U(x) + O(X^{1/2}), \tag{38}$$

where

$$U(x) = \sum_{X/2<n\le X} \Lambda(n)e(n^c x).$$

Using Abel's transformation we get

$$\begin{aligned} U(x) = &- \int\limits_{X/2}^{X} (\Psi(t) - \Psi(X/2)) \frac{d}{dt}(e(t^c x))\, dt \\ &+ (\Psi(X) - \Psi(X/2))e(X^c x). \end{aligned}$$

According to Lemma 4 we obtain

$$\begin{aligned} U(x) = &- \int\limits_{X/2}^{X} \bigg(t - \frac{X}{2} - \sum_{|\gamma|\le T} \frac{t^\varrho - (X/2)^\varrho}{\varrho} \\ &\qquad\qquad + O\bigg(\frac{X\log^2 X}{T}\bigg)\bigg) \frac{d}{dt}(e(t^c x))\, dt \\ &+ \bigg(X - \frac{X}{2} - \sum_{|\gamma|\le T} \frac{X^\varrho - (X/2)^\varrho}{\varrho} + O\bigg(\frac{X\log^2 X}{T}\bigg)\bigg) e(X^c x) \\ = &- \int\limits_{X/2}^{X} \bigg(t - \frac{X}{2} - \sum_{|\gamma|\le T} \frac{t^\varrho - (X/2)^\varrho}{\varrho}\bigg) \frac{d}{dt}(e(t^c x))\, dt \\ &+ \bigg(X - \frac{X}{2} - \sum_{|\gamma|\le T} \frac{X^\varrho - (X/2)^\varrho}{\varrho}\bigg) e(X^c x) + O\bigg(\frac{\tau X^{1+c}}{T}\log^2 X\bigg). \end{aligned}$$

Using partial integration we get

$$\begin{aligned} U(x) &= \int\limits_{X/2}^{X} e(t^c x)\Big(1 - \sum_{|\gamma|\le T} t^{\varrho-1}\Big)\, dt + O\bigg(\frac{\tau X^{1+c}}{T}\log^2 X\bigg) \\ &= I(x) - J(x) + O\bigg(\frac{\tau X^{1+c}}{T}\log^2 X\bigg). \end{aligned}$$

The conclusion follows from formulas (3), (4), (38) and Lemma 13.

P r o o f o f t h e T h e o r e m. It is sufficient to show that the sum

$$B = \sum_{\substack{X/2<p_1,p_2,p_3\le X \\ |p_1^c+p_2^c+p_3^c-N|<\varepsilon}} \log p_1 \log p_2 \log p_3$$

tends to infinity as X tends to infinity. Set

$$B_1 = \sum_{X/2<p_1,p_2,p_3\le X} \log p_1 \log p_2 \log p_3 \cdot \varphi(p_1^c + p_2^c + p_3^c - N)\,.$$

By the definition of $\varphi(y)$ we have

$$B \ge B_1\,. \tag{39}$$

The Fourier transformation formula gives us

$$B_1 = \sum_{X/2<p_1,p_2,p_3\le X} \log p_1 \log p_2 \log p_3 \int_{-\infty}^{\infty} e((p_1^c + p_2^c + p_3^c - N)x)\Phi(x)\,dx$$
$$= \int_{-\infty}^{\infty} S^3(x) e(-Nx)\Phi(x)\,dx\,.$$

Let us represent B_1 in the form

$$B_1 = D_1 + D_2 + D_3\,, \tag{40}$$

where

$$D_1 = \int_{-\tau}^{\tau} S^3(x)e(-Nx)\Phi(x)\,dx\,,$$
$$D_2 = \int_{\tau<|x|<K} S^3(x)e(-Nx)\Phi(x)\,dx\,,$$
$$D_3 = \int_{|x|>K} S^3(x)e(-Nx)\Phi(x)\,dx\,.$$

Using (5)–(8) and Lemma 1 we have

$$|D_3| \ll \int_K^{\infty} |S^3(x)|\,|\Phi(x)|\,dx \tag{41}$$
$$\ll X^3 \int_K^{\infty} \frac{1}{x}\left(\frac{r}{2\pi\Delta x}\right)^r dx \ll X^3\left(\frac{r}{2\pi\Delta K}\right)^r \ll 1\,.$$

We next treat D_2:

$$|D_2| \ll \int_{\tau}^{K} |S^3(x)|\,|\Phi(x)|\,dx \tag{42}$$
$$\ll (\max_{\tau\le x\le K}|S(x)|)\int_{\tau}^{K} |S^2(x)|\,|\Phi(x)|\,dx\,.$$

According to Lemma 1 we get

$$\int\limits_{\tau}^{K} |S^2(x)|\,|\Phi(x)|\,dx \ll \varepsilon \int\limits_{\tau}^{1/\varepsilon} |S^2(x)|\,dx + \int\limits_{1/\varepsilon}^{K} |S^2(x)|\frac{dx}{x}$$

$$\ll \varepsilon \sum_{0\le n\le 1/\varepsilon} \int\limits_{n}^{n+1} |S^2(x)|\,dx + \sum_{1/\varepsilon-1\le n\le K} \frac{1}{n}\int\limits_{n}^{n+1} |S^2(x)|\,dx\,.$$

The last estimate and Lemma 7(iii) give

$$\int\limits_{\tau}^{K} |S^2(x)|\,|\Phi(x)|\,dx \ll X\log^4 X\,.$$

Therefore, by (42) and by Lemma 10 we have

$$|D_2| \ll \varepsilon\frac{X^{3-c}}{\log X}\,. \tag{43}$$

We now find an asymptotic formula for D_1. Set

$$H_1 = \int\limits_{-\tau}^{\tau} I^3(x)e(-Nx)\Phi(x)\,dx\,.$$

Using Lemmas 7(i), 7(ii) and 14 we have

$$\begin{aligned} |D_1 - H_1| &\ll \int\limits_{-\tau}^{\tau} |S^3(x) - I^3(x)|\,|\Phi(x)|\,dx \\ &\ll \varepsilon \int\limits_{-\tau}^{\tau} |S(x) - I(x)|\,(|S^2(x)| + |I^2(x)|)\,dx \\ &\ll \varepsilon X e^{-(\log X)^{1/5}}\Big(\int\limits_{-\tau}^{\tau} |S^2(x)|\,dx + \int\limits_{-\tau}^{\tau} |I^2(x)|\,dx\Big) \\ &\ll \varepsilon X^{3-c} e^{-(\log X)^{1/6}}\,. \end{aligned} \tag{44}$$

Define

$$H = \int\limits_{-\infty}^{\infty} I^3(x)e(-Nx)\Phi(x)\,dx\,.$$

Then we have

$$|H - H_1| \ll \int\limits_{\tau}^{\infty} |I^3(x)|\,|\Phi(x)|\,dx\,. \tag{45}$$

By Lemma 2 we get $|I(x)| \ll 1/(|x|X^{c-1})$. Therefore, by (45) and Lemma 1 we obtain

$$|H-H_1| \ll \frac{1}{X^{3(c-1)}} \int\limits_{\tau}^{\infty} |\Phi(x)| \frac{dx}{x^3} \ll \frac{\varepsilon}{X^{3(c-1)}\tau^2} \ll \varepsilon \frac{X^{3-c}}{\log X}\,. \tag{46}$$

It now follows from formulas (40), (41), (43)–(46) that

$$B_1 = H + O\left(\frac{\varepsilon X^{3-c}}{\log X}\right).$$

Hence by Lemma 6 we have $B_1 \gg \varepsilon X^{3-c}$. Together with (40) this implies that $B \gg \varepsilon X^{3-c}$, and so $B \to \infty$ as $X \to \infty$.

The Theorem is proved.

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References

[1] S. A. Gritsenko, *On a problem of I. M. Vinogradov*, Mat. Zametki 39 (1987), 625–640 (in Russian).

[2] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. 15 (1972), 164–170.

[3] A. A. Karatsuba, *Principles of Analytic Number Theory*, Nauka, Moscow 1983 (in Russian).

[4] I. I. Piatetski-Shapiro, *On a variant of Waring–Goldbach's problem*, Mat. Sb. 30 (72) (1) (1952), 105–120 (in Russian).

[5] B. I. Segal, *On a theorem analogous to Waring's theorem*, Dokl. Akad. Nauk SSSR (N. S.) 2 (1933), 47–49 (in Russian).

[6] E. G. Titchmarsh, *The Theory of the Riemann Zeta-function* (revised by D. R. Heath-Brown), Clarendon Press, Oxford 1986.

[7] D. I. Tolev, *Diophantine inequalities involving prime numbers*, Thesis, Moscow University, 1990 (in Russian).

[8] R. C. Vaughan, *On the distribution of αp modulo* 1, Mathematika 24 (1977), 135–141.

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