A note on some expansions of $p$-adic functions

by

GRZEGORZ SZKIBIEL (Szczecin)

Introduction. Recently J. Rutkowski (see [3]) has defined the $p$-adic analogue of the Walsh system, which we shall denote by $(\phi_m)_{m \in \mathbb{N}_0}$. The system $(\phi_m)_{m \in \mathbb{N}_0}$ is defined in the space $C(\mathbb{Z}_p, \mathbb{C}_p)$ of $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$. J. Rutkowski has also considered some questions concerning expansions of functions from $C(\mathbb{Z}_p, \mathbb{C}_p)$ with respect to $(\phi_m)_{m \in \mathbb{N}_0}$.

This paper is a remark to Rutkowski's paper. We define another system $(h_n)_{n \in \mathbb{N}_0}$ in $C(\mathbb{Z}_p, \mathbb{C}_p)$, investigate its properties and compare it to the system defined by Rutkowski. The system $(h_n)_{n \in \mathbb{N}_0}$ can be viewed as a $p$-adic analogue of the well-known Haar system of real functions (see [1]). It turns out that in general functions are expanded much easier with respect to $(h_n)_{n \in \mathbb{N}_0}$ than to $(\phi_m)_{m \in \mathbb{N}_0}$. Moreover, a function in $C(\mathbb{Z}_p, \mathbb{C}_p)$ has an expansion with respect to $(h_n)_{n \in \mathbb{N}_0}$ if it has an expansion with respect to $(\phi_m)_{m \in \mathbb{N}_0}$. At the end of this paper an example is given of a function which has an expansion with respect to $(h_n)_{n \in \mathbb{N}_0}$ but not with respect to $(\phi_m)_{m \in \mathbb{N}_0}$.

Throughout the paper the ring of $p$-adic integers, the field of $p$-adic numbers and the completion of its algebraic closure will be denoted by $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ respectively ($p$ prime). In addition, we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $E = \{0, 1, \ldots, p - 1\}$.

The author would like to thank Jerzy Rutkowski for fruitful comments and remarks that permitted an improvement of the presentation.

Definition and basic properties. Let $p$ be a fixed prime number and $n \in \mathbb{N}_0$. If $n \neq 0$ then for some $k \in \mathbb{N}_0$ we have $n = n_0 + n_1 p + \ldots + n_k p^k$, where $n_i \in E$ for $i \in \{0, 1, \ldots, k\}$ and $n_k \neq 0$. Define $n_- = n_0 + n_1 p + \ldots + n_{k-1} p^{k-1}$, $n_- = n_k$, $n_+ = n_0$, $n_+ = p^k$. Let $\zeta$ be a primitive $p$-root of unity in $\mathbb{C}_p$.

The functions $h_0, h_1, \ldots$ are defined as follows: $h_0 \equiv 1$ and for $n > 0$ we put

$$h_n(x) := \begin{cases} n_+ \zeta^{n-x} & \text{if } x \in n_- + n_0 \mathbb{Z}_p, \\ 0 & \text{otherwise}, \end{cases}$$
where \( x = x_0 + x_1p + \ldots + x_kp^k + \ldots \) is a \( p \)-adic integer number in Hensel’s form (i.e. \( x_i \in E \)).

Before proving some properties of \( (h_n)_{n \in \mathbb{N}_0} \), we shall introduce some notation. For \( f \in C(\mathbb{Z}_p, \mathbb{C}_p) \) define

\[
\overline{f}(x) := \begin{cases} f(x)^{-1} & \text{if } f(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}
\]

The function \( \langle \cdot, \cdot \rangle : C(\mathbb{Z}_p, \mathbb{C}_p) \times C(\mathbb{Z}_p, \mathbb{C}_p) \to \mathbb{C}_p \) defined by

\[
\langle f, g \rangle := \int_{\mathbb{Z}_p} f \overline{g} = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} f(j)\overline{g}(j)
\]

has some properties of the inner product. We shall see that the system \( (h_n)_{n \in \mathbb{N}_0} \) is orthogonal with respect to the above defined “inner product”.

Moreover, we define

\[
V_k := \{ f \in C(\mathbb{Z}_p, \mathbb{C}_p) : \forall x, y \in \mathbb{Z}_p (x \equiv y \pmod{p^k} \Rightarrow f(x) = f(y)) \}.
\]

Observe that \( V_k \) is a \( p^k \)-dimensional vector space over \( \mathbb{C}_p \). Now we shall prove

**Theorem 1.** Let \( x = x_0 + x_1p + \ldots + x_kp^k + \ldots \in \mathbb{Z}_p \) \( (x_i \in E) \). The functions \( h_0, h_1, \ldots \) have the following properties:

(a) \( |h_0(x)|_p = 1, \ |h_n(x)|_p = \begin{cases} n^{-1} & \text{if } x \in n_+n_p\mathbb{Z}_p, \\ 0 & \text{otherwise,} \end{cases} \)

where \( | \cdot |_p \) denotes the \( p \)-adic norm;

(b) \( \sum_{j=0}^{p^k-1} h_{n+n_0+j}(x) = n_p^{n_+x} \);   

(c) \( h_n \) is continuous for all \( n \in \mathbb{N}_0 \);

(d) \( \langle h_n, h_m \rangle = \begin{cases} n^{-1} & \text{if } n = m, \\ 0 & \text{otherwise;} \end{cases} \)

(e) \( h_0, h_1, \ldots, h_{p^k-1} \) form a basis in the vector space \( V_k \) over \( \mathbb{C}_p \).

**Proof.** Properties (a)–(c) are easy to verify. Let \( n = n_0 + n_1p + \ldots + n_rp^r, m = m_0 + m_1p + \ldots + m_sp^s \) and \( j = j_0 + j_1p + \ldots + j_{k-1}p^{k-1}, \) where \( n_r \neq 0, m_s \neq 0 \) and all coefficients are in \( E \). To prove (d) consider the following sum for \( k > \max\{s, r\} \):

\[
S = \sum_{j=0}^{p^k-1} h_n(j)\overline{h}_m(j).
\]

Assume \( r > s \). Then

\[
h_n(j)\overline{h}_m(j) \neq 0 \quad \text{iff} \quad j \equiv n \pmod{p^r} \quad \text{and} \quad m \equiv n \pmod{p^{s+1}},
\]

and

\[
h_n(j)\overline{h}_m(j) = 0 \quad \text{otherwise.}
\]

Hence, \( n \) and \( m \) are orthogonal with respect to the inner product \( \langle \cdot, \cdot \rangle \). Now, let \( k > r, s \). We shall show that the sum

\[
S = \sum_{j=0}^{p^k-1} h_n(j)\overline{h}_m(j)
\]

is different from zero. If \( k = r, s \), then \( S = 0 \). For \( k > r, s \), we have

\[
S = \sum_{j=0}^{p^k-1} h_n(j)\overline{h}_m(j) = \sum_{j=0}^{p^k-1} \sum_{j_0=0}^{p^r-1} \sum_{j_1=0}^{p^s-1} \overline{h}_n(j_0 + j_1p^r)\overline{h}_m(j_1 + j_2p^s).
\]

Since \( n \equiv m \pmod{p^r} \quad \text{and} \quad m \equiv n \pmod{p^{s+1}} \), we have

\[
\sum_{j_0=0}^{p^r-1} \sum_{j_1=0}^{p^s-1} \overline{h}_n(j_0 + j_1p^r)\overline{h}_m(j_1 + j_2p^s) = \sum_{j_0=0}^{p^r-1} \sum_{j_1=0}^{p^s-1} \overline{h}_n(j_0 + j_1p^r)\overline{h}_m(j_1 + j_2p^s) = \sum_{j_0=0}^{p^r-1} \sum_{j_1=0}^{p^s-1} \overline{h}_n(j_0 + j_1p^r)\overline{h}_m(j_1 + j_2p^s).
\]

Hence, \( S \neq 0 \). This completes the proof of (d).
so

\[
S = \sum_{j_r=0}^{p-1} \sum_{i=0}^{p^{k-r-1}-1} h_n(j_r p^r + ip^{r+1}) \\
\times \bar{h}_m(n_s p^s + n_{s+1} p^{s+1} + \ldots + n_{r-1} p^{r-1} + j_r p^r + ip^{r+1}) \\
= p^{r-s} \zeta^{-m_n} \sum_{i=0}^{p^{k-r-1}-1} p^{k-r-1-1} \sum_{j_r=0}^{p-1} \left( \sum_{n} \zeta^{n_r j_r} \right) = 0.
\]

Reasoning similarly for \( r < s \) one also obtains \( S = 0 \). If \( r = s \) then

\[ h_n(j) \bar{h}_m(j) \neq 0 \text{ iff } j \equiv n \pmod{p^r} \text{ and } m \equiv n \pmod{p^r}. \]

If \( n_r \neq m_r \) then

\[
S = \sum_{j_r=0}^{p-1} \sum_{i=0}^{p^{k-r-1}-1} h_n(j_r p^r + ip^{r+1}) \bar{h}_m(j_r p^r + ip^{r+1}) \\
= p^{k-r-1} \sum_{j_r=0}^{p-1} \zeta^{(n_r-m_r)j_r} = 0.
\]

Otherwise (i.e. when \( n_r = m_r \)) one obtains \( S = p^{k-r} = p^k n_p^{-1} \). Therefore (d) holds.

(e) Observe that \( h_0, h_1, \ldots, h_{p^k-1} \) belong to \( V_k \). It now suffices to show that if \( f \in V_k \) then

\[
f = \left( p^{-k} \sum_{j=0}^{p^k-1} f(j) \right) h_0 \\
+ \sum_{n=1}^{p^k-1} \left( p^{-k} \sum_{j=0}^{p^{k-1} n_p^{-1} - 1} \sum_{p-1}^{p-1} \zeta^{-n^+} f(j pm_p + sn_p + n_-) \right) h_n.
\]

Denote the right side by \( g \). It suffices to show that \( f(r) = g(r) \) for \( r \in \{0, 1, \ldots, p^k-1\} \), because for each \( x \in \mathbb{Z}_p \) there exists \( r \in \{0, 1, \ldots, p^k-1\} \) such that \( x \equiv r \pmod{p^k} \) and \( f, g \in V_k \). Set

\[
S_i = \sum_{n=p^i}^{p^{k-1}} \sum_{j=0}^{p^k} \sum_{s=0}^{p^{k-1} n_p^{-1} - 1} \sum_{p-1}^{p-1} \zeta^{-n^+} f(j pm_p + sn_p + n_-) h_n(r),
\]

where \( i \in \{0, 1, \ldots, k-1\} \).

Let \( r = r_0 + r_1 p + \ldots + r_{k-1} p^{k-1} \), where \( r_0, r_1, \ldots, r_{k-1} \in E \). Then
one has
\[ g(r) = g(r_0 + r_1 p + \ldots + r_{k-1} p^{k-1}) \]
\[ = p^{-k} \sum_{s=0}^{p-1} \sum_{j=0}^{p^{k-1}-1} f(jp + s) \]
\[ + \sum_{n=1}^{p^{-1} p^{k-1} - 1} \sum_{j=0}^{p^{k-1}-1} \sum_{s=0}^{p-1} \zeta^{-n s} f(jp + s) h_n(r_0 + r_1 p + \ldots + r_{k-1} p^{k-1}) + S_1 \]
\[ = p^{-k} \sum_{n=0}^{p-1} \sum_{s=0}^{n \cdot (r_0 - s)} \zeta^{n (r_0 - s)} f(jp + s) + S_1. \]

Observe that \( \sum_{n=0}^{p-1} \zeta^{n (r_0 - s)} \neq 0 \iff s = r_0 \), therefore
\[ g(r) = p^{-k} \sum_{j=0}^{p^{k-1}-1} p f(jp + r_0) + S_1. \]

Reasoning in the same way one obtains
\[ g(r) = p^{-k} \sum_{j=0}^{p^{k-1}-1} p^{k-1} f(jp^{k-1} + r_{k-2} p^{k-2} + \ldots + r_1 p + r_0) + S_{k-1} \]
\[ = p^{-k} \sum_{s=0}^{p-1} p^{k-1} f(sp^{k-1} + r_{k-2} p^{k-2} + \ldots + r_1 p + r_0) \]
\[ + \sum_{n=p^{k-1}}^{p^{k-1}} p^{-k} \sum_{s=0}^{n \cdot (r_0 - s)} \zeta^{n \cdot (r_0 - s)} f(jp^k + sp^{k-1} + n \cdot h_n(r_0 + r_1 p + \ldots + r_{k-2} p^{k-2} + r_{k-1} p^{k-1})). \]

But if \( r_0 + r_1 p + \ldots + r_{k-2} p^{k-2} \neq n_- \) then \( h_n(r) = 0 \) so one gets
\[ g(r) = p^{-k} \sum_{n=0}^{p-1} \sum_{s=0}^{p^{k-1}} p^{k-1} \zeta^{n \cdot (r_0 - s)} f(sp^{k-1} + r_{k-2} p^{k-2} + \ldots + r_1 p + r_0). \]

If \( r_{k-1} \neq s \) then \( \sum_{n=0}^{p-1} \zeta^{n \cdot (r_0 - s)} = 0 \) so finally one obtains
\[ g(r) = f(r_{k-1} p^{k-1} + r_{k-2} p^{k-2} + \ldots + r_1 p + r_0) = f(r). \]

Expansion of functions with respect to the system \((h_n)_{n \in \mathbb{N}_0}\). We start with some notations. The sequence \((x^{(k)})_{k \in \mathbb{N}}\) where \(x^{(k)} = x_0 + x_1 p + \ldots + x_{k-1} p^{k-1}\) is called the standard sequence of the element \(x = x_0 + x_1 p + \ldots \in \mathbb{Z}_p\). The sequence \((f^{(k)})_{k \in \mathbb{N}}\) where \(f^{(k)}(x) = f(x^{(k)})\) is
Expansions of $p$-adic functions

133

called the standard sequence of the function $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$. It is easy to see that

$$
\lim_{k \to \infty} x^{(k)} = x, \quad \lim_{k \to \infty} f^{(k)}(x) = f(x)
$$

for all $x \in \mathbb{Z}_p$ and that $f^{(k)} \in V_k$. So one may apply the formula (1) to $f^{(k)}$ and obtain

$$
f^{(k)} = \sum_{n=0}^{p^{k-1}-1} f_n^{(k)} h_n,
$$

where

$$
\begin{align*}
  f_0^{(k)} &= p^{-k} \sum_{j=0}^{p^{k-1}-1} f(j), \\
  f_n^{(k)} &= p^{-k} \sum_{j=0}^{p^{k-1}-1} \sum_{s=0}^{p-1} \zeta^{-n+s} f(jp^n + sn + n) \quad \text{if } 0 < n < p^k.
\end{align*}
$$

(If $p^{k-1}n_p^{-1} - 1 < 0$ then put $f_n^{(k)} = 0$.)

**Definition 1.** A function $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ has an expansion with respect to the system $(h_n)_{n \in \mathbb{N}_0}$ if the following conditions are satisfied:

1. (E1) for any $n \in \mathbb{N}_0$ the limit $f_n := \lim_{k \to \infty} f_n^{(k)}$ exists;
2. (E2) $\lim_{n \to \infty} n_p f_n = 0$.

Observe that (E2) implies the convergence of $\sum_{n=0}^{\infty} f_n h_n$. This series is called the expansion of $f$ with respect to $(h_n)_{n \in \mathbb{N}_0}$. We write $f \sim \sum_{n=0}^{\infty} f_n h_n$.

**Remark.** The series $\sum_{n=0}^{\infty} f_n h_n$ is also convergent if the sequence $(|f_n|_p)_{n \in \mathbb{N}_0}$ is bounded. Indeed, if there exists $M \in \mathbb{R}$ such that for any $n \in \mathbb{N}_0$ we have $|f_n|_p \leq M$ then

$$
0 \leq |f_n n_p|_p \leq M |n_p|_p \quad \text{and} \quad \lim_{n \to \infty} |n_p|_p = 0,
$$

so (E2) holds and the series $\sum_{n=0}^{\infty} f_n h_n$ is convergent.

The next theorem follows immediately from the above definition.

**Theorem 2.** The set of all functions which have an expansion with respect to $(h_n)_{n \in \mathbb{N}_0}$ is a vector space over $\mathbb{C}_p$.

The following result describes a class of functions which have an expansion with respect to $(h_n)_{n \in \mathbb{N}_0}$.
Theorem 3. If there exist constants \(d_0, d_1, \ldots \in \mathbb{C}_p\) such that \(f = \sum_{m=0}^{\infty} d_m h_m\) then \(f\) has an expansion with respect to \((h_n)_{n \in \mathbb{N}_0}\) and \(f \sim \sum_{m=0}^{\infty} d_m h_m\).

Proof. It is sufficient to compute \(f^{(k)}_n\) where \(k, n \in \mathbb{N}_0\), and to show that \(\lim_{k \to \infty} f^{(k)}_n = d_n\). For \(n = 0\) one has
\[
f^{(k)}_0 = p^{-k} \sum_{j=0}^{p^k-1} d_0 + \sum_{m=1}^{\infty} d_m \left(p^{-k} \sum_{j=0}^{p^k-1} h_{m}(j) \bar{h}_0(j)\right) = d_0,
\]
by virtue of (2) and the proof of Theorem 1(d). For \(n > 0\), consider the sum
\[
S = \sum_{j=0}^{p^{k-1}n-1} \sum_{s=0}^{p-1} \zeta^{-n+s} h_{m}(jpn + sn + n)\).
\]
Using the definition of \((h_n)_{n \in \mathbb{N}_0}\) and the properties of roots of unity one obtains \(S = 0\) if \(n \neq m\) and \(S = p^k\) if \(n = m\).

Finally, one has
\[
f^{(k)}(n) = \sum_{m \neq n} d_m \left(p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{p-1} \zeta^{-n+s} h_{m}(jpn + sn + n)\right) + d_n p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{p-1} \zeta^{-n+s} h_{n}(jpn + sn + n) = d_n.
\]
Now one can see that \(\lim_{k \to \infty} f^{(k)}_n = d_n\) for \(n \in \mathbb{N}_0\), so (E1) holds. By convergence of \(\sum_{m=0}^{\infty} d_m h_m\), (E2) also holds. •

From the above theorem one can deduce the following two corollaries:

Corollary 4. If \(f = \sum_{m=0}^{\infty} d_m h_m = \sum_{m=0}^{\infty} d'_m h_m\) then \(d_m = d'_m\). •

Corollary 5. If the expansions of \(f, g \in C(\mathbb{Z}_p, \mathbb{C}_p)\) with respect to \((h_n)_{n \in \mathbb{N}_0}\) are convergent to those functions, then \(fg\) has an expansion with respect to \((h_n)_{n \in \mathbb{N}_0}\).

Proof. Let \(f = \sum_{n=0}^{\infty} f_n h_n\) and \(g = \sum_{n=0}^{\infty} g_n h_n\). These series are absolutely convergent so their product is also absolutely convergent. Hence one may change the order of its terms. Because the product \(h_n h_m\) is again \(h_s\) or \(\lambda h_s\) (for some \(s \in \mathbb{N}_0, \lambda \in \mathbb{C}_p\)) one can obtain the series \(fg = \sum_{s=0}^{\infty} d_s h_s\) as a product \((\sum_{n=0}^{\infty} f_n h_n)(\sum_{n=0}^{\infty} g_n h_n)\). To finish the proof it is enough to apply Theorem 3. •

Now we shall give a few examples of expansions with respect to \((h_n)_{n \in \mathbb{N}_0}\).
Examples of expansions. (a) Identity on $\mathbb{Z}_p$ ($f(x) = x$). Employing formulas (2) and Definition 1 one obtains

$$f_0 = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} j = \lim_{k \to \infty} (p^{-k} 2^{-1} (p^k - 1)p^k) = -2^{-1},$$

$$f_n = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{p-1} \zeta^{-n+s}(jpn_p + sn_p + n_-)$$

$$= p^{-1} \sum_{s=0}^{p-1} s \zeta^{-n+s} \text{ for } n > 0.$$

Note that if $p = 2$ one gets $f_n = -2^{-1}$ for all $n \in \mathbb{N}_0$.

(b) Quadratic function ($f(x) = x^2$). It follows by direct computation that

$$f_0 = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} j^2 = \lim_{k \to \infty} (p^{-k} 6^{-1} (p^k - 1)p^k(2p^k - 1)) = 6^{-1},$$

$$f_n = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{p-1} \zeta^{-n+s}(jpn_p + sn_p + n_-)^2$$

$$= \lim_{k \to \infty} \sum_{s=0}^{p-1} s \zeta^{-n+s}(p^{-1}nps + p^{-1}n_- + n_p(p^k - 1))$$

$$= \sum_{s=0}^{p-1} s \zeta^{-n+s}(p^{-1}nps + p^{-1}n_- - n_p) \text{ for } n > 0.$$

(c) Exponential function ($f(x) = \exp(ax)$ where $|a|_p < p^{1/(p-1)}$). In this case, using the properties of the function $\exp$, one gets

$$f_0 = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} \exp(aj) = a(\exp(a) - 1)^{-1},$$

$$f_n = \lim_{k \to \infty} \left( p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{p-1} \zeta^{-n+s} \exp(ajpn_p) \exp(asn_p) \exp(an_-) \right)$$

$$= \lim_{k \to \infty} \left( p^{-k} \exp(an_-)(\exp(ap^k) - 1)(\exp(apm_p) - 1)^{-1} \right.$$  

$$\left. \times \sum_{s=0}^{p-1} \zeta^{-n+s} \exp(asn_p) \right)$$
\[
\begin{aligned}
&= a \exp(an_\cdot) \exp(apn_p) - 1 \sum_{s=0}^{p-1} \zeta^{-n+s} \exp(asn_p) \quad \text{for } n > 0.
\end{aligned}
\]

(d) *Trigonometric functions* \((f(x) = \sin(ax), g(x) = \cos(ax), \text{where } |a| p < p^{1/(p-1)})\). Applying well-known formulas one obtains

\[
f_0 = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^{k-1}} \sin(aj) = \lim_{k \to \infty} \left( p^{-k} \sin(2^{-1}a) \right)^{-1} \sin(2^{-1}a(p^k - 1) \sin(2^{-1}ap^k)) = -2^{-1}a,
\]

\[
f_n = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^{k-1}n_p^{-1}-1} \sum_{s=0}^{p-1} \zeta^{-n+s} \sin(ajpn + asn_p + an_\cdot)
= \lim_{k \to \infty} \left( p^{-k} \sin(2^{-1}apn_p) \right)^{-1} \sin(2^{-1}apn_p(p^{k-1}n_p^{-1} - 1))
\times \sin(2^{-1}ap^k) \sum_{s=0}^{p-1} \zeta^{-n+s} \cos(asn_p + an_\cdot)
+ p^{-k} \sin(2^{-1}apn_p) \cos(2^{-1}apn_p(p^{k-1}n_p^{-1} - 1))
\times \sin(2^{-1}ap^k) \sum_{s=0}^{p-1} \zeta^{-n+s} \sin(asn_p + an_\cdot)
\]

\[
= -2^{-1}a \left( \sum_{s=0}^{p-1} \zeta^{-n+s} \cos(asn_p + an_\cdot) + \tan(2^{-1}apn_p)^{-1} \sum_{s=0}^{p-1} \zeta^{-n+s} \sin(asn_p + an_\cdot) \right) \quad \text{for } n > 0.
\]

Reasoning in the same way one gets \(g_0 = 2^{-1}a \cdot \tan(2^{-1}a)\) and

\[
g_n = 2^{-1}a \left( \tan(2^{-1}apn_p)^{-1} \sum_{s=0}^{p-1} \zeta^{-n+s} \cos(asn_p + an_\cdot) + \sum_{s=0}^{p-1} \zeta^{-n+s} \sin(asn_p + an_\cdot) \right) \quad \text{for } n > 0.
\]

(e) *Characteristic function of a coset of the residue class field.* Let \(A = t + p^n \mathbb{Z}_p\), where \(0 \leq t \leq p^{-1}t\). Then

\[
\chi_A(x) = \begin{cases} 
1, & x \in A, \\
0, & x \notin A.
\end{cases}
\]
Without any difficulty one obtains

\[(\chi_A)_0 = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^{k-1}-1} \chi_A(j) = \lim_{k \to \infty} p^{-k} \sum_{i=0}^{p^{k-r}-1} \chi_A(t + ip^r) = p^{-r}.
\]

For \(n > 0\) assume \(n = n_0 + n_1 p + \ldots + n_a p^a\), \(t = t_0 + t_1 p + \ldots + t_{r-1} p^{r-1}\) and consider two cases \(a < r\) and \(a \geq r\). In the first case one gets

\[
(\chi_A)_n = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^{k-a-1}-1} \sum_{s=0}^{p-1} \zeta^{-n_s} \chi_A(jp^a + sp^n + n_0).
\]

Considering the second case, note that if \(n_0 \neq t \pmod{p^a}\) then \((\chi_A)_n = 0\). Otherwise one obtains

\[
(\chi_A)_n = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^{k-a-1}-1} \sum_{s=0}^{p-1} \zeta^{-n_s} \chi_A(jp^a + sp^n + n_0) = p^{-a-1} \sum_{s=0}^{p-1} \zeta^{-n_s} = 0.
\]

**Relationship between** \((h_n)_{n \in \mathbb{N}_0}\) **and** \((\phi_m)_{m \in \mathbb{N}_0}\). The aim of this section is to show that \(f\) has an expansion with respect to \((h_n)_{n \in \mathbb{N}_0}\) if it has one with respect to the system \((\phi_m)_{m \in \mathbb{N}_0}\) defined by Rutkowski (see [3]), and to give an example of a function which has an expansion with respect to \((h_n)_{n \in \mathbb{N}_0}\) but not with respect to \((\phi_m)_{m \in \mathbb{N}_0}\). First recall the definition and basic properties of \((\phi_m)_{m \in \mathbb{N}_0}\). For \(m = m_0 + m_1 p + \ldots + m_s p^s \in \mathbb{N}\) define

\[
\phi_m(x) = \phi_m(x_0 + x_1 p + \ldots + x_s p^s + \ldots) = \zeta^{x_0 m_0 + x_1 m_1 + \ldots + x_s m_s},
\]

\[
\phi_0(x) \equiv 1.
\]

It follows immediately that

\[
\phi_m(x_0 + x_1 p + \ldots + x_r p^r + x_{r+1} p^{r+1} + \ldots) = \phi_m(x_0 + x_1 p + \ldots + x_r p^r) \phi_m(x_{r+1} p^{r+1} + \ldots).
\]

The system \((\phi_m)_{m \in \mathbb{N}_0}\) is orthonormal in the sense of the definition given before Theorem 1. The functions \(\phi_0, \phi_1, \ldots, \phi_{p^{k-1}}\) form a basis in the vector space \(V_k\) (see Theorem 1(e)). For \(f \in C(\mathbb{Z}_p, \mathbb{C}_p)\), elements of its standard
sequence can be represented in the form
\[
f^{(k)} = \sum_{m=0}^{p^k-1} \left( p^{-k} \sum_{j=0}^{p^k-1} f(j) \overline{\phi}_m(j) \right) \phi_m, \quad \text{where } \overline{\phi}_m(j) = \phi_m(j)^{-1}.
\]

Define \( \widehat{f}^{(k)} = p^{-k} \sum_{j=0}^{p^k-1} f(j) \overline{\phi}_m(j) \). Making use of the above notations we introduce

**Definition 2.** A function \( f \in C(\mathbb{Z}_p, \mathbb{C}_p) \) has an expansion with respect to the system \( (\phi_m)_{m \in \mathbb{N}_0} \) if the following holds:

(I) for any \( m \in \mathbb{N}_0 \) the limit \( \widehat{f}^{(k)} = \lim_{k \to \infty} \widehat{f}^{(k)}_m \) exists;

(II) \( \lim_{m \to \infty} \widehat{f}^{(k)}_m = 0 \).

Note that (II) guarantees the convergence of the series \( \sum_{m=0}^{\infty} \widehat{f}^{(k)}_m \phi_m \), called the expansion of \( f \) with respect to \( (\phi_m)_{m \in \mathbb{N}_0} \). We write \( f \sim \sum_{m=0}^{\infty} \widehat{f}^{(k)}_m \phi_m \). Now we prove the main theorem of this section.

**Theorem 6.** A function \( f \) has an expansion with respect to \( (h_n)_{n \in \mathbb{N}_0} \) if it has one with respect to \( (\phi_m)_{m \in \mathbb{N}_0} \).

**Proof.** First we transform the formulas for the coefficients \( f^{(k)}_n \) in the expansion of \( f \) with respect to \( (h_n)_{n \in \mathbb{N}_0} \). For \( n > 0 \) and \( k \) large enough,

\[
f^{(k)}_n = p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{n_p-1} \zeta^{-n+s} f(jp^n + sn + n_-)
= p^{-k} \sum_{j=0}^{p^k-1} f(jp^n + n_-) \overline{h}_n(jp^n + n_-)n_p
= n_p p^{-k} \sum_{j=0}^{p^k-1} \sum_{r=0}^{n_p-1} f(jp^n + r) \overline{h}_n(jp^n + r)
= n_p p^{-k} \sum_{j=0}^{p^k-1} f(j) \overline{h}_n(j).
\]

One can check that

\[
\overline{h}_n = n_p^{-2} \sum_{r=0}^{n_p-1} \overline{\phi}_m(n_-) \phi_{m+n_p+r}
\]

(4)

(\text{where } -n_+ \equiv m_+ \pmod{p} \text{ and } n > 0),

\[
\overline{h}_0 \equiv \overline{\phi}_0.
\]
Applying (4) to (3) one obtains
\[ f_n^{(k)} = n_p^{-1} \sum_{r=0}^{n_p-1} \phi_r(n-) p^{-k} \sum_{j=0}^{p^k-1} f(j) \phi_{n+p+r}(j) \]
\[ = n_p^{-1} \sum_{r=0}^{n_p-1} \phi_r(n-) \hat{f}_{n+p+r}^{(k)}. \]

The limit \( f_n = \lim_{k \to \infty} f_n^{(k)} \) exists because \( \hat{f}_{n+p+r} = \lim_{k \to \infty} \hat{f}_{n+p+r}^{(k)} \) exists by Definition 2 and \( f_n = n_p^{-1} \sum_{r=0}^{n_p-1} \phi_r(n-) \hat{f}_{n+p+r} \), so condition (E1) of Definition 1 is satisfied. Now,
\[ |n_p f_n|_p = \left| \sum_{r=0}^{n_p-1} \phi_r(n-) \hat{f}_{n+p+r} \right|_p \leq \max\{ |f_{n+p+r}|_p : 0 \leq r \leq n_p \}. \]

But \( \lim_{n \to \infty} \hat{f}_n = 0 \) so \( \max\{ |f_{n+p+r}|_p : 0 \leq r \leq n_p \} \to 0 \) as \( n \to \infty \). Thus \( \lim_{n \to \infty} n_p f_n = 0 \) and condition (E2) of Definition 1 is also satisfied.

Applying the above theorem and the result proved in [3], we immediately obtain the following

**Corollary 7.** (a) There exists a function \( f \in C(Z_p, \mathbb{C}_p) \) which has an expansion with respect to \( (h_n)_{n \in \mathbb{N}_0} \) and \( f \neq 0, f \sim 0 \).

(b) Every uniformly differentiable function has an expansion with respect to \( (h_n)_{n \in \mathbb{N}_0} \).

(c) There exists a differentiable function which does not have an expansion with respect to \( (h_n)_{n \in \mathbb{N}_0} \).

Now we will show that the system \( (h_n)_{n \in \mathbb{N}_0} \) is more general than \( (\phi_m)_{m \in \mathbb{N}_0} \).

**Example.** Consider the function \( f : Z_p \to \mathbb{C}_p \) given by \( f(0) = 0 \) and \( f(xa p^a + x_{a+1} p^{a+1} + \ldots) = p^{a+1} c^{a+1} \), where \( x_a \) is non-zero. One can check that \( f \) is continuous. We shall show that \( f \) has an expansion with respect to \( (h_n)_{n \in \mathbb{N}_0} \), but the sequence \( (\hat{f}_{n^{(p^a)})})_{n \in \mathbb{N}} \) is convergent to \( 2(p-1) \) so statement (II) of Definition 2 fails. We first prove the following facts:

(i) for \( s \in \mathbb{N}, x \in Z_p \) one has \( f(p^s x) = p^s f(x) \);
(ii) \( f(xa p^a + x_{a+1} p^{a+1} + \ldots + x_{a+r} p^{a+r} \ldots) = f(xa p^a + x_{a+1} p^{a+1} + \ldots + x_{a+r} p^{a+r}) \), where \( x_a \) is non-zero and \( r \geq 1 \);
(iii) \( \sum_{j=0}^{p^k-1} f(\alpha p^j + j p^k) = 0 \) for \( s \in \mathbb{N}_0, k \in \mathbb{N}, \alpha \in E \setminus \{0\} \);
(iv) \( \sum_{j=0}^{p^k-1} f(j) = (p-1)p^k \) for \( k \in \mathbb{N} \).

The properties (i), (ii) are easy to verify and we get (iii) immediately by
direct computations:

\[ \sum_{j=0}^{p^k-1} f(\alpha p^s + j p^{s+1}) = \sum_{i=0}^{p-1} \sum_{j=0}^{p^k-1} f(\alpha p^s + ip^{s+1} + j p^{s+2}) \]

\[ = p^{k-1} p^{s+1} \sum_{i=0}^{p-1} \zeta^i = 0. \]

To prove (iv) write

\[ \sum_{j=0}^{p^k-1} f(j) = \sum_{j_0=1}^{p-1} \sum_{i=0}^{p^k-1} f(j_0 + ip) + \sum_{j_1=1}^{p-1} \sum_{i=0}^{p^k-1} f(j_1p + ip^2) + \ldots \]

\[ + \sum_{j_{k-2}=1}^{p-1} \sum_{i=0}^{p^k-1} f(j_{k-2}p^{k-2} + ip^{k-1}) + \sum_{i=1}^{p-1} f(ip^{k-1}) + f(0). \]

The last two terms are 0 and \((p-1)p^k\) respectively by definition of \(f\) while the others are zero by (iii).

Now we are ready to compute the coefficients \(f_n\). Using (iv), (2) and Definition 1, one obtains \(f_0 = p - 1\). For \(n > 0\) we consider three cases: 1° \(n_+ = 0\), 2° \(n_+ = an_p p^{-1}\) where \(a \in E \setminus \{0\}\) and 3° \(n_+ \neq 0\) or \(n_- \neq an_p p^{-1}\). In the first case one gets

\[ f_n = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{p-1} \zeta^{-n+s} f(jp + sn_p) \]

\[ = \lim_{k \to \infty} p^{-k} p^{p^k-1} p^{-n_-} \sum_{s=1}^{p-1} \zeta^{-n_+} \sum_{j=0}^{p-1} f(s + j p) + \sum_{j=0}^{p-1} f(j). \]

Here we have used (i). Applying (iii) and (iv) one can check that \(f_n = p - 1\).

Consider the second case:

\[ f_n = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{p-1} \zeta^{-n+s} f(jp + sn_p + an_p^{-1}) \]

\[ = \lim_{k \to \infty} p^{-k} p^{p^k-1} p^{-n_-} \sum_{s=0}^{p-1} p \zeta^{(1-n_+)} \zeta^s. \]

Here if \(n_+ = 1\) then \(f_n = 1\) and otherwise \(f_n = 0\). Finally, if neither the first nor the second case holds then using (ii) one has
Expansions of $p$-adic functions

$$f_n = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} \sum_{s=0}^{p-1} \zeta^{-n+s} f(jpn_p + sn_p + n_-)$$

$$= \lim_{k \to \infty} p^{-k} p^{k-1} p^{1-n} f(n_-) \sum_{s=0}^{p-1} \zeta^{-n+s} = 0.$$

Since $f_n \in \mathbb{Z}_p$ for all $n \in \mathbb{N}_0$ the function $f$ has an expansion with respect to $(h_n)_{n \in \mathbb{N}_0}$ by the remark after Definition 1.

Let us compute the coefficients $\hat{f}_{p^s}$ (where $s \in \mathbb{N}$):

$$\hat{f}_{p^s} = \lim_{k \to \infty} p^{-k} \sum_{j=0}^{p^k-1} f(j) \bar{\varphi}_{p^s}(j)$$

$$= \lim_{k \to \infty} p^{-k} \sum_{i=0}^{p^{s-1}-1} \sum_{a=0}^{p^k-1} \sum_{b=0}^{p^k-1} \sum_{c=0}^{p^k-2} \sum_{j=0}^{p^k-1} f(i + ap^{s-1} + bp^s + cp^{s+1} + jp^{s+2}) \zeta^{-b}$$

$$= \lim_{k \to \infty} p^{-k} \left( \sum_{i=1}^{p^{s-1}-1} \sum_{a=0}^{p^k-1} f(i + ap^{s-1}) \sum_{b=0}^{p^k-1} \zeta^{-b} \right.$$  
$$+ p^{k-s-1} \sum_{a=1}^{p^k-1} \sum_{b=0}^{p^k-1} f(ap^{s-1} + bp^s) \zeta^{-b}$$

$$+ p^{k-s-2} \sum_{b=1}^{p^k-1} \sum_{c=0}^{p^k-1} f(bp^s + cp^{s+1}) \zeta^{-b}$$

$$+ \sum_{c=1}^{p^k-1} \sum_{j=0}^{p^k-1} f(cp^{s+1} + jp^{s+2}) + \sum_{j=0}^{p^k-1} f(jp^{s+2}) \right).$$

The first sum is zero. Applying (iii) one finds that the third and fourth sums are also zero. Using (i) and (iv) one shows that the fifth sum equals $p^{k}(p-1)$. Finally, applying the definition of $f$, one concludes that the second sum is $p^{k}(p-1)$. So $\hat{f}_{p^s} = 2(p-1)$.

References

