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A note on some expansions of p-adic functions

by

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Introduction. Recently J. Rutkowski (see [3]) has defined the p-adic analogue of the Walsh system, which we shall denote by $(\phi_m)_{m\in\mathbb{N}_0}$. The system $(\phi_m)_{m\in\mathbb{N}_0}$ is defined in the space $C(\mathbb{Z}_p,\mathbb{C}_p)$ of $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$. J. Rutkowski has also considered some questions concerning expansions of functions from $C(\mathbb{Z}_p,\mathbb{C}_p)$ with respect to $(\phi_m)_{m\in\mathbb{N}_0}$.

This paper is a remark to Rutkowski's paper. We define another system $(h_n)_{n\in\mathbb{N}_0}$ in $C(\mathbb{Z}_p,\mathbb{C}_p)$, investigate its properties and compare it to the system defined by Rutkowski. The system $(h_n)_{n\in\mathbb{N}_0}$ can be viewed as a p-adic analogue of the well-known Haar system of real functions (see [1]). It turns out that in general functions are expanded much easier with respect to $(h_n)_{n\in\mathbb{N}_0}$ than to $(\phi_m)_{m\in\mathbb{N}_0}$. Moreover, a function in $C(\mathbb{Z}_p,\mathbb{C}_p)$ has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$ if it has an expansion with respect to $(\phi_m)_{m\in\mathbb{N}_0}$. At the end of this paper an example is given of a function which has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$ but not with respect to $(\phi_m)_{m\in\mathbb{N}_0}$.

Throughout the paper the ring of p-adic integers, the field of p-adic numbers and the completion of its algebraic closure will be denoted by $\mathbb{Z}_p,\mathbb{Q}_p$ and $\mathbb{C}_p$ respectively (p prime). In addition, we write $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $E=\{0,1,\ldots,p-1\}$.

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Definition and basic properties. Let p be a fixed prime number and $n\in\mathbb{N}_0$. If $n\neq 0$ then for some $k\in\mathbb{N}_0$ we have $n=n_0+n_1p+\ldots+n_kp^k$, where $n_i\in E$ for $i\in\{0,1,\ldots,k\}$ and $n_k\neq 0$. Define $n_-=n_0+n_1p+\ldots+n_{k-1}p^{k-1}$, $n_+=n_k$, $n_p=p^k$. Let ζ be a primitive p-root of unity in $\mathbb{C}_p$. The functions $h_0,h_1,\ldots$ are defined as follows: $h_0\equiv 1$ and for $n>0$ we put

$$h_n(x):=\begin{cases} n_p\zeta^{n_+x_k} & \text{if } x\in n_-+n_p\mathbb{Z}_p,\\ 0 & \text{otherwise,}\end{cases}$$

where $x = x_0 + x_1 p + \ldots + x_k p^k + \ldots$ is a p-adic integer number in Hensel's form (i.e. $x_i \in E$).

Before proving some properties of $(h_n)_{n\in\mathbb{N}_0}$, we shall introduce some notation. For $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ define

$$\overline{f}(x) := \begin{cases} f(x)^{-1} & \text{if } f(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\langle \cdot, \cdot \rangle : C(\mathbb{Z}_p, \mathbb{C}_p) \times C(\mathbb{Z}_p, \mathbb{C}_p) \to \mathbb{C}_p$ defined by

$$\langle f, g \rangle := \int_{\mathbb{Z}_p} f\overline{g} = \lim_{k\to\infty} p^{-k} \sum_{j=0}^{p^k-1} f(j)\overline{g}(j)$$

has some properties of the inner product. We shall see that the system $(h_n)_{n\in\mathbb{N}_0}$ is orthogonal with respect to the above defined "inner product". Moreover, we define

$$V_k := \{f \in C(\mathbb{Z}_p, \mathbb{C}_p) : \forall x, y \in \mathbb{Z}_p \ (x \equiv y \ (\operatorname{mod} p^k) \Rightarrow \ f(x) = f(y))\}\,.$$

Observe that V_k is a p^k-dimensional vector space over $\mathbb{C}_p$. Now we shall prove

THEOREM 1. *Let $x = x_0 + x_1 p + \ldots + x_k p^k + \ldots \in \mathbb{Z}_p$ $(x_i \in E)$. The functions $h_0, h_1, \ldots$ have the following properties*:

(a) $|h_0(x)|_p = 1, \quad |h_n(x)|_p = \begin{cases} n_p^{-1} & \textit{if } x \in n_- + n_p\mathbb{Z}_p, \\ 0 & \textit{otherwise}, \end{cases}$

where $|\cdot|_p$ denotes the p-adic norm;

(b) $\sum_{j=0}^{p^k-1} h_{n_+ n_p + j}(x) = n_p \zeta^{n_+ x_k}$;

(c) *h_n is continuous for all $n \in \mathbb{N}_0$*;

(d) $\langle h_n, h_m \rangle = \begin{cases} n_p^{-1} & \textit{if } n = m, \\ 0 & \textit{otherwise}; \end{cases}$

(e) *$h_0, h_1, \ldots, h_{p^k-1}$ form a basis in the vector space V_k over $\mathbb{C}_p$.*

Proof. Properties (a)–(c) are easy to verify. Let $n = n_0 + n_1 p + \ldots + n_r p^r$, $m = m_0 + m_1 p + \ldots + m_s p^s$ and $j = j_0 + j_1 p + \ldots + j_{k-1} p^{k-1}$, where $n_r \neq 0$, $m_s \neq 0$ and all coefficients are in E. To prove (d) consider the following sum for $k > \max\{s, r\}$:

$$S = \sum_{j=0}^{p^k-1} h_n(j)\overline{h}_m(j)\,.$$

Assume $r > s$. Then

$$h_n(j)\overline{h}_m(j) \neq 0 \quad \text{iff} \quad j \equiv n \ (\operatorname{mod} p^r) \text{ and } m \equiv n \ (\operatorname{mod} p^{s+1})\,,$$

so

$$S=\sum_{j_r=0}^{p-1}\sum_{i=0}^{p^{k-r-1}-1} h_n(j_rp^r+ip^{r+1})$$
$$\times\,\overline{h}_m(n_sp^s+n_{s+1}p^{s+1}+\ldots+n_{r-1}p^{r-1}+j_rp^r+ip^{r+1})$$
$$=p^{r-s}\zeta^{-m_sn_s}\sum_{i=0}^{p^{k-r-1}-1}\Big(\sum_{j_r=0}^{p-1}\zeta^{n_rj_r}\Big)=0\,.$$

Reasoning similarly for $r<s$ one also obtains $S=0$. If $r=s$ then

$$h_n(j)\overline{h}_m(j)\neq 0 \quad \text{iff} \quad j\equiv n \ (\mathrm{mod}\, p^r) \text{ and } m\equiv n \ (\mathrm{mod}\, p^r)\,.$$

If $n_r\neq m_r$ then

$$S=\sum_{j_r=0}^{p-1}\sum_{i=0}^{p^{k-r-1}-1} h_n(j_rp^r+ip^{r+1})\overline{h}_m(j_rp^r+ip^{r+1})$$
$$=p^{k-r-1}\sum_{j_r=0}^{p-1}\zeta^{(n_r-m_r)j_r}=0\,.$$

Otherwise (i.e. when $n_r=m_r$) one obtains $S=p^{k-r}=p^kn_p^{-1}$. Therefore (d) holds.

(e) Observe that $h_0,h_1,\ldots,h_{p^k-1}$ belong to V_k. It now suffices to show that if $f\in V_k$ then

$$(1)\qquad f=\Big(p^{-k}\sum_{j=0}^{p^k-1}f(j)\Big)h_0$$
$$+\sum_{n=1}^{p^k-1}\Big(p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}f(jpn_p+sn_p+n_-)\Big)h_n.$$

Denote the right side by g. It suffices to show that $f(r)=g(r)$ for $r\in\{0,1,\ldots,p^k-1\}$, because for each $x\in\mathbb{Z}_p$ there exists $r\in\{0,1,\ldots,p^k-1\}$ such that $x\equiv r \ (\mathrm{mod}\, p^k)$ and $f,g\in V_k$. Set

$$S_i=\sum_{n=p^i}^{p^k-1}p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}f(jpn_p+sn_p+n_-)h_n(r)\,,$$

where $i\in\{0,1,\ldots,k-1\}$.

Let $r=r_0+r_1p+\ldots+r_{k-1}p^{k-1}$, where $r_0,r_1,\ldots,r_{k-1}\in E$. Then one has

$$\begin{aligned}
g(r) &= g(r_0 + r_1 p + \ldots + r_{k-1}p^{k-1}) \\
&= p^{-k}\sum_{s=0}^{p-1}\sum_{j=0}^{p^{k-1}-1} f(jp+s) \\
&\quad + \sum_{n=1}^{p-1} p^{-k}\sum_{j=0}^{p^{k-1}-1}\sum_{s=0}^{p-1}\zeta^{-ns} f(jp+s)h_n(r_0 + r_1 p + \ldots + r_{k-1}p^{k-1}) + S_1 \\
&= p^{-k}\sum_{n=0}^{p-1}\sum_{s=0}^{p-1}\sum_{j=0}^{p^{k-1}-1}\zeta^{n(r_0-s)} f(jp+s) + S_1\,.
\end{aligned}$$

Observe that $\sum_{n=0}^{p-1}\zeta^{n(r_0-s)} \neq 0$ iff $s = r_0$, therefore

$$g(r) = p^{-k}\sum_{j=0}^{p^{k-1}-1} pf(jp+r_0) + S_1\,.$$

Reasoning in the same way one obtains

$$\begin{aligned}
g(r) &= p^{-k}\sum_{j=0}^{p-1} p^{k-1} f(jp^{k-1} + r_{k-2}p^{k-2} + \ldots + r_1 p + r_0) + S_{k-1} \\
&= p^{-k}\sum_{s=0}^{p-1} p^{k-1} f(sp^{k-1} + r_{k-2}p^{k-2} + \ldots + r_1 p + r_0) \\
&\quad + \sum_{n=p^{k-1}}^{p^k-1} p^{-k}\sum_{j=0}^{p^0-1}\sum_{s=0}^{p-1}\zeta^{-n_+ s} f(jp^k + sp^{k-1} + n_-)h_n(r_0 + r_1 p + \ldots \\
&\qquad \ldots + r_{k-2}p^{k-2} + r_{k-1}p^{k-1})\,.
\end{aligned}$$

But if $r_0 + r_1 p + \ldots + r_{k-2}p^{k-2} \neq n_-$ then $h_n(r) = 0$ so one gets

$$g(r) = p^{-k}\sum_{n_+=0}^{p-1}\sum_{s=0}^{p-1} p^{k-1}\zeta^{n_+(r_{k-1}-s)} f(sp^{k-1} + r_{k-2}p^{k-2} + \ldots + r_1 p + r_0)\,.$$

If $r_{k-1} \neq s$ then $\sum_{n_+=0}^{p-1}\zeta^{n_+(r_{k-1}-s)} = 0$ so finally one obtains

$$g(r) = f(r_{k-1}p^{k-1} + r_{k-2}p^{k-2} + \ldots + r_1 p + r_0) = f(r)\,. \ \blacksquare$$

Expansion of functions with respect to the system $(h_n)_{n\in\mathbb{N}_0}$. We start with some notations. The sequence $(x^{(k)})_{k\in\mathbb{N}}$ where $x^{(k)} = x_0 + x_1 p + \ldots + x_{k-1}p^{k-1}$ is called the *standard sequence* of the element $x = x_0 + x_1 p + \ldots \in \mathbb{Z}_p$. The sequence $(f^{(k)})_{k\in\mathbb{N}}$ where $f^{(k)}(x) = f(x^{(k)})$ is called the *standard sequence* of the function $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$. It is easy to see that

$$\lim_{k\to\infty} x^{(k)} = x\,, \qquad \lim_{k\to\infty} f^{(k)}(x) = f(x)$$

for all $x \in \mathbb{Z}_p$ and that $f^{(k)} \in V_k$. So one may apply the formula (1) to $f^{(k)}$ and obtain

$$f^{(k)} = \sum_{n=0}^{p^k-1} f_n^{(k)} h_n\,,$$

where

$$\begin{aligned} f_0^{(k)} &= p^{-k} \sum_{j=0}^{p^k-1} f(j)\,, \\ f_n^{(k)} &= p^{-k} \sum_{j=0}^{p^{k-1}n_p^{-1}-1} \sum_{s=0}^{p-1} \zeta^{-n+s} f(jpn_p + sn_p + n_-) \qquad \text{if } 0 < n < p^k\,. \end{aligned} \tag{2}$$

(If $p^{k-1}n_p^{-1} - 1 < 0$ then put $f_n^{(k)} = 0$.)

Definition 1. A function $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ *has an expansion with respect to the system* $(h_n)_{n\in\mathbb{N}_0}$ if the following conditions are satisfied:

(E1) for any $n \in \mathbb{N}_0$ the limit $f_n := \lim_{k\to\infty} f_n^{(k)}$ exists;
(E2) $\lim_{n\to\infty} n_p f_n = 0$.

Observe that (E2) implies the convergence of $\sum_{n=0}^{\infty} f_n h_n$. This series is called the *expansion of f with respect to* $(h_n)_{n\in\mathbb{N}_0}$. We write $f \sim \sum_{n=0}^{\infty} f_n h_n$.

Remark. The series $\sum_{n=0}^{\infty} f_n h_n$ is also convergent if the sequence $(|f_n|_p)_{n\in\mathbb{N}_0}$ is bounded. Indeed, if there exists $M \in \mathbb{R}$ such that for any $n \in \mathbb{N}_0$ we have $|f_n|_p \le M$ then

$$0 \le |f_n n_p|_p \le M|n_p|_p \quad \text{and} \quad \lim_{n\to\infty} |n_p|_p = 0\,,$$

so (E2) holds and the series $\sum_{n=0}^{\infty} f_n h_n$ is convergent.

The next theorem follows immediately from the above definition.

Theorem 2. *The set of all functions which have an expansion with respect to* $(h_n)_{n\in\mathbb{N}_0}$ *is a vector space over* $\mathbb{C}_p$. ■

The following result describes a class of functions which have an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$.

THEOREM 3. *If there exist constants* $d_0, d_1, \ldots \in \mathbb{C}_p$ *such that* $f = \sum_{m=0}^{\infty} d_m h_m$ *then* f *has an expansion with respect to* $(h_n)_{n\in\mathbb{N}_0}$ *and* $f \sim \sum_{m=0}^{\infty} d_m h_m$.

Proof. It is sufficient to compute $f_n^{(k)}$ where $k, n \in \mathbb{N}_0$, and to show that $\lim_{k\to\infty} f_n^{(k)} = d_n$. For $n = 0$ one has

$$f_0^{(k)} = p^{-k}\sum_{j=0}^{p^k-1} d_0 + \sum_{m=1}^{\infty} d_m\Big(p^{-k}\sum_{j=0}^{p^k-1} h_m(j)\overline{h}_0(j)\Big) = d_0\,,$$

by virtue of (2) and the proof of Theorem 1(d). For $n > 0$, consider the sum

$$S = \sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1} \zeta^{-n_+ s} h_m(jpn_p + sn_p + n_-)\,.$$

Using the definition of $(h_n)_{n\in\mathbb{N}_0}$ and the properties of roots of unity one obtains $S = 0$ if $n \neq m$ and $S = p^k$ if $n = m$.

Finally, one has

$$\begin{aligned} f_n^{(k)} = \sum_{m\neq n} d_m\Big(p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1} \zeta^{-n_+ s} h_m(jpn_p + sn_p + n_-)\Big) \\ + d_n p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1} \zeta^{-n_+ s} h_n(jpn_p + sn_p + n_-) = d_n\,. \end{aligned}$$

Now one can see that $\lim_{k\to\infty} f_n^{(k)} = d_n$ for $n \in \mathbb{N}_0$, so (E1) holds. By convergence of $\sum_{m=0}^{\infty} d_m h_m$, (E2) also holds. ∎

From the above theorem one can deduce the following two corollaries:

COROLLARY 4. *If* $f = \sum_{m=0}^{\infty} d_m h_m = \sum_{m=0}^{\infty} d'_m h_m$ *then* $d_m = d'_m$. ∎

COROLLARY 5. *If the expansions of* f, $g \in C(\mathbb{Z}_p, \mathbb{C}_p)$ *with respect to* $(h_n)_{n\in\mathbb{N}_0}$ *are convergent to those functions*, *then* fg *has an expansion with respect to* $(h_n)_{n\in\mathbb{N}_0}$.

Proof. Let $f = \sum_{n=0}^{\infty} f_n h_n$ and $g = \sum_{n=0}^{\infty} g_n h_n$. These series are absolutely convergent so their product is also absolutely convergent. Hence one may change the order of its terms. Because the product $h_n h_m$ is again h_s or λh_s (for some $s \in \mathbb{N}_0$, $\lambda \in \mathbb{C}_p$) one can obtain the series $fg = \sum_{s=0}^{\infty} d_s h_s$ as a product $(\sum_{n=0}^{\infty} f_n h_n)(\sum_{n=0}^{\infty} g_n h_n)$. To finish the proof it is enough to apply Theorem 3. ∎

Now we shall give a few examples of expansions with respect to $(h_n)_{n\in\mathbb{N}_0}$.

EXAMPLES OF EXPANSIONS. (a) *Identity on* $\mathbb{Z}_p$ ($f(x)=x$). Employing formulas (2) and Definition 1 one obtains

$$f_0 = \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^k-1} j = \lim_{k\to\infty}(p^{-k}2^{-1}(p^k-1)p^k) = -2^{-1},$$

$$\begin{aligned} f_n &= \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}(jpn_p+sn_p+n_-)\\ &= p^{-1}\sum_{s=0}^{p-1}s\zeta^{-n_+s} \quad \text{for } n>0\,. \end{aligned}$$

Note that if $p=2$ one gets $f_n=-2^{-1}$ for all $n\in\mathbb{N}_0$.

(b) *Quadratic function* ($f(x)=x^2$). It follows by direct computation that

$$f_0 = \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^k-1} j^2 = \lim_{k\to\infty}(p^{-k}6^{-1}(p^k-1)p^k(2p^k-1)) = 6^{-1},$$

$$\begin{aligned} f_n &= \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}(jpn_p+sn_p+n_-)^2\\ &= \lim_{k\to\infty}\sum_{s=0}^{p-1}s\zeta^{-n_+s}(p^{-1}n_ps+p^{-1}n_-+n_p(p^{k-1}-1))\\ &= \sum_{s=0}^{p-1}s\zeta^{-n_+s}(p^{-1}n_ps+p^{-1}n_--n_p) \quad \text{for } n>0\,. \end{aligned}$$

(c) *Exponential function* ($f(x)=\exp(ax)$ where $|a|_p<p^{1/(p-1)}$). In this case, using the properties of the function exp, one gets

$$f_0 = \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^k-1}\exp(aj) = a(\exp(a)-1)^{-1},$$

$$\begin{aligned} f_n &= \lim_{k\to\infty}\Big(p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}\exp(ajpn_p)\exp(asn_p)\exp(an_-)\Big)\\ &= \lim_{k\to\infty}\Big(p^{-k}\exp(an_-)(\exp(ap^k)-1)(\exp(apn_p)-1)^{-1}\\ &\qquad\qquad \times\sum_{s=0}^{p-1}\zeta^{-n_+s}\exp(asn_p)\Big) \end{aligned}$$

$$= a\exp(an_-)(\exp(apn_p)-1)^{-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}\exp(asn_p) \quad \text{for } n>0\,.$$

(d) *Trigonometric functions* ($f(x)=\sin(ax)$, $g(x)=\cos(ax)$, where $|a|_p<p^{1/(p-1)}$). Applying well-known formulas one obtains

$$\begin{aligned} f_0 &= \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^k-1}\sin(aj) \\ &= \lim_{k\to\infty}(p^{-k}(\sin(2^{-1}a))^{-1}\sin(2^{-1}a(p^k-1))\sin(2^{-1}ap^k)) = -2^{-1}a\,, \\ f_n &= \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}\sin(ajpn_p+asn_p+an_-) \\ &= \lim_{k\to\infty}\Big(p^{-k}(\sin(2^{-1}apn_p))^{-1}\sin(2^{-1}apn_p(p^{k-1}n_p^{-1}-1)) \\ &\qquad\qquad \times\sin(2^{-1}ap^k)\sum_{s=0}^{p-1}\zeta^{-n_+s}\cos(asn_p+an_-) \\ &\quad + p^{-k}(\sin(2^{-1}apn_p))^{-1}\cos(2^{-1}apn_p(p^{k-1}n_p^{-1}-1)) \\ &\qquad\qquad \times\sin(2^{-1}ap^k)\sum_{s=0}^{p-1}\zeta^{-n_+s}\sin(asn_p+an_-)\Big) \\ &= -\,2^{-1}a\Big(\sum_{s=0}^{p-1}\zeta^{-n_+s}\cos(asn_p+an_-) \\ &\quad +\,(\tan(2^{-1}apn_p))^{-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}\sin(asn_p+an_-)\Big) \quad \text{for } n>0\,. \end{aligned}$$

Reasoning in the same way one gets $g_0=2^{-1}a\cdot\tan(2^{-1}a)$ and

$$\begin{aligned} g_n = 2^{-1}a\Big((\tan(2^{-1}apn_p))^{-1}&\sum_{s=0}^{p-1}\zeta^{-n_+s}\cos(asn_p+an_-) \\ &+\sum_{s=0}^{p-1}\zeta^{-n_+s}\sin(asn_p+an_-)\Big) \quad \text{for } n>0\,. \end{aligned}$$

(e) *Characteristic function of a coset of the residue class field.* Let $A=t+p^r\mathbb{Z}_p$, where $0\le t\le p^{r-1}$. Then

$$\chi_A(x)=\begin{cases}1, & x\in A\,,\\ 0, & x\notin A\,.\end{cases}$$

Without any difficulty one obtains

$$(\chi_A)_0 = \lim_{k\to\infty} p^{-k} \sum_{j=0}^{p^k-1} \chi_A(j) = \lim_{k\to\infty} p^{-k} \sum_{i=0}^{p^{k-r}-1} \chi_A(t+ip^r) = p^{-r}\,.$$

For $n > 0$ assume $n = n_0 + n_1 p + \ldots + n_a p^a$, $t = t_0 + t_1 p + \ldots + t_{r-1}p^{r-1}$ and consider two cases $a < r$ and $a \ge r$. In the first case one gets

$$\begin{aligned}(\chi_A)_n &= \lim_{k\to\infty} p^{-k} \sum_{j=0}^{p^{k-a-1}-1} \sum_{s=0}^{p-1} \zeta^{-n_a s}\chi_A(jp^{a+1} + sp^a + n_-) \\ &= \begin{cases} 0 & \text{if } n_- \not\equiv t \pmod{p^a}, \\ p^{-r}\zeta^{-n_a t_a} & \text{if } n_- \equiv t \pmod{p^a}. \end{cases}\end{aligned}$$

Considering the second case, note that if $n_- \not\equiv t \pmod{p^r}$ then $(\chi_A)_n = 0$. Otherwise one obtains

$$\begin{aligned}(\chi_A)_n &= \lim_{k\to\infty} p^{-k} \sum_{j=0}^{p^{k-a-1}-1} \sum_{s=0}^{p-1} \zeta^{-n_+ s}\chi_A(jp^{a+1} + sp^a + n_-) \\ &= p^{-a-1} \sum_{s=0}^{p-1} \zeta^{-n_+ s} = 0\,.\end{aligned}$$

Relationship between $(h_n)_{n\in\mathbb{N}_0}$ and $(\phi_m)_{m\in\mathbb{N}_0}$. The aim of this section is to show that f has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$ if it has one with respect to the system $(\phi_m)_{m\in\mathbb{N}_0}$ defined by Rutkowski (see [3]), and to give an example of a function which has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$ but not with respect to $(\phi_m)_{m\in\mathbb{N}_0}$. First recall the definition and basic properties of $(\phi_m)_{m\in\mathbb{N}_0}$. For $m = m_0 + m_1 p + \ldots + m_s p^s \in \mathbb{N}$ define

$$\begin{aligned}&\phi_m(x) = \phi_m(x_0 + x_1 p + \ldots + x_s p^s + \ldots) = \zeta^{x_0 m_0 + x_1 m_1 + \ldots + x_s m_s}\,, \\ &\phi_0(x) \equiv 1\,.\end{aligned}$$

It follows immediately that

$$\begin{aligned}&\phi_m(x_0 + x_1 p + \ldots + x_r p^r + x_{r+1}p^{r+1} + \ldots) \\ &\qquad = \phi_m(x_0 + x_1 p + \ldots + x_r p^r)\phi_m(x_{r+1}p^{r+1} + \ldots)\,.\end{aligned}$$

The system $(\phi_m)_{m\in\mathbb{N}_0}$ is orthonormal in the sense of the definition given before Theorem 1. The functions $\phi_0, \phi_1, \ldots, \phi_{p^k-1}$ form a basis in the vector space V_k (see Theorem 1(e)). For $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$, elements of its standard sequence can be represented in the form

$$f^{(k)} = \sum_{m=0}^{p^k-1} \Big(p^{-k} \sum_{j=0}^{p^k-1} f(j)\overline{\phi}_m(j)\Big)\phi_m, \qquad \text{where } \overline{\phi}_m(j) = \phi_m(j)^{-1}\,.$$

Define $\widehat{f}_m^{(k)} = p^{-k}\sum_{j=0}^{p^k-1} f(j)\overline{\phi}_m(j)$. Making use of the above notations we introduce

DEFINITION 2. A function $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ *has an expansion with respect to the system* $(\phi_m)_{m\in\mathbb{N}_0}$ if the following holds:

(I) for any $m \in \mathbb{N}_0$ the limit $\widehat{f}_m = \lim_{k\to\infty} \widehat{f}_m^{(k)}$ exists;

(II) $\lim_{m\to\infty} \widehat{f}_m = 0$.

Note that (II) guarantees the convergence of the series $\sum_{m=0}^{\infty} \widehat{f}_m\phi_m$, called the *expansion of* f *with respect to* $(\phi_m)_{m\in\mathbb{N}_0}$. We write $f \sim \sum_{m=0}^{\infty} \widehat{f}_m\phi_m$. Now we prove the main theorem of this section.

THEOREM 6. *A function f has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$ if it has one with respect to $(\phi_m)_{m\in\mathbb{N}_0}$.*

Proof. First we transform the formulas for the coefficients $f_n^{(k)}$ in the expansion of f with respect to $(h_n)_{n\in\mathbb{N}_0}$. For $n > 0$ and k large enough,

$$\begin{aligned} f_n^{(k)} &= p^{-k} \sum_{j=0}^{p^{k-1}n_p^{-1}-1} \sum_{s=0}^{p-1} \zeta^{-n_+ s} f(jpn_p + sn_p + n_-) \\ &= p^{-k} \sum_{i=0}^{p^k n_p^{-1}-1} f(in_p + n_-)\overline{h}_n(in_p + n_-)n_p \\ &= n_p p^{-k} \sum_{i=0}^{p^k n_p^{-1}-1} \sum_{r=0}^{n_p-1} f(in_p + r)\overline{h}_n(in_p + r) \\ &= n_p p^{-k} \sum_{j=0}^{p^k-1} f(j)\overline{h}_n(j)\,. \end{aligned} \tag{3}$$

One can check that

$$\begin{aligned} &\overline{h}_n = n_p^{-2} \sum_{r=0}^{n_p-1} \overline{\phi}_r(n_-)\phi_{m_+n_p+r} \\ &\qquad\qquad (\text{where } -n_+ \equiv m_+ \ (\operatorname{mod} p) \text{ and } n > 0)\,, \\ &\overline{h}_0 \equiv \overline{\phi}_0\,. \end{aligned} \tag{4}$$

Applying (4) to (3) one obtains

$$f_n^{(k)} = n_p^{-1} \sum_{r=0}^{n_p-1} \overline{\phi}_r(n_-) p^{-k} \sum_{j=0}^{p^k-1} f(j) \overline{\phi}_{n_+ n_p + r}(j)$$
$$= n_p^{-1} \sum_{r=0}^{n_p-1} \overline{\phi}_r(n_-) \widehat{f}^{(k)}_{n_+ n_p + r}\,.$$

The limit $f_n = \lim_{k\to\infty} f_n^{(k)}$ exists because $\widehat{f}_{n_+ n_p + r} = \lim_{k\to\infty} \widehat{f}^{(k)}_{n_+ n_p + r}$ exists by Definition 2 and $f_n = n_p^{-1} \sum_{r=0}^{n_p-1} \overline{\phi}_r(n_-) \widehat{f}_{n_+ n_p + r}$, so condition (E1) of Definition 1 is satisfied. Now,

$$|n_p f_n|_p = \Big| \sum_{r=0}^{n_p-1} \overline{\phi}_r(n_-) \widehat{f}_{n_+ n_p + r} \Big|_p \le \max\{|\widehat{f}_{n_+ n_p + r}|_p : 0 \le r \le n_p\}\,.$$

But $\lim_{n\to\infty} \widehat{f}_n = 0$ so $\max\{|\widehat{f}_{n_+ n_p + r}|_p : 0 \le r \le n_p\} \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} n_p f_n = 0$ and condition (E2) of Definition 1 is also satisfied. ∎

Applying the above theorem and the result proved in [3], we immediately obtain the following

COROLLARY 7. (a) *There exists a function $f \in C(\mathbb{Z}_p, \mathbb{C}_p)$ which has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$ and $f \ne 0$, $f \sim 0$.*

(b) *Every uniformly differentiable function has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$.*

(c) *There exists a differentiable function which does not have an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$.*

Now we will show that the system $(h_n)_{n\in\mathbb{N}_0}$ is more general than $(\phi_m)_{m\in\mathbb{N}_0}$.

EXAMPLE. Consider the function $f : \mathbb{Z}_p \to \mathbb{C}_p$ given by $f(0) = 0$ and $f(x_a p^a + x_{a+1} p^{a+1} + \ldots) = p^{a+1} \zeta^{a+1}$, where x_a is non-zero. One can check that f is continuous. We shall show that f has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$, but the sequence $(\widehat{f}_{p^s})_{s\in\mathbb{N}}$ is convergent to $2(p-1)$ so statement (II) of Definition 2 fails. We first prove the following facts:

(i) for $s \in \mathbb{N}$, $x \in \mathbb{Z}_p$ one has $f(p^s x) = p^s f(x)$;

(ii) $f(x_a p^a + x_{a+1} p^{a+1} + \ldots + x_{a+r} p^{a+r} + \ldots) = f(x_a p^a + x_{a+1} p^{a+1} + \ldots + x_{a+r} p^{a+r})$, where x_a is non-zero and $r \ge 1$;

(iii) $\sum_{j=0}^{p^k-1} f(\alpha p^s + j p^{s+1}) = 0$ for $s \in \mathbb{N}_0$, $k \in \mathbb{N}$, $\alpha \in E\backslash\{0\}$;

(iv) $\sum_{j=0}^{p^k-1} f(j) = (p-1)p^k$ for $k \in \mathbb{N}$.

The properties (i), (ii) are easy to verify and we get (iii) immediately by direct computations:

$$\sum_{j=0}^{p^k-1} f(\alpha p^s + jp^{s+1}) = \sum_{i=0}^{p-1}\sum_{j=0}^{p^{k-1}-1} f(\alpha p^s + ip^{s+1} + jp^{s+2})$$
$$= p^{k-1}p^{s+1}\sum_{i=0}^{p-1}\zeta^i = 0\,.$$

To prove (iv) write

$$\sum_{j=0}^{p^k-1} f(j) = \sum_{j_0=1}^{p-1}\sum_{i=0}^{p^{k-1}-1} f(j_0+ip) + \sum_{j_1=1}^{p-1}\sum_{i=0}^{p^{k-1}-1} f(j_1p+ip^2)+\ldots$$
$$+\sum_{j_{k-2}=1}^{p-1}\sum_{i=0}^{p^{k-1}-1} f(j_{k-2}p^{k-2}+ip^{k-1}) + \sum_{i=1}^{p-1} f(ip^{k-1}) + f(0)\,.$$

The last two terms are 0 and $(p-1)p^k$ respectively by definition of f while the others are zero by (iii).

Now we are ready to compute the coefficients f_n. Using (iv), (2) and Definition 1, one obtains $f_0 = p-1$. For $n>0$ we consider three cases: 1° $n_- = 0$, 2° $n_- = an_pp^{-1}$ (where $a \in E\backslash\{0\}$) and 3° $n_- \neq 0$ or $n_- \neq an_pp^{-1}$. In the first case one gets

$$f_n = \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}f(jpn_p+sn_p)$$
$$=\lim_{k\to\infty} p^{-k}n_p\Big(\sum_{s=1}^{p-1}\zeta^{-n_+s}\sum_{j=0}^{p^{k-1}n_p^{-1}-1} f(s+jp) + p\sum_{j=0}^{p^{k-1}n_p^{-1}-1} f(j)\Big)\,.$$

Here we have used (i). Applying (iii) and (iv) one can check that $f_n = p-1$. Consider the second case:

$$f_n = \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^{k-1}n_p^{-1}-1}\sum_{s=0}^{p-1}\zeta^{-n_+s}f(jpn_p+sn_p+an_pp^{-1})$$
$$=\lim_{k\to\infty} p^{-k}n_pp^{-1}p^{k-1}n_p^{-1}\sum_{s=0}^{p-1}p\zeta^{(1-n_+)s}\,.$$

Here if $n_+ = 1$ then $f_n = 1$ and otherwise $f_n = 0$. Finally, if neither the first nor the second case holds then using (ii) one has

$$f_n = \lim_{k\to\infty} p^{-k} \sum_{j=0}^{p^{k-1}n_p^{-1}-1} \sum_{s=0}^{p-1} \zeta^{-n_+ s} f(jpn_p + sn_p + n_-)$$
$$= \lim_{k\to\infty} p^{-k} p^{k-1} n_p^{-1} f(n_-) \sum_{s=0}^{p-1} \zeta^{-n_+ s} = 0\,.$$

Since $f_n \in \mathbb{Z}_p$ for all $n \in \mathbb{N}_0$ the function f has an expansion with respect to $(h_n)_{n\in\mathbb{N}_0}$ by the remark after Definition 1.

Let us compute the coefficients $\widehat{f}_{p^s}$ (where $s \in \mathbb{N}$):

$$\widehat{f}_{p^s} = \lim_{k\to\infty} p^{-k} \sum_{j=0}^{p^k-1} f(j)\overline{\phi}_{p^s}(j)$$
$$= \lim_{k\to\infty} p^{-k}$$
$$\times \sum_{i=0}^{p^{s-1}-1} \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \sum_{j=0}^{p^{k-s-2}-1} f(i + ap^{s-1} + bp^s + cp^{s+1} + jp^{s+2})\zeta^{-b}$$
$$= \lim_{k\to\infty} p^{-k} \Big(p^{k-s-1} \sum_{i=1}^{p^{s-1}-1} \sum_{a=0}^{p-1} f(i + ap^{s-1}) \sum_{b=0}^{p-1} \zeta^{-b}$$
$$+ p^{k-s-1} \sum_{a=1}^{p-1}\sum_{b=0}^{p-1} f(ap^{s-1} + bp^s)\zeta^{-b}$$
$$+ p^{k-s-2} \sum_{b=1}^{p-1}\sum_{c=0}^{p-1} f(bp^s + cp^{s+1})\zeta^{-b}$$
$$+ \sum_{c=1}^{p-1} \sum_{j=0}^{p^{k-s-2}-1} f(cp^{s+1} + jp^{s+2}) + \sum_{j=0}^{p^{k-s-2}-1} f(jp^{s+2}) \Big)\,.$$

The first sum is zero. Applying (iii) one finds that the third and fourth sums are also zero. Using (i) and (iv) one shows that the fifth sum equals $p^k(p-1)$. Finally, applying the definition of f, one concludes that the second sum is $p^k(p-1)$. So $\widehat{f}_{p^s} = 2(p-1)$.

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