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# Systems of linear forms and covers for star bodies

by

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In this paper Diophantine approximation involving general distance and error functions will be considered for systems of linear forms. *Distance functions* will be denoted by $F : \mathbb{R}^n \to \mathbb{R}$ and are continuous, non-negative and satisfy

$$F(u\boldsymbol{x}) = uF(\boldsymbol{x})$$

for all $u \ge 0$ ([3], p. 103). An *error function* $\psi : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ is positive and satisfies $\psi(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$, where $|\boldsymbol{x}| = \max\{|x_1|, \ldots, |x_n|\}$. An upper bound depending on a cover for a star body associated with the distance function $F$ (see [8]) and a restriction $\psi_1$ of the error function $\psi$ for the Hausdorff dimension of sets of linear forms (regarded as points in $\mathbb{R}^{mn}$) is obtained. The Hausdorff dimension is also expressed in terms of the Hausdorff dimension of sets of vectors (in $\mathbb{R}^n$) and of the lower order (defined below) of the error function $\psi_1$ (corresponding to the case $m = 1$) when the functions $\psi$ and $\psi_1$ satisfy suitable decay conditions. This combines and extends the results of [4], [5] and [13]. Definitions and properties of Hausdorff dimension are given in [7].

First some notation and definitions are needed. Throughout $m, n$ and $N$ will be positive integers and $\boldsymbol{q} \in \mathbb{Z}^m$. The system

$$a_1 x_{1j} + \ldots + a_m x_{mj}, \quad 1 \le j \le n,$$

of $n$ real linear forms in $m$ real variables $a_1, \ldots, a_m$ will be written more concisely as $\boldsymbol{a}X$ where $\boldsymbol{a} \in \mathbb{R}^m$ and where $X$ is an $m \times n$ real matrix. The set of real $m \times n$ matrices will be identified with $\mathbb{R}^{mn}$ in the usual way (i.e., by identifying the matrix $X = (x_{ij})$ with the point $(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{mn})$ in $\mathbb{R}^{mn}$). For each $\boldsymbol{x}$ in $\mathbb{R}^n$, denote the symmetrised fractional part of $\boldsymbol{x}$ by $\langle \boldsymbol{x} \rangle = \boldsymbol{x} - \boldsymbol{k}_{\mathsf{x}}$, where $\boldsymbol{k}_{\mathsf{x}}$ is the unique integer vector such that $\boldsymbol{x} - \boldsymbol{k}_{\mathsf{x}} \in (-\frac{1}{2}, \frac{1}{2}]^n$. Thus

$$|\langle \boldsymbol{x} \rangle| = \max\{\|x_1\|, \ldots, \|x_n\|\},$$

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where for each real $u$, $\|u\| = \min\{|u-k| : k \in \mathbb{Z}\}$. Given an error function $\psi : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^+$, define $\psi_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}^+$ by

$$\psi_1(u) = \psi(u, 0, \ldots, 0).$$

Thus the function $\psi_1$ is essentially the restriction of $\psi$ to $(\mathbb{R} \setminus \{0\}) \times \{0\} \times \ldots \times \{0\}$, so that $\psi_1$ is positive and $\psi_1(x) \to 0$ as $x \to \infty$. Functions $\Delta : [0, \infty) \to (0, \infty)$, where $\Delta(0) = 1$ and $\Delta(x) \to \infty$ monotonically as $x \to \infty$, which satisfy the growth conditions

$$\frac{\log \Delta(x)}{x} \to 0 \text{ as } x \to \infty \quad \text{and} \quad \int_1^\infty \frac{\log \Delta(x)}{x^2}\, dx < \infty,$$

characterise a large class of functions arising in small denominator problems and are called approximation functions ([11], p. 95). In the applications considered below, $\psi_1(x) = x^{-\tau}$; its reciprocal $1/\psi_1$ is essentially an approximation function in the sense of [11] (strictly speaking $(x+1)^\tau$ is an approximation function). The *lower order* $\lambda = \lambda(f)$ of the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\lambda = \liminf_{x \to \infty} \frac{\log f(x)}{\log x}$$

([9], p. 16), so that given $\varepsilon > 0$, $f(x) \geq x^{\lambda - \varepsilon}$ for all sufficiently large $x$.

Suppose that for each positive $r$, $\mathcal{C}_r$ is a collection of hypercubes $C$ in $\mathbb{R}^n$ with $\ell(C) \ll R$ (as usual $\ll$ denotes an inequality with an unspecified positive constant factor). The $s$-volume $\ell^s(\mathcal{C})$, where $s \geq 0$, of a collection $\mathcal{C}$ of hypercubes $C$ is given by

$$\ell^s(\mathcal{C}) = \sum_{C \in \mathcal{C}} \ell(C)^s$$

(see [7]). For each $s$ in $[0, n]$, the (*upper*) *order* $\omega(s)$ of the $s$-volume $\ell^s(\mathcal{C}_r)$ of $\mathcal{C}_r$ will be defined as

$$\omega(s) = \liminf_{r \to 0+} \frac{\log \ell^s(\mathcal{C}_r)}{\log r},$$

so that given $\varepsilon > 0$, $\ell^s(\mathcal{C}_r) \leq r^{\omega(s) - \varepsilon}$ for all sufficiently small $r$. It is shown in Lemma 1 of [5] that $\omega(s)$ increases with $s$ and is concave (i.e., $\omega$ satisfies $\omega(ta + (1-t)b) \geq t\omega(a) + (1-t)\omega(b)$ for each $a, b$ in $(0, n)$ and $t$ in $[0, 1]$) and hence continuous on $(0, n)$.

The Hausdorff dimension $\dim W(F, \psi; m, n)$ of the set

$$W(F, \psi; m, n) = \{X \in \mathbb{R}^{mn} : F(\langle \boldsymbol{q} X \rangle) < \psi(\boldsymbol{q}) \text{ for infinitely many } \boldsymbol{q} \in \mathbb{Z}^m\}$$

will be estimated in terms of the lower order $\lambda_1$ of $1/\psi_1$ and the lower order $\omega(s)$ of the $s$-volume of a cover for the star body

$$S_r = F^{-1}([0, r)) \cap I^n = \{\boldsymbol{x} \in I^n : F(\boldsymbol{x}) < r\},$$

where $r > 0$ and $I = [-\frac{1}{2}, \frac{1}{2}]$. It will then be determined in terms of the dimension of $W(F, \psi_1; 1, n)$. This extends the simultaneous Diophantine approximation results of [5] to systems of linear forms and generalizes the results of [4] and [13]. It will also be used to determine the Hausdorff dimension of the set of real $m \times n$ matrices $X$ for which the inequality

$$\prod_{j=1}^{n} \Big| \Big\langle \sum_{i=1}^{m} q_i x_{ij} \Big\rangle \Big|^{\alpha_j/A} < \prod_{i=1}^{m} (\overline{q}_i)^{-\tau},$$

where $\alpha_1, \ldots, \alpha_n > 0$, $A = \alpha_1 + \ldots + \alpha_n$ and $\overline{q} = \max\{1, |q|\}$, holds for infinitely many $\boldsymbol{q} \in \mathbb{Z}^m$ (in [13], $\alpha_j = 1$ for each $j = 1, \ldots, n$).

Since

$$\dim\{A_1 \cup A_2 \cup \ldots\} = \sup\{\dim A_k : k = 1, 2, \ldots\}, \tag{1}$$

in order to determine the Hausdorff dimension of $W(F, \psi; m, n)$ it suffices to consider the dimension of the set

$$\begin{aligned} \widetilde{W}(F, \psi; m, n) &= \{X \in I^{mn} : F(\langle \boldsymbol{q}X \rangle) < \psi(\boldsymbol{q}) \text{ for infinitely many } \boldsymbol{q} \in \mathbb{Z}^m\} \\ &= W(F, \psi; m, n) \cap I^{mn}. \end{aligned}$$

THEOREM 1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a distance function and suppose that for each $r > 0$, the starbody $F^{-1}([0, r)) \cap I^n$ has a cover $\mathcal{C}_r$ of $n$-dimensional hypercubes $C$ with sides of length $\ell(C) \ll r$. Let $\omega(s)$ be the order of the $s$-volume $\ell^s(\mathcal{C}_r)$. Let $\psi : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be an error function and suppose that the reciprocal $1/\psi_1$ of $\psi_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}^+$ has lower order $\lambda_1$ satisfying*

$$\lim_{s \to 0+} \frac{\omega(s)}{n+1} < \frac{1}{\lambda_1} < \lim_{s \to n-} \omega(s).$$

*Suppose further that for every $v > \omega(s_0)$,*

$$\sum_{|\mathsf{q}| = q} \psi(\boldsymbol{q})^v \ll \psi_1(q)^v. \tag{2}$$

*Then*

$$\dim W(F, \psi; m, n) \le (m-1)n + s_0 = mn + 1 - \lambda_1 \omega(s_0),$$

*where $s_0 \in (0, n)$ is the unique solution of*

$$\lambda_1 \omega(s) = n - s + 1.$$

Proof. Choose $t$ such that $(m-1)n + s_0 < t < mn$ and put $s = t - (m-1)n$ so that $s_0 < s < n$. The number $\eta = \lambda_1 \omega(s) - n + s - 1 > 0$ since $\omega(s)$ increases with $s$ and $s > s_0$. Let $\varepsilon$ be an arbitrary number satisfying

$$0 < \varepsilon < \eta / 2(\lambda_1 + \omega(s)).$$

The set $\widetilde{W}(F,\psi;m,n)$ can be expressed in the 'lim-sup' form

$$\widetilde{W}(F,\psi;m,n)=\bigcap_{N=1}^{\infty}\bigcup_{q=N}^{\infty}\bigcup_{|\mathsf{q}|=q}\bigcup_{\mathsf{p}}B(\boldsymbol{p},\boldsymbol{q};\psi(\boldsymbol{q}))\,,$$

where $\boldsymbol{p}$ runs through all integer vectors in $\boldsymbol{q}(-\frac{1}{2},\frac{1}{2}]^{mn}$ (i.e., $\boldsymbol{p}=\boldsymbol{q}X$ for some $X$ in $(-\frac{1}{2},\frac{1}{2}]^{mn}$) and for each $\boldsymbol{p}\in\mathbb{Z}^n$, $\boldsymbol{q}\in\mathbb{Z}^m$ and $r>0$,

$$B(\boldsymbol{p},\boldsymbol{q};r)=\{X\in I^{mn}:F(\boldsymbol{q}X-\boldsymbol{p})<r,\boldsymbol{q}X-\boldsymbol{p}\in(-\tfrac{1}{2},\tfrac{1}{2}]^{n}\}\,.$$

Now $B(\boldsymbol{p},q;r)=\{\boldsymbol{x}\in I^n:F(q\boldsymbol{x}-\boldsymbol{p})<r,q\boldsymbol{x}-\boldsymbol{p}\in(-\frac{1}{2},\frac{1}{2}]^n\}$, where $q>0$, is a starbody centred at $\boldsymbol{p}/q$ and is contained in $S_r/q+\boldsymbol{p}/q$, since if $y\in qB(\boldsymbol{p},q;r)-\boldsymbol{p}$, then $F(\boldsymbol{y})=F(q\boldsymbol{x}-\boldsymbol{p})<r$. The cover $\mathcal{C}_r$ for $S_r$ gives a cover $\mathcal{B}(\boldsymbol{p},q;r)$ for $B(\boldsymbol{p},q;r)$ consisting of $n$-dimensional hypercubes $C$ with $\ell(C)\ll r/q$ and, by the definition of $\omega(s)$, with $s$-volume $\ell^s(\mathcal{B}(\boldsymbol{p},q;r))$ satisfying

$$\ell^s(\mathcal{B}(\boldsymbol{p},q;r))\ll\ell^s(\mathcal{C}_r)q^{-s}\ll r^{\omega(s)-\varepsilon}q^{-s}\tag{3}$$

for any $\varepsilon>0$ and sufficiently small $r$. By Lemma 2 of [4] or by [13] there is a cover $\mathcal{H}(\boldsymbol{p},\boldsymbol{q};r)$ say of $B(\boldsymbol{p},\boldsymbol{q};r)$ by $mn$-dimensional hypercubes $H$ with $\ell(H)\ll r/|\boldsymbol{q}|$ and $t$-volume

$$\ell^t(\mathcal{H}(\boldsymbol{p},\boldsymbol{q};r))\ll\ell^s(\mathcal{B}(\boldsymbol{p},|\boldsymbol{q}|;r))\,.$$

Hence for each $N=1,2,\ldots,$ $\widetilde{W}(F,\psi;m,n)$ has a cover $\Lambda_N$ of hypercubes $C$ with $\ell(C)\ll 1/N$ and with $t$-volume $\ell^t(\Lambda_N)$ satisfying

$$\begin{aligned}\ell^s(\Lambda_N)&\ll\sum_{q=N}^{\infty}\sum_{|\mathsf{q}|=q}\sum_{|\mathsf{p}|\le|\mathsf{q}|}\ell^t(\mathcal{H}(\boldsymbol{p},\boldsymbol{q};\psi(\boldsymbol{q})))\\&\ll\sum_{q=N}^{\infty}\sum_{|\mathsf{q}|=q}\sum_{|\mathsf{p}|\le|\mathsf{q}|}\ell^s(\mathcal{B}(\boldsymbol{p},|\boldsymbol{q}|;\psi(\boldsymbol{q})))\\&\ll\sum_{q=N}^{\infty}\sum_{|\mathsf{q}|=q}\sum_{|\mathsf{p}|\le|\mathsf{q}|}|\boldsymbol{q}|^{-s}\psi(\boldsymbol{q})^{\omega(s)-\varepsilon}\end{aligned}$$

by (3) with $r=\psi(\boldsymbol{q})$ and $q=|\boldsymbol{q}|$. Hence for all $t>(m-1)n+s_0$ (or $s>s_0$) and $N$ sufficiently large,

$$\begin{aligned}\ell^t(\Lambda_N)&\ll\sum_{q=N}^{\infty}q^{-s}\sum_{|\mathsf{q}|=q}\psi(\boldsymbol{q})^{\omega(s)-\varepsilon}\sum_{|\mathsf{p}|\le|\mathsf{q}|}1\\&\ll\sum_{q=N}^{\infty}q^{n-s}\sum_{|\mathsf{q}|=q}\psi(\boldsymbol{q})^{\omega(s)-\varepsilon}\ll\sum_{q=N}^{\infty}q^{n-s}\psi_1(q)^{\omega(s)-\varepsilon}\end{aligned}$$

by (2). Hence by the definition of the lower order $\lambda_1$ of $1/\psi_1$, for sufficiently large $N$,

$$\ell^t(\Lambda_N) \ll \sum_{q=N}^{\infty} q^{n-s} q^{-(\lambda_1-\varepsilon)(\omega(s)-\varepsilon)}$$
$$\ll \sum_{q=N}^{\infty} q^{n-s-\lambda_1\omega(s)-\varepsilon(\lambda_1+\omega(s))} \ll \sum_{q=N}^{\infty} q^{-1-\eta/2}$$

by the choice of $\eta$ and $\varepsilon$. Thus $\ell^t(\Lambda_N) \to 0$ as $N \to \infty$ and it follows that $\dim W(F,\psi;m,n) \le (m-1)n + s_0$.

By using the above result and the following general lower bound, the Hausdorff dimension of $W(F,\psi;m,n)$ can be expressed in terms of the dimension of $W(F,\psi_1;1,n)$. To do this a slight additional restriction on the order $\omega$ is needed. It suffices to assume that $\omega(s)$ is strictly increasing, or that in the cover $\mathcal{C}_r$, $\ell(C) \gg r$, or that since $\omega(s)$ is a convex function of $s$, for some $\delta > 0$, $\omega(s)$ is not constant for $s > s_0 - \delta$ or that $\ell(C)^s \ll r^{\omega(s)}$. In the applications made, each of these conditions is fulfilled.

LEMMA 1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a distance function and let $\psi : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be any function. Then*

$$\dim W(F,\psi;m,n) \ge (m-1)n + \dim W(F,\psi_1;1,n)\,.$$

Proof. [4], Lemma 1.

THEOREM 2. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a distance function and suppose that for each $r > 0$, the starbody $F^{-1}([0,1)) \cap I^n$ has a cover $\mathcal{C}_r$ of $n$-dimensional hypercubes $C$ with sides of length $\ell(C) \ll r$. Let $\omega(s)$ be the lower order of the $s$-volume $\ell^s(\mathcal{C}_r)$ and suppose that $\omega(s)$ is strictly increasing. Let $\psi : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be an error function and suppose that for $s > \dim W(F,\psi_1;1,n)$,*

$$\sum_{|\mathbf{q}|=q} \psi(q)^{\omega(s)} \ll \psi_1(q)^{\omega(s)} \tag{4}$$

*and*

$$\sum_{q=1}^{\infty} q^{n-s}\psi_1(q)^{\omega(s)} < \infty\,. \tag{5}$$

*Then*

$$\dim W(F,\psi;m,n) = (m-1)n + \dim W(F,\psi_1;1,n)\,.$$

Proof. The proof is similar to that of Theorem 1 except that the lower bound of the lemma coincides with the upper bound. Write $h = \dim W(F,\psi;m,n)$ and assume without loss of generality that $h < n$. Choose $t$ such that $(m-1)n+h < t < mn$ and put $s = t-(m-1)n$ so that $h < s < n$. Since $\omega(s)$ increases strictly with $s$, $\omega(s) - \omega(h) > 0$. Let $\varepsilon$ be an arbitrary number satisfying $0 < \varepsilon < \omega(s) - \omega(h)$.

As in Theorem 1, for each $N = 1, 2, \ldots$, $\widetilde{W}(F, \psi; m, n)$ has a cover $\Lambda_N$ consisting of hypercubes $C$ with $\ell(C) \ll 1/N$ and with $t$-volume satisfying

$$\ell^t(\Lambda_N) \ll \sum_{q=N}^{\infty} \sum_{|\mathsf{q}|=q} \sum_{|\mathsf{p}|\le|\mathsf{q}|} |\boldsymbol{q}|^{-s}\psi(\boldsymbol{q})^{\omega(s)-\varepsilon} .$$

Hence for all $t > (m-1)n + h$ (i.e., $s > h$) and $N$ sufficiently large,

$$\ell^t(\Lambda_N) \ll \sum_{q=N}^{\infty} q^{n-s}\psi_1(q)^{\omega(s)-\varepsilon} \ll \sum_{q=N}^{\infty} q^{n-s'}\psi_1(q)^{\omega(s')} ,$$

where $\omega(s') = \omega(s) - \varepsilon$, so that by the choice of $\varepsilon$ and since $\omega$ is strictly increasing, $h < s' < s$. Thus $\ell^t(\Lambda_N) \to 0$ as $N \to \infty$ by (5) and it follows that $\dim W(F, \psi; m, n) \le (m-1)n + h$.

The complementary inequality $\dim W(F, \psi; m, n) \ge (m-1)n + h$, and hence the theorem, follow from the lemma.

From now on take $\psi_1(x) = x^{-\tau}$ for $x, \tau > 0$, so that $\lambda_1 = \tau$, and write

$$W(F, \psi_1; 1, n) = W_F(\tau; n) .$$

It is shown in [5] that when $\omega(s)\tau > 1$, $\dim W_F(\tau; n) \le s_0$, where $\tau\omega(s_0) = n - s_0 + 1$. When $F(\boldsymbol{x}) = |\boldsymbol{x}|$ or more generally when $F$ is a gauge function (i.e., when $F^{-1}(0) = \{\mathbf{0}\}$, see [8], p. 8), then $\omega(s) = s$ and $\dim W_F(\tau; n) = (n+1)/(\tau+1)$ for $\tau > 1/n$. Since $(n+1)/(\tau+1)$ is a continuous decreasing function of $\tau$ which equals $n$ when $\tau = 1/n$, since $\dim W_F(\tau; n) \le n$ and since $\tau \ge \tau'$ implies that $W_F(\tau; n) \subseteq W_F(\tau'; n)$, it follows that

$$\dim W_F(\tau; n) = (n+1)/(\tau+1)$$

holds for $\tau \ge 1/n$. For if $\dim W_F(1/n; n) = \theta < n$, then for $\tau - 1/n > 0$ and sufficiently small, $W_F(\tau; n) > \theta$, a contradiction. The estimate (4) is essential for Theorem 2 and does not hold for the error function $\psi(\boldsymbol{x}) = |\boldsymbol{x}|^{-\tau}$ when $m \ge 2$; indeed, in this case the Hausdorff dimension is larger and $\dim W(F, \psi; m, n) = (m-1)n + (m+n)/(\tau+1)$ when $\tau \ge m/n$ [2].

Corollary 1. *Suppose that* $F : \mathbb{R}^n \to \mathbb{R}$ *is a gauge function and that* $\psi : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ *satisfies* $\psi_1(x) = \psi(x, 0, \ldots, 0) = x^{-\tau}$ *and*

$$\sum_{|\mathsf{q}|=q} \psi(\boldsymbol{q})^s \ll q^{-\tau s}$$

*for each* $s > (n+1)(\tau+1)$. *Then for* $\tau \ge 1/n$,

$$\dim W(F, \psi; m, n) = (m-1)n + \frac{n+1}{\tau+1} .$$

Proof. By [5], $\omega(s) = s$ and by [6] or [10] and by the above remarks on the continuity of the dimension, $\dim W_F(\tau; n) = (n+1)/(\tau+1)$ when

$\tau \geq 1/n$. The estimate (5) holds since when $s > (n+1)/(\tau+1)$,

$$\sum_{q=1}^{\infty} q^{n-s} q^{-\tau s} < \infty$$

and the corollary follows.

This result was obtained in [4] but the condition corresponding to (5) was omitted; this does not affect the applications made there. Note that since $\tau s_0 = n - s_0 + 1$ when $s_0 = (n+1)(\tau+1)$, it follows from Theorem 1 and the above remarks on the continuity of the dimension that $\dim W(F, \psi; m, n) \leq (m-1)n + (n+1)/(\tau+1)$ when $\tau \geq 1/n$.

The function $F$ given by $F(\boldsymbol{x}) = \prod_{j=1}^{n} |x_j|^{1/n}$ is evidently a distance function; $F$ is also of interest in the geometry of numbers ([8], Chapter 4) and Diophantine approximation ([12], p. 69). In [13], Yu extended [1] to systems of linear forms and showed (using different notation) that when $\psi(\boldsymbol{x}) = \prod_{i=1}^{m} (\overline{x}_i)^{-\tau}$, where $\tau > 1/n$ and where for each real $x$, $\overline{x} = \max\{|x|, 1\}$, then

$$\dim W(F, \psi; m, n) = mn - 1 + \frac{2}{\tau n + 1}\,.$$

Using Theorem 2, we now extend Yu's result.

Corollary 2. *Let $\alpha_1, \ldots, \alpha_n$ be positive numbers with $\alpha = \max\{\alpha_j : 1 \leq j \leq n\}$ and $A = \alpha_1 + \ldots + \alpha_n$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$\Phi(\boldsymbol{x})^A = \prod_{j=1}^{n} |x_j|^{\alpha_j}$$

*and $\psi : \mathbb{R}^n \to \mathbb{R}$ by*

$$\psi(\boldsymbol{x}) = \prod_{i=1}^{m} (\overline{x}_i)^{-\tau}\,.$$

*Then if $\tau \geq \alpha/A$, the set*

$$W_\Phi(\tau; m, n) = \{X \in \mathbb{R}^{mn} : \Phi(\langle \boldsymbol{q} X \rangle) < \psi(\boldsymbol{q}) \text{ for infinitely many } \boldsymbol{q} \in \mathbb{Z}^m\}$$

*has Hausdorff dimension*

$$\dim W_\Phi(\tau; m, n) = mn - 1 + \frac{2\alpha}{\tau A + \alpha}\,.$$

Proof. Evidently $\Phi$ is a distance function and the restriction $\psi_1$ of $\psi$ is given by $\psi_1(q) = q^{-\tau}$, $q \in \mathbb{N}$. Now $\omega(s) = A/(s - n + 1)/\alpha$ for $n-1 < s < n$ ([5], Lemma 2) and is strictly increasing and $\dim W_\Phi(\tau; 1, n) = n - 1 + 2\alpha/(\tau A + \alpha)$ when $\tau > \alpha/A$ ([5], Theorem 2). By continuity this formula also holds when $\tau = \alpha/A$. The sum corresponding to (5) is

$$\sum_{q=1}^{\infty} q^{n-s} q^{-\tau A(s-n+1)/\alpha},$$

which is finite since $\tau A(s-n+1)/\alpha > n-s+1$ when $s > n-1+2\alpha/(\tau A+\alpha)$. Finally, it has to be shown that the inequality

$$\sum_{|\mathsf{q}|=q} \prod_{i=1}^{m} |\overline{q}_i|^{-\tau A(s-n+1)/\alpha} \ll q^{-\tau A(s-n+1)/\alpha} \tag{6}$$

corresponding to (4) holds when $s > n-1+2\alpha/(\tau A+\alpha)$. Now for each $q \in \mathbb{N}$,

$$\sum_{|\mathsf{q}|=q} \prod_{i=1}^{m} |\overline{q}_i|^{-\tau A(s-n+1)/\alpha} \ll q^{-\tau A(s-n+1)/\alpha} \sum_{i=1}^{m} \prod_{j\neq i} \sum_{q_j=-q}^{q} |\overline{q}_j|^{-\tau A(s-n+1)/\alpha}$$
$$\ll q^{-\tau A(s-n+1)/\alpha} \Big(\sum_{k=1}^{\infty} k^{-\tau A(s-n+1)/\alpha}\Big)^{m-1}.$$

But $s > n-1+2\alpha/(\tau A+\alpha)$ implies that $\tau A(s-n+1)/\alpha > 2\tau A/(\tau A+\alpha) \ge 1$ when $\tau \ge A/\alpha$ and so (6) holds. Hence by Theorem 2 and continuity, when $\tau \ge A/\alpha$,

$$\dim W_{\Phi}(\tau; m, n) = (m-1)n + n - 1 + \frac{2\alpha}{\tau A+\alpha} = mn - 1 + \frac{2\alpha}{\tau A+\alpha}.$$

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