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On semi-strong U-numbers

by

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Dedicated to Professor Orhan Ş. İçen
on his seventieth birthday

In this paper we shall define irregular semi-strong U_m-numbers and semi-strong U_m-numbers and investigate some properties of such numbers.

DEFINITION 1 ([1]). Let $\gamma \in \mathbb{C}$ and $k \in \mathbb{Z}^+$. If there are infinitely many polynomials $P_n(x) \in \mathbb{Z}[x]$ $(\deg P_n(x) = m_n \le k)$ such that

(a) $$0 < |P_n(\gamma)| = H(P_n)^{-w(n)} \quad (n = 1, 2, \ldots), \qquad \lim_{n\to\infty} w(n) = \infty\,,$$

(b) $$|P_n(\gamma)| < H(P_{n+1})^{-\varrho} \quad \text{for some fixed } \varrho > 0\,,$$

then we say that γ is an *irregular semi-strong U-number.* If $\liminf_{n\to\infty} m_n = \lim_{n\to\infty} m_n$, we call γ is a *semi-strong U-number.* By Theorem 4 in [5] we see that if $\liminf_{n\to\infty} m_n = m$ then $\gamma \in U_m$. Thus the number $\zeta^{1/m}$ in Theorem 5 in [5] is a semi-strong U_m-number. Furthermore, U_m-numbers in [1] and [2] are also semi-strong.

In the sequel U_m^{is} and U_m^{s} will denote the set of all irregular semi-strong U_m-numbers and the set of all semi-strong U_m-numbers respectively.

We shall now collect some lemmas:

LEMMA 1. *Let α_1, α_2 be two algebraic numbers with different minimal polynomials. Then*

$$|\alpha_1 - \alpha_2| \ge 2^{-\max(n_1,n_2)+1}(n_1+1)^{-n_2}(n_2+1)^{-n_1}H(\alpha_1)^{-n_2}H(\alpha_2)^{-n_1}$$

where n_1, n_2 are the degrees and $H(\alpha_1), H(\alpha_2)$ are the heights of α_1, α_2 respectively. (See Güting [3], Th. 7.)

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([1]) We note that the U_m-numbers obtained by LeVeque's method in [5] are here called "irregular semi-strong U_m-numbers".

Lemma 2. *Let α_1, α_2 be conjugate algebraic numbers. Then*

$$|\alpha_1-\alpha_2|\ge(4n)^{1-n/2}(n+1)^{1-n/2}H(\alpha_1)^{-n+1/2}$$

where n is the degree of α_1. (See Güting [3], Th. 8.)

Lemma 3. *Let $P(x)\in\mathbb{Z}[x]$ be a polynomial of degree $\le n$ with height $\le H$, and let α be a root of $P(x)=0$. If ξ is a complex number with $|\xi-\alpha|<1$ then*

$$|\xi-\alpha|\ge n^{-2}(1+\xi)^{-n+1}H^{-1}|P(\xi)|\,.$$

(See Schneider [6], Lemme 15, p. 74.)

Lemma 4. *Let $P(x)$ be a polynomial of degree $\le n, H(P)\le H$, and assume that $P(x)=0$ has only simple roots. Then*

$$|\xi-\alpha_0|\le c_0|a_0^{-1}|H^{n-1}|P(\xi)|$$

where $\xi\in\mathbb{C}$, c_0 is a positive constant depending only on n, a_0 is the leading coefficient of $P(x)$ and α_0 is the root of $P(x)=0$ which is nearest to ξ. (See Schneider [6], Lemme 18, p. 78.)

Lemma 5. *Let $\alpha_1,\alpha_2,\ldots,\alpha_k$ $(k\ge1)$ be algebraic numbers which belong to an algebraic number field K of degree g, and let $F(y,x_1,x_2,\ldots,x_k)$ be a polynomial with rational integral coefficients and with degree at least one in y. If η is an algebraic number such that $F(\eta,\alpha_1,\ldots,\alpha_k)=0$, then the degree of $\eta\le dg$ and $h_\eta\le3^{2dg+(l_1+l_2+\ldots+l_k)\cdot g}H^gh_{\alpha_1}^{l_1g}\ldots h_{\alpha_k}^{l_kg}$, where h_η is the height of η, H is the maximum of the absolute values of the coefficients of F, l_i is the degree of F in x_i $(i=1,\ldots,k), d$ is the degree of F in y, and h_{α_i} is the height of α_i $(i=1,\ldots,k)$.* (See İçen [4].)

Theorem 1. *Let $\{\alpha_i\}$ be a sequence of algebraic numbers with*

$$\deg\alpha_i=m_i\le k,\qquad\lim_{i\to\infty}H(\alpha_i)=\infty\,, \tag{1}$$

$$0<|\alpha_{i+1}-\alpha_i|=H(\alpha_i)^{-w(i)}\quad\text{where }\lim_{i\to\infty}w(i)=\infty\,, \tag{2}$$

$$|\alpha_{i+1}-\alpha_i|\le H(\alpha_{i+1})^{-\varrho}\quad\text{for some }\varrho>0\,. \tag{3}$$

Then

$$\lim_{i\to\infty}\alpha_i\in U_m^{\rm is}\quad\text{where } m=\liminf_{i\to\infty}m_i\,.$$

Proof. It follows from Lemma 1, Lemma 2, (1) and (2) that $H(\alpha_{i+1})>H(\alpha_i)^2$ if i is sufficiently large. Let m, n $(m>n)$ be integers. By (1)

$$\begin{aligned}|\alpha_m-\alpha_n|&\le\sum_{i=n}^{m-1}|\alpha_{i+1}-\alpha_i|\\&<\sum_{i=n}^{\infty}H(\alpha_i)^{-w(i)}<c_1H(\alpha_n)^{-w(n)}\quad(n\text{ large})\end{aligned} \tag{4}$$

where c_1 is a positive constant not depending on $H(\alpha_n)$. Since $H(\alpha_n)^{-w(n)} \to 0$ as $n \to \infty$, (4) shows that $\{\alpha_i\}$ is a Cauchy sequence and so $\lim_{i\to\infty} \alpha_i$ exists. Set $\lim_{i\to\infty} \alpha_i = \gamma$ and let i be a positive integer. Since $\alpha_i \to \gamma$, there is an α_s $(s > i)$ such that

$$|\gamma - \alpha_s| \le H(\alpha_i)^{-w(i)}.$$

Using this and (4) we have

$$0 < |\gamma - \alpha_i| \le |\alpha_s - \alpha_i| + |\gamma - \alpha_s| < H(\alpha_i)^{-w(i)+1} \quad (i \text{ large}). \tag{5}$$

Hence applying Lemma 3 (and using (5)) yields

$$0 < |P_i(\gamma)| < H(P_i)^{-w(i)/2} \quad (i \text{ large}) \tag{6}$$

where P_i is the minimal polynomial of α_i. On the other hand, a combination of (6), (2) and (3) gives us

$$|P_i(\gamma)| < H(P_i)^{-w(i)/2} = |\alpha_{i+1} - \alpha_i|^{1/2} \le H(P_{i+1})^{-\varrho/2} \quad (i \text{ large}). \tag{7}$$

Thus (6) and (7) show that $\gamma \in U_m^{\mathrm{is}}$. Conversely, if $\gamma \in U_m^{\mathrm{is}}$, one can show, using Lemma 4, that there exists a sequence of algebraic numbers $\{\alpha_i\}$ satisfying the relations in the theorem for some $\varrho > 0$ and a sequence $\{w(i)\}$.

THEOREM 2. *Let $m \in \mathbb{Z}^+$ and let $P(x) \in \mathbb{Z}[x]$ be a polynomial of degree ≥ 1. Then there exist infinitely many $\gamma \in U_m^{\mathrm{s}}$ such that $P(\gamma) \in U_m^{\mathrm{s}}$.*

Proof. Let α be an algebraic number of degree m and let $\alpha^{(1)} = \alpha, \alpha^{(2)}, \ldots, \alpha^{(m)}$ denote the field conjugates of α. Let $n \in \mathbb{Z}^+$, $P(x) = \sum_{i=0}^{k} b_i x^i$ $(b_k \ne 0)$. We consider the equations

$$P(\alpha^{(i)} + y) = P(\alpha^{(j)} + y) \quad (1 \le i, j \le m,\ i \ne j), \tag{8}$$

where $y = n^{-1}$. For fixed i, j, (8) is equivalent to a polynomial equation $a_{k-1}y^{k-1} + \ldots + a_0 = 0$. Since $a_{k-1} = b_k(\alpha^{(i)} - \alpha^{(j)}) \ne 0$, (8) has only finitely many solutions in y. Therefore if n is sufficiently large then $\deg P(\alpha + n^{-1}) = m$. Let $\{w(i)\}$ be a sequence of real numbers with $w(i) \to \infty$ as $i \to \infty$. Now we define algebraic numbers α_i $(i = 1, 2, \ldots)$ as follows:

$$\begin{cases} \alpha_1 = \alpha + n_1^{-1} & \text{where } n_1 \in \mathbb{Z}^+ \text{ with} \\ & \qquad \deg P(\alpha + n_1^{-1}) = m,\ n_1 > 3^{3m}, \\ \alpha_{i+1} = \alpha_i + n_{i+1}^{-1} & (i \ge 1) \end{cases} \tag{9}$$

where n_{i+1} is a positive integer satisfying the conditions

$$\text{(a) } \deg P(\alpha_i + n_{i+1}^{-1}) = m, \quad \text{(b) } H(\alpha_i)^{w(i)} \le n_{i+1}, \quad \text{(c) } n_i^2 < n_{i+1}. \tag{10}$$

By (9) we have $\alpha_{i+1} = \alpha + \sum_{k=1}^{i+1} n_k^{-1}$. On the other hand, it is clear that

$$H\Big(\sum_{k=1}^{i+1} n_k^{-1}\Big) \le \prod_{k=1}^{i+1} n_k \,.$$

Using this and (10)(c) in Lemma 5 we obtain

$$(11) \qquad H(\alpha_{i+1}) \le n_{i+1}^{2m+2} \quad (i \text{ large})\,.$$

A combination of (9) and (11) gives us

$$(12) \qquad |\alpha_{i+1} - \alpha_i| = n_{i+1}^{-1} \le H(\alpha_{i+1})^{-1/(2m+2)} \quad (i \text{ large})\,.$$

Next it follows from (9) and (10)(b) that

$$(13) \qquad |\alpha_{i+1} - \alpha_i| \le H(\alpha_i)^{-w(i)}\,,$$

so we have $\gamma = \lim_{i\to\infty} \alpha_i \in U_m^{\mathrm{s}}$ by Theorem 1. To prove $P(\gamma) \in U_m^{\mathrm{s}}$, we put $P(\alpha_i) = \beta_i$ $(i = 1, 2, \ldots)$. It is well known that

$$|\beta_{i+1} - \beta_i| = |P(\alpha_{i+1}) - P(\alpha_i)| = |\alpha_{i+1} - \alpha_i|\,|P'(t)| \quad (i = 1, 2, \ldots)$$

where $\alpha_i < t < \alpha_{i+1}$ and $P'(x)$ is the derivative of $P(x)$. Since $\alpha_i \to \gamma$ as $i \to \infty$, there is a constant $c_2 > 0$ depending only on γ and $P(x)$ such that $|P'(t)| < c_2$. Thus we have

$$(14) \qquad |\beta_{i+1} - \beta_i| < |\alpha_{i+1} - \alpha_i| H(\alpha_i) \quad (i \text{ large})\,.$$

On the other hand, applying Lemma 5 (using $(11)_i$) we find

$$(15) \qquad H(\beta_i) \le H(\alpha_i)^{km+1} \quad (i \text{ large})\,.$$

Hence a combination of (13), (14) and (15) shows that

$$0 < |\beta_{i+1} - \beta_i| < H(\beta_i)^{(-w(i)+1)/(km+1)} \quad (i \text{ large})\,.$$

Next writing (15) for $i+1$ and combining this with (12) and (14) we find

$$|\beta_{i+1} - \beta_i| < |\alpha_{i+1} - \alpha_i|^{1/2} < H(\beta_{i+1})^{-1/\delta}$$

where $\delta = 2(2m+2)(km+1)$. So by Theorem 1 we have $\lim_{i\to\infty} \beta_i = P(\lim_{i\to\infty} \alpha_i) = P(\gamma) \in U_m^{\mathrm{s}}$.

The following can be obtained by using the arguments in Theorem 1.

Corollary 1. *Let* $\gamma \in U_m^{\mathrm{s}}$ *and* $P(x) \in \mathbb{Z}[x]$ *with* $\deg P(x) \ge 1$. *Then* $P(\gamma) \in U_n^{\mathrm{s}}$, *where* $n \mid m$.

Corollary 2. *Let* p *be a prime*, $\gamma \in U_p^{\mathrm{s}}$ *and* $P(x) \in \mathbb{Z}[x]$ *with* $1 \le \deg P(x) < p$. *Then* $P(\gamma) \in U_p^{\mathrm{s}}$.

Theorem 3. *Let* $m \in \mathbb{Z}^+$ *and let* $\{P_n(x)\}$ *be a sequence of polynomials in* $\mathbb{Z}[x]$ *with* $\deg P_n(x) \ge 1$ $(n = 1, 2, \ldots)$. *Then there are infinitely many* $\gamma \in U_m^{\mathrm{s}}$ *such that* $P_n(\gamma) \in U_m^{\mathrm{s}}$ $(n = 1, 2, \ldots)$.

P r o o f. Let $\alpha > 1$ be algebraic of degree m and let $\{w(i)\}$ be a sequence of positive real numbers with $\lim_{i\to\infty} w(i) = \infty$.

We shall construct $N_k \in \mathbb{Z}^+$ as follows:

Let N_1 be a positive integer satisfying

$$\deg P_1(\alpha + N_1^{-1}) = m\,, \qquad N_1 > 3^{3m}\,. \tag{16}$$

Then we define N_k $(k \ge 2)$ as an integer satisfying the conditions

$$\begin{aligned} &(\mathrm{a})_{i,k} && \deg P_i\Big(\alpha + \sum_{j=1}^{k} N_j^{-1}\Big) = m \quad (i = 1, 2, \ldots, k)\,, \\ (17)\quad &(\mathrm{b})_k && H\Big(\alpha + \sum_{j=1}^{k-1} N_j^{-1}\Big)^{w(k-1)} < N_k\,, \\ &(\mathrm{c}) && N_{k-1}^2 < N_k\,. \end{aligned}$$

Now set $\alpha_1 = \alpha + N_1^{-1}$ and $\alpha_{i+1} = \alpha_i + N_{i+1}^{-1}$ for $i \ge 1$. Using Theorem 2 and $(17)(\mathrm{b})_k$ one can show that $\gamma := \lim_{i\to\infty} \alpha_i = \alpha + \sum_{i=1}^{\infty} N_i^{-1} \in U_m^{\mathrm{s}}$.

Next, let $n \ge 1$ be an integer. We define algebraic numbers β_i as

$$\beta_1 = \alpha + \sum_{j=1}^{n} N_j^{-1}, \qquad \beta_{i+1} = \beta_i + N_{n+i}^{-1} \quad (i = 1, 2, \ldots)\,.$$

It is clear that $\lim_{i\to\infty} \beta_i = \lim_{i\to\infty} \alpha_i = \gamma$. On the other hand, by $(17)(\mathrm{a})_{i=n,k=n}$ we deduce $\deg P_n(\beta_1) = m$ and by $(17)(\mathrm{a})_{i=n,k=n+j}$, $(17)(\mathrm{b})_{k=n+j}$, and $(17)(\mathrm{c})$ we have

$$\begin{aligned} \deg P_n(\beta_j) = m\,, \quad H(\beta_j)^{w(n+j-1)} &\le N_{n+j}\,, \\ N_{n+j}^2 &< N_{n+j+1} \quad (j = 2, 3, \ldots)\,, \end{aligned}$$

that is, $\{\beta_j\}$ and $P_n(x)$ satisfy the conditions in Theorem 2. Thus $P_n(\gamma) \in U_m^{\mathrm{s}}$ $(n = 1, 2, \ldots)$.

Definition 2. Let $\{x_i\}$ be a sequence of positive integers with

$$\lim_{i\to\infty} \frac{\log x_{i+1}}{\log x_i} = \infty$$

and let $\gamma \in U_m^{\mathrm{is}}$ with convergents $\{\alpha_i\}$ as in Theorem 1. If there exist a subsequence $\{x_{n_i}\}$ of $\{x_i\}$ and positive real numbers k_1, k_2 such that

$$x_{n_i}^{k_1} \le H(\alpha_i) \le x_{n_i}^{k_2} \quad (i = 1, 2, \ldots) \tag{18}$$

then we say that the sequence $\{H(\alpha_i)\}$ is *comparable with* $\{x_i\}$.

Theorem 4. *Let* $\{x_i\}$ *be as in Definition* 2. *Then the set*

$$F = A \cup \{\gamma \in U_m^{\mathrm{is}} \mid \{H(\alpha_i)\} \text{ is comparable with } \{x_i\},\ \text{where } \alpha_i \to \gamma, m \in \mathbb{Z}^+\}$$

is an uncountable subfield of $\mathbb{C}$ *which is algebraically closed.*

P r o o f. Let $y_1, y_2 \in F$. Assume that $y_1 \in U_r^{\text{is}}$, $y_2 \in U_t^{\text{is}}$. Then there are positive real numbers k_1, k_2, k_3, k_4, ϱ_1, ϱ_2 and sequences of algebraic numbers $\{\alpha_i\}$, $\{\beta_i\}$ ($\deg \alpha_i$, $\deg \beta_i \le k$, where $k \ge \max(r,t)$) such that

$$\begin{gathered} 0 < |y_1 - \alpha_i| = H(\alpha_i)^{-w(i)} < H(\alpha_{i+1})^{-\varrho_1}\,, \\ \lim_{i\to\infty} w(i) = \infty\,, \qquad \lim_{i\to\infty} H(\alpha_i) = \infty\,, \end{gathered} \tag{19}$$

$$\begin{gathered} 0 < |y_2 - \beta_i| = H(\beta_i)^{-w_2(i)} < H(\beta_{i+1})^{-\varrho_2}\,, \\ \lim_{i\to\infty} w_2(i) = \infty\,, \qquad \lim_{i\to\infty} H(\beta_i) = \infty\,, \end{gathered} \tag{20}$$

and subsequences $\{x_{n_i}\}$, $\{x_{m_i}\}$ of $\{x_i\}$ satisfying

$$x_{n_i}^{k_1} \le H(\alpha_i) \le x_{n_i}^{k_2} \qquad (i = 1, 2, \ldots)\,, \tag{21}$$

$$x_{m_i}^{k_3} \le H(\beta_i) \le x_{m_i}^{k_4} \qquad (i = 1, 2, \ldots)\,. \tag{22}$$

Let $\{x_{r_i}\}$ denote the monotonic union sequence formed from $\{x_{n_i}\}$, $\{x_{m_i}\}$. Assume that $x_{r_{i_0}} > \max(H(\alpha_1), H(\beta_1))$. We define positive integers $j(i)$, $t(i)$ and then algebraic numbers δ_i as

$$n_{j(i)} = \max\{n_\nu \mid n_\nu \le r_i\}\,, \qquad m_{t(i)} = \max\{m_\nu \mid m_\nu \le r_i\}\,, \qquad i > i_0\,, \tag{23}$$

$$\delta_i = \alpha_{j(i)} + \beta_{t(i)}\,. \tag{24}$$

Consider the set $B = \{\delta_i \mid i \ge i_0\}$. If B contains only finitely many algebraic numbers, there is a subsequence of $\{i\}$, say $\{i_k\}$, and an algebraic number δ which belongs to B such that

$$\delta = \alpha_{j(i_k)} + \beta_{t(i_k)} \qquad (k = 1, 2, \ldots)\,.$$

In this equality taking limit as $k \to \infty$ we obtain $y_1 + y_2 = \delta \in A \subset F$. Secondly, assume that B contains infinitely many algebraic numbers. Hence there is a subsequence $\{i_k\}$ of $\{i\}$ with

$$\delta_{i_k} = \alpha_{j(i_k)} + \beta_{t(i_k)} \tag{25}$$

($i_1 > i_0, \delta_{i_r} \ne \delta_{i_s}$ if $r \ne s, k = 1, 2, \ldots$, $i_r < i_s$ for $r < s$, $\delta_{i_k} = \delta_j$ for $j = i_k + 1, i_k + 2, \ldots, i_{k+1} - 1$).

On the other hand, by Lemma 5, we have

$$H(\delta_{i_k}) \le 3^{2k^2} H(\alpha_{j(i_k)})^{k^2} H(\beta_{t(i_k)})^{k^2} \qquad (k = 1, 2, \ldots)\,. \tag{26}$$

Next by (21) and (22) we get

$$H(\alpha_{j(i_k)}) \le x_{n_{j(i_k)}}^{k_2}\,, \qquad H(\beta_{t(i_k)}) \le x_{m_{t(i_k)}}^{k_4}\,.$$

Finally, by (23), we obtain

$$H(\alpha_{j(i_k)})\,,\ H(\beta_{t(i_k)}) \le x_{r_{i_k}}^{\max(k_2,k_4)}\,.$$

Thus using this in (25) and putting $k_5 = k^2 \max(k_2, k_4) + 1$ yields

$$H(\delta_{i_k}) \le x_{r_{i_k}}^{k_5} \qquad (k \text{ large})\,. \tag{27}$$

On the other hand, a combination of (21), (22) and (23) gives us

$$(28)\qquad \begin{aligned} H(\alpha_{j(i_{k+1}-1)+1}) &\geq x^{k_1}_{n_{j(i_{k+1}-1)+1}} \geq x^{k_1}_{r_{i_{k+1}}}\,, \\ H(\beta_{t(i_{k+1}-1)+1}) &\geq x^{k_3}_{m_{j(i_{k+1}-1)+1}} \geq x^{k_3}_{r_{i_{k+1}}}\,. \end{aligned}$$

Using (19), (20) and (28) shows that

$$(29)\qquad \begin{aligned} |y_1+y_2-\delta_{i_k}| &= |y_1+y_2-\delta_{i_{k+1}-1}| \\ &\leq |y_1-\alpha_{j(i_{k+1}-1)}| + |y_2-\beta_{t(i_{k+1}-1)}|\,, \\ H(\alpha_{j(i_{k+1}-1)+1})^{-\varrho_1} &+ H(\beta_{t(i_{k+1}-1)+1})^{-\varrho_2} \leq 2x^{-\varrho}_{r_{i_{k+1}}}\,, \end{aligned}$$

where $\varrho = \min(\varrho_1 k_1, \varrho_2 k_3)$.

Next, writing (27) with k replaced by $k+1$ and using this in (29) we have

$$(30)\qquad |y_1+y_2-\delta_{i_k}| \leq H(\delta_{i_{k+1}})^{-\varrho/2k_5} \quad (k \text{ large})\,.$$

Furthermore, it follows from (27) and (29) that

$$|y_1+y_2-\delta_{i_k}| \leq H(\delta_{i_k})^{-w(i_k)} \quad (k \text{ large})$$

where $w(i_k) = \varrho \log x_{r_{i_{k+1}}}/2k_5 \log x_{r_{i_k}}$. It is clear that $w(i_k) \to \infty$ as $k \to \infty$, so we have $y_1+y_2 \in U^{\text{is}}_m$ for some $m \leq k^2$.

Now we show that $\{H(\delta_{i_k})\}$ is comparable with $\{x_i\}$. Using (29) and $x_{r_{i_{k+1}}} > x_{r_{i_k}}$ in the inequality $|\delta_{i_{k+1}}-\delta_{i_k}| \leq |y_1+y_2-\delta_{i_{k+1}}| + |y_1+y_2-\delta_{i_k}|$ we obtain

$$(31)\qquad |\delta_{i_{k+1}}-\delta_{i_k}| \leq x^{-\varrho/2}_{r_{i_{k+1}}} \quad (k \text{ large})\,.$$

Next, by Lemma 1,

$$|\delta_{i_{k+1}}-\delta_{i_k}| \geq H(\delta_{i_{k+1}})^{-3k^2} \quad (k \text{ large})\,.$$

Combining this with (31) gives

$$(32)\qquad H(\delta_{i_{k+1}}) > x^{-\varrho/6k^2}_{r_{i_{k+1}}} \quad (k \text{ large})\,.$$

Thus (27) and (32) show that $\{H(\delta_{i_k})\}$ is comparable with $\{x_i\}$, that is, $y_1+y_2 \in F$.

Now we show that $y_1y_2 \in F$. For this we shall approximate y_1y_2 by algebraic numbers δ'_i defined as

$$(33)\qquad \delta'_i = \alpha_{j(i)} \cdot \beta_{t(i)} \quad (i > i_0)\,.$$

If $B = \{\delta'_i \mid i > i_0\}$ contains only finitely many algebraic numbers, then it follows from (33) that $y_1y_2 \in A \subset F$. If not, there is a subsequence $\{i_k\}$ of $\{i\}$ such that

$$(34)\qquad \delta'_{i_k} = \alpha_{j(i_k)} \cdot \beta_{t(i_k)}$$

($i_1 > i_0$, $\delta_{i_r} \neq \delta_{i_s}$ if $r \neq s$, $k = 1, 2, \ldots$, $i_r < i_s$ for $r < s$, $\delta_j = \delta_{i_k}$ for $j = i_k + 1, i_k + 2, \ldots, i_{k+1} - 1$). Using (19), (20) and (28) we obtain

$$
\begin{aligned}
(35)\quad |y_1 y_2 - \delta'_{i_k}| &= |y_1 y_2 - \delta'_{i_{k+1}-1}| = |y_1 y_2 - \alpha_{j(i_{k+1}-1)} \cdot \beta_{t(i_{k+1}-1)}| \\
&\le |y_1|\,|y_2 - \beta_{t(i_{k+1}-1)}| + |\beta_{t(i_{k+1}-1)}|\,|y_1 - \alpha_{j(i_{k+1}-1)}| \\
&\le M x_{r_{i_{k+1}}}^{-\varrho}
\end{aligned}
$$

where $M = 2\max\{|y_1|, |y_2| + 1\}$.

On the other hand, using similar arguments to the previous steps, we obtain

$$
(36)\qquad H(\delta'_{i_k}) \le x_{r_{i_k}}^{k_5} \quad (k \text{ large}).
$$

Hence, using (35) and (36), we get

$$
|y_1 y_2 - \delta'_{i_k}| \le H(\delta'_{i_{k+1}})^{-\varrho/2k_5} \le H(\delta'_{i_k})^{w(i_k)} \quad (k \text{ large})
$$

where $w(i_k) \to \infty$ as $k \to \infty$, which shows that $y_1 y_2 \in U_m^{\text{is}}$ for some $m \le k^2$. Next by using Lemma 1 and (35), one can show that $\{H(\delta'_{i_k})\}$ is comparable with $\{x_i\}$ and so we have $y_1 y_2 \in F$.

Finally let $\alpha \in A$. Then using similar arguments to the proof of the fact that $y_1 + y_2$, $y_1 y_2 \in F$, and approximating αy_1, $\alpha + y_1$, $-y_1$, y_1^{-1} by $\{\alpha\alpha_i\}$, $\{\alpha + \alpha_i\}$, $\{-\alpha_i\}$, $\{\alpha_i^{-1}\}$ respectively, one can show that αy_1, $\alpha + y_1$, $-y_1$, $y_1^{-1} \in F$.

Now we show that F is algebraically closed. Consider the equation

$$
f(x) = a_0 + a_1 x + \ldots + a_k x^k = 0 \quad (k \ge 1, a_k \neq 0)
$$

where $a_i \in F$. We may assume that $a_\nu \in U_m^{\text{is}}$ ($\nu = 0, 1, \ldots, k$); only trivial changes are required if some are algebraic. Hence there are sequences of algebraic numbers $\alpha_i^{(\nu)}$ ($\nu = 0, 1, \ldots, k$), subsequences $\{x_{n_i^{(\nu)}}\}$ ($\nu = 0, 1, \ldots, k$) and positive real numbers ϱ_ν, $k_1^{(\nu)}$, $k_2^{(\nu)}$, t_ν ($\nu = 0, 1, \ldots, k$) with the following properties:

$$
(37)\qquad |a_\nu - \alpha_i^{(\nu)}| = H(\alpha_i^{(\nu)})^{-w_\nu(i)} < H(\alpha_{i+1}^{(\nu)})^{-\varrho_\nu}
$$

($\deg \alpha_i^{(\nu)} \le t_\nu$, $\nu = 0, 1, \ldots, k$, $i = 1, 2, \ldots$),

$$
(38)\qquad x_{n_i^{(\nu)}}^{k_1^{(\nu)}} \le H(\alpha_i^{(\nu)}) \le x_{n_i^{(\nu)}}^{k_2^{(\nu)}} \quad (\nu = 0, 1, \ldots, k,\ i = 1, 2, \ldots).
$$

We may also assume that all roots of $f(x) = 0$ are simple.

Let $f(\gamma) = 0$ for some $\gamma \in \mathbb{C}$ and let $\{x_{r_i}\}$ be the monotonic union sequence formed from $\{x_{n_i^{(0)}}\}$, $\{x_{n_i^{(1)}}\}, \ldots, \{x_{n_i^{(k)}}\}$. Let r_{i_0} be a positive integer with $x_{r_{i_0}} \ge \max_{\nu=0,1,\ldots,k} H(\alpha_1^{(\nu)})$. For $i \ge i_0$ we define integers $j_\nu(i)$ and polynomials $F_i(x)$ as

$$
(39)\qquad j_\nu(i) = \max\{n_r^{(\nu)} \mid n_r^{(\nu)} \le r_i\} \quad (\nu = 0, 1, \ldots, k),
$$

(40) $$F_i(x) = \alpha^{(0)}_{j_0(i)} + \alpha^{(1)}_{j_1(i)}x + \ldots + \alpha^{(k)}_{j_k(i)}x^k \quad (i \geq i_0)\,.$$

Since $\alpha^{(\nu)}_{j_\nu(i)} \to a_\nu$ as $i \to \infty$ $(\nu = 0, 1, \ldots, k)$, there is a sequence of algebraic numbers $\{\delta_i\}$ such that $F_i(\delta_i) = 0$ and $\delta_i \to \gamma$ as $i \to \infty$. Now if $\gamma \in A$ then there is nothing to prove. Therefore we may suppose that $\gamma \notin A$. This yields that the set $\{\delta_i \mid i \geq i_0\}$ is infinite. Furthermore, we shall assume that $\delta_r \neq \delta_s$ if $r \neq s$ (if not, the proof can be completed using the arguments in (24), (25)).

It is well known that

(41) $$f(\gamma) - f(\delta_i) = \eta(\gamma - \delta_i)f'(\theta_i)$$

where $\eta \in \mathbb{C}$ with $0 \leq |\eta| \leq 1$ and θ_i is a complex number on the segment $\overline{\gamma\delta_i}$. Since γ is a simple root of $f(x) = 0$, we have $f'(\gamma) \neq 0$. Furthermore, since $\delta_i \to \gamma$ as $i \to \infty$, there is a constant c_3 such that $|f'(\theta_i)| > c_3$ for large i. Thus, using this and $f(\gamma) = 0$ in (41), we obtain

(42) $$|\gamma - \delta_i| < (|\eta|c_3)^{-1}|f(\delta_i)| \quad (i \text{ large})\,.$$

Now we give an upper bound for $|f(\delta_i)|$. Using (37) we obtain

$$\begin{aligned}|f(\delta_i)| = \Big|\sum_{t=0}^{k}(a_t - \alpha^{(t)}_{j_t(i)} + \alpha^{(t)}_{j_t(i)})\delta_i^t\Big| &\leq \sum_{t=0}^{k}|a_t - \alpha^{(t)}_{j_t(i)}|\,|\delta_i^t| \\ \leq \{H(\alpha^{(0)}_{j_0(i)+1})^{-\varrho_0} + H(\alpha^{(1)}_{j_1(i)+1})^{-\varrho_1} + \ldots \\ \ldots + H(\alpha^{(k)}_{i_k(i)+1})^{-\varrho_k}\}\max(1, |\delta_i|)^k\end{aligned}$$

and so

(43) $$|f(\delta_i)| \leq c_4\{\min_{\nu=0,1,\ldots,k} H(\alpha^{(\nu)}_{j_t(i)+1})\}^{-\varrho}$$

where $c_4 = (k+1)(|\gamma|+1)^k$ and $\varrho = \min_{\nu=0,1,\ldots,k}\{\varrho_\nu\}$.

On the other hand, by (38) and (39) we have

$$\min_{\nu=0,1,\ldots,k}\{H(\alpha^{(\nu)}_{j_\nu(i)+1})\} \geq x^{k_1^{(\nu)}}_{n_{j_\nu(i)+1}} \geq x^{k_1^{(\nu)}}_{r_{i+1}} \geq x^{k_6}_{r_{i+1}} \quad (i \text{ large})$$

where $k_6 = \min_{\nu=0,1,\ldots,k}\{k_1^{(\nu)}\}$. Combining this with (42) and (43) we obtain

(44) $$|\gamma - \delta_i| \leq c_4(c_3|\eta|)^{-1}x^{-k_6}_{r_{i+1}} < x^{-k_6\varrho/2}_{r_{i+1}} \quad (i \text{ large})\,.$$

Next, applying Lemma 5 (using (39) and (40)), we get

$$H(\delta_i) \leq x^{k_7}_{r_i} \quad (i \text{ large})$$

where $k_7 > 0$ is a fixed real number. Using this in (44) we obtain

$$|\gamma - \delta_i| = H(\delta_i)^{w_4(i)} < H(\delta_{i+1})^{(-k_6\varrho/2)k_7} \quad (i \text{ large})$$

where $w_4(i) \to \infty$ as $i \to \infty$, which shows $\gamma \in U_m^{\text{is}}$ for some $m \in \mathbb{Z}^+$. Finally, using similar arguments to the previous steps, one can show that $H(\delta_i)$ is comparable with $\{x_i\}$ and this completes the proof.

As a consequence of Theorem 4 we have

COROLLARY 3. *Let* $\{x_i\}$ *be a sequence as in Definition* 1. *Then the set of all semi-strong Liouville numbers comparable with* $\{x_i\}$, *together with the rationals*, *forms an uncountable subfield of* $\mathbb{R}$.

Furthermore, the following can be obtained by using arguments in Theorem 4:

COROLLARY 4. *Let* $F(y, x_1, x_2, \ldots, x_k)$ *be a polynomial with algebraic coefficients*, $\gamma \in U$ *and* $\gamma_i \in U_{m_i}^{\text{is}}$ $(i = 1, \ldots, k)$. *Then* $F(\gamma, \gamma_1, \ldots, \gamma_k) \in U \cup A$.

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