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Lower bounds for a certain class of error functions

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1. Introduction. An arithmetical function f that does not deviate too largely from the identity function $I : n \mapsto n$ frequently satisfies an asymptotic relation

$$\sum_{n\le x} f(n) = C_f x^2 + R_f(x),$$

in which the error term $R_f(x)$ is the primary object of interest.

A quite thoroughly investigated example is provided by Euler's totient φ. For instance, A. Walfisz's [17] well known upper bound

$$R_\varphi(x) = \sum_{n\le x} \varphi(n) - \frac{3}{\pi^2}x^2 \ll x(\log x)^{2/3}(\log\log x)^{4/3}$$

has superseded F. Mertens' elementary estimate [12]

$$R_\varphi(x) \ll x\log x,$$

and in the opposite direction there are the results due to S. S. Pillai and S. D. Chowla [14]

$$R_\varphi(x) = \Omega(x\log\log\log x) \tag{1.1}$$

and P. Erdős and H. N. Shapiro [4]

$$R_\varphi(x) = \Omega_\pm(x\log\log\log\log x). \tag{1.2}$$

Subsequently J. H. Proschan [15] applied the techniques of [4] and [14] to obtain Ω-results for the remainder term $R_f(x)$ corresponding to arithmetical functions $f = I * (\mu\cdot g)$, where μ is the Möbius function and g is a positive integer valued completely multiplicative function that satisfies certain growth conditions.

In this paper we will show how a method that has recently been used by H. L. Montgomery [13] to improve (1.1) and (1.2) to

$$R_\varphi(x) = \Omega_\pm(x\sqrt{\log\log x}) \tag{1.3}$$

can be extended to a class of arithmetical functions that is considerably larger than that which was treated in [15].

Moreover, our estimates are as a rule much sharper than Proschan's, typically improving his $\Omega_{\pm}(x \log\log\log\log x)$ to $\Omega_{\pm}(x(\log\log x)^{\delta})$ for an appropriate positive constant $\delta = \delta(f)$.

Our results are applicable to many generalizations of Euler's φ-function, e.g. the totients of Schemmel and Nagell (cf. [16]) and the function φ_F defined with respect to an irreducible polynomial $F \in \mathbb{Z}[x]$ by

$$\varphi_F(n) := n \prod_{p|n} \left(1 - \frac{\varrho(p)}{p}\right)$$

where $\varrho(p)$ denotes the number of zeros of $F(x) \pmod p$.

2. Definitions and statement of main results. The members of the class of functions that we investigate are of the form $f = I * h$, where h is an arithmetical function that has certain properties in common with the Möbius function.

However, the similarity between h and μ need not be too close, since h is allowed to be unbounded, for example. The precise conditions that are to be fulfilled by h are summarized in the following

DEFINITION 2.1. For real $r \geq 0$ and a positive integer k the class $\mathcal{C}(r,k)$ consists of all real-valued multiplicative arithmetical functions h which satisfy

(2.1) $$\sum_{n\leq x} |h(n)| \ll x(\log x)^r;$$

(2.2) $$c(h) := \sum_{n=1}^{\infty} h(n)n^{-2} \neq 0;$$

(2.3) there exists an integer $B \geq 1$ such that $h(p^i) = 0$ for primes p not dividing B and $1 \leq i < k$;

(2.4) if n is a k-full integer then $h(n) = \mu(\alpha(n))|h(n)|$, where $\alpha(n) := \prod_{p|n} p$ is the squarefree kernel of n;

(2.5) the series $\sum_p |h(p^k)|p^{-k}$ diverges;

(2.6) the series $\sum_p |h(p^k)|^2 p^{-2k}$ converges.

Remarks. (a) Throughout the letter p denotes a prime.

(b) Note that (2.1) implies that $\sum_{n\geq 1} h(n)n^{-1-\varepsilon}$ converges absolutely for every $\varepsilon > 0$.

(c) The Möbius function is in $\mathcal{C}(0,1)$.

Our primary result is

THEOREM 2.2. *Let $f := I * h$ where $h \in \mathcal{C}(r,k)$. Suppose there is a monotonically decreasing function ξ, defined for $x > 0$, which has the following properties*:

$$\sup_{y>x}\Big|\sum_{x<n\le y}\frac{h(n)}{n}\Big| \le \xi(x) \qquad (x>0)\,; \tag{2.7}$$

(2.8) $\xi(x)(\log x)^r$ *is decreasing for sufficiently large* x *and*

$$\lim_{x\to\infty}\xi(x)(\log x)^r = 0\,;$$

$$\frac{\xi(x-1)}{\xi(x)} \to 1 \quad \textit{and} \quad x\xi(x) \gg (\log x)^{r+1} \quad \textit{as } x\to\infty\,. \tag{2.9}$$

Furthermore, assume there is an integer $M \ge 3$ for which the congruence $x^k \equiv -1 \pmod M$ has $\Delta\varphi(M) \ge 1$ solutions $\pmod M$ and such that for integers a, relatively prime to M,

$$\sum_{\substack{p\le x\\ p\equiv a \,(\mathrm{mod}\, M)}} |h(p^k)|p^{-k} = \frac{1}{\varphi(M)}\Theta(x) + O(1) \tag{2.10}$$

where

$$\Theta(x) := \sum_{p\le x}|h(p^k)|p^{-k}\,. \tag{2.11}$$

Set

$$L(x) := ((\log x)^r \cdot \xi(x(\log x)^{-r}))^{-1}\,. \tag{2.12}$$

Then we have

$$\sum_{n\le x}\frac{f(n)}{n} = c(h)x + E(x)\,, \tag{2.13}$$

where

$$E(x) \ll (\log x)^{r+1} \tag{2.14}$$

and

$$E(x) = \Omega_{\pm}(\exp(\Delta\cdot\Theta((2\Delta k)^{-1}\log L(\sqrt{x})))). \tag{2.15}$$

In most cases the conclusion of the theorem carries over to the perhaps more natural error term

$$R(x) = \sum_{n\le x} f(n) - \tfrac{1}{2}c(h)x^2. \tag{2.16}$$

This is the subject of the first of the next two corollaries, for which we retain the notation and assumptions of Theorem 2.2.

Corollary 2.3. *We have*

$$R(x) \ll x(\log x)^{r+1} \tag{2.17}$$

and, if additionally $\xi(x)\log x \ll 1$, *then*

$$R(x) = \Omega_{\pm}(x \cdot \exp(\Delta \cdot \Theta((2\Delta k)^{-1} \log L(\sqrt{x})))) \,. \tag{2.18}$$

Corollary 2.4. *If* $\lim_{x\to\infty} \xi(x)\log x = 0$ *then*

$$\sum_{n\le x} E(n) \sim \tfrac{1}{2}(c(h) - b(h))x \tag{2.19}$$

and

$$\sum_{n\le x} R(n) \sim \tfrac{1}{4}c(h)x^2 \tag{2.20}$$

where

$$b(h) := \sum_{n=1}^{\infty} \frac{h(n)}{n}.$$

3. Proof of Theorem 2.2. It follows from $f = I * h$ and Abel's inequality (cf. [11], Satz 140) that

$$\begin{aligned} E(x) &= -x \sum_{n>x} h(n)n^{-2} - \sum_{n\le x} \frac{h(n)}{n}\left\{\frac{x}{n}\right\} \\ &= -\sum_{n\le x} \frac{h(n)}{n}\left\{\frac{x}{n}\right\} + O(\xi(x)) \,. \end{aligned} \tag{3.1}$$

Here $\{t\}$ denotes the fractional part of the real number t.

From (3.1) we deduce that for all positive x and y

$$E(x) = -\sum_{n\le y} \frac{h(n)}{n}\left\{\frac{x}{n}\right\} + O(\xi(x)) + O\left(\frac{x}{y}\xi(y/2)\right). \tag{3.2}$$

This is because for $y \le x$ we have

$$\begin{aligned} \Big| \sum_{y<n\le x} \frac{h(n)}{n}\left\{\frac{x}{n}\right\}\Big| &= \Big| \sum_{1\le k\le x/y} \sum_{\substack{x/(k+1)<n\le x/k \\ n>y}} \frac{h(n)}{n}\left\{\frac{x}{n}\right\}\Big| \\ &\le \sum_{k\le x/y} \xi(x/(k+1)) \le \frac{x}{y}\xi(y/2) \,, \end{aligned}$$

and for $y > x$

$$\Big| \sum_{x<n\le y} \frac{h(n)}{n}\left\{\frac{x}{n}\right\}\Big| \le \xi(x) \,.$$

Following Montgomery [13] we introduce the function

$$s(t) := \begin{cases} \frac{1}{2} - \{t\} & \text{if } t \notin \mathbb{Z}, \\ 0 & \text{if } t \in \mathbb{Z} \end{cases}$$

into formula (3.2) and use the convergence of $\sum_{n=1}^{\infty} h(n)n^{-1}$ to obtain for $y > 0$ and nonintegral $x > 0$

$$E(x) = \sum_{n \le y} \frac{h(n)}{n} s\left(\frac{x}{n}\right) + O\left(\frac{x}{y}\xi(y/2)\right) + O(1)\,. \tag{3.3}$$

For natural numbers d, q and N and nonintegral β, $0 < \beta < q$, we have (cf. [13], Lemma 3)

$$\sum_{n=1}^{N} s\left(\frac{nq+\beta}{d}\right) = (d,q)s\left(\frac{\beta}{(d,q)}\right)\frac{N}{d} + O(d)\,,$$

which along with (3.3) and (2.1) yields (upon inverting the order of summation) for $y > 0$

$$\begin{aligned} \sum_{n=1}^{N} E(nq+\beta) = N \sum_{d \le y} \frac{h(d)}{d^2}(d,q)s\left(\frac{\beta}{(d,q)}\right) + O(N) \\ + O(y(\log y)^r) + O(N^2 q y^{-1}\xi(y/2))\,. \end{aligned} \tag{3.4}$$

The above formula (3.4) suggests a closer investigation of

$$\Sigma(y,q,\beta) := \sum_{d \le y} \frac{h(d)}{d^2}(d,q)s\left(\frac{\beta}{(d,q)}\right). \tag{3.5}$$

Since h is multiplicative and each natural number d may be written uniquely as $d = uv$ where $\alpha(u)|q$ and $(v,q) = 1$, we have

$$\Sigma(y,q,\beta) = \sum_{\substack{u \le y \\ \alpha(u)|q}} \frac{h(u)}{u^2}(u,q)s\left(\frac{\beta}{(u,q)}\right) \sum_{\substack{v \le y/u \\ (v,q)=1}} \frac{h(v)}{v^2}\,. \tag{3.6}$$

For the sake of convenience set

$$\Phi_q := \sum_{\substack{v \ge 1 \\ (v,q)=1}} h(v)v^{-2}$$

and note that (2.1) and partial summation imply that

$$\Phi_q = \sum_{\substack{v \le y/u \\ (v,q)=1}} h(v)v^{-2} + O\left(\frac{u}{y}(\log y)^r\right). \tag{3.7}$$

Since (again by partial summation)

$$\sum_{\substack{u\le y\\ \alpha(u)|q}} \frac{|h(u)|}{u}(u,q) \le q\sum_{u\le y}\frac{|h(u)|}{u} \ll q(\log y)^{r+1}\,,$$

formulas (3.6) and (3.7) give

$$\Sigma(y,q,\beta) = \Phi_q \sum_{\substack{u\le y\\ \alpha(u)|q}} \frac{h(u)}{u^2}(u,q)s\left(\frac{\beta}{(u,q)}\right) + O\left(\frac{q}{y}(\log y)^{2r+1}\right). \tag{3.8}$$

Recall (cf. (2.3)) the existence of an integer B such that $h(p^i) = 0$ whenever $1 \le i < k$ and $(p, B) = 1$, and choose for a given $y \ge 1$ a squarefree natural number Q satisfying

$$(Q,B) = 1 \quad \text{and} \quad q := Q^k \le y. \tag{3.9}$$

Taking into account that $h(u) = 0$ whenever $\alpha(u)|q$, unless u is k-full, we may parametrize the integers u in (3.8) by $u = a^k b$, where a is a (necessarily squarefree) divisor of Q and $\alpha(b)|a$. Thus we obtain

$$\Sigma(y,q,\beta) = \Phi_q \sum_{a|Q} \frac{\mu(a)}{a^k} s\left(\frac{\beta}{a^k}\right) \sum_{\substack{b\le y/a^k\\ \alpha(b)|a}} \frac{|h(a^k b)|}{b^2} + O\left(\frac{q}{y}(\log y)^{2r+1}\right), \tag{3.10}$$

where we have used (2.4).

Now set $m := \Delta\varphi(M)$ and denote by $r_1,\dots,r_m$ representatives of the distinct residue classes $x \pmod M$ which satisfy $x^k \equiv -1 \pmod M$.

Let $t \ge t_0$ be a real parameter, and define

$$Q := \prod_{\substack{p\le t\\ (p,B)=1\\ p\equiv r_1,\dots,r_m \,(\mathrm{mod}\, M)}} p\,. \tag{3.11}$$

Determine N as the smallest natural number such that

$$N \ge 2 \quad \text{and} \quad L(N-1) < q = Q^k \le L(N)\,. \tag{3.12}$$

As (2.8) ensures that $\lim_{x\to\infty} L(x) = \infty$, N is well defined provided t_0 is large enough. With

$$y := 2N(\log N)^{-r} \tag{3.13}$$

it follows from (2.9) that $q \le y$ for large t, i.e. (3.9) is satisfied, and thus

(3.4), (3.5) and (3.10) may be combined to yield

$$(3.14)\qquad \sum_{n\le N} E(nq+\beta) = N\Phi_q \sum_{a\mid Q} \frac{\mu(a)}{a^k} s\Big(\frac{\beta}{a^k}\Big) \sum_{\substack{b\le y/a^k\\ \alpha(b)\mid a}} \frac{|h(a^k b)|}{b^2} + O(N)\,.$$

The influence of the factor Φ_q on the size and the sign of the right side of (3.14) is negligible since

$$|\Phi_q| \ge \Big|\sum_{n\ge 1}\frac{h(n)}{n^2}\Big|\Big(\sum_{n\ge 1}\frac{|h(n)|}{n^2}\Big)^{-1},$$

and the sign of Φ_q is constant for large t, as one sees upon consideration of the relevant Euler factors $\sum_{i\ge 0} h(p^i)p^{-2i}$. Thus without loss of generality we may suppose that Φ_q remains larger than a fixed positive constant.

To obtain the Ω_+-result for $E(x)$ we restrict the parameter t to the range of values for which $\mu(Q)=1$. With $\beta = q/M$ the conditions $0<\beta<q$ and $\beta\notin\mathbb{Z}$ are trivially satisfied.

If a divides Q then

$$\frac{\beta}{a^k} = \Big(\frac{Q}{a}\Big)^k \frac{1}{M} \quad\text{and}\quad \Big(\frac{Q}{a}\Big)^k \equiv \mu(a) \pmod{M}\,,$$

which implies that

$$\mu(a)s(\beta/a^k) = 1/2 - 1/M \ge 1/6\,.$$

Hence we deduce from (3.14) that

$$\begin{aligned}\sum_{n\le N} E(nq+\beta) &\gg N\sum_{a\mid Q} a^{-k} \sum_{\substack{b\le y/a^k\\ \alpha(b)\mid a}} |h(a^k b)| b^{-2} + O(N)\\ &\gg N\sum_{a\mid Q} |h(a^k)|a^{-k} + O(N)\,,\end{aligned}$$

whence

$$(3.15)\qquad \sum_{n\le N} E(nq+\beta) \gg N\prod_{p\mid Q}(1+|h(p^k)|p^{-k}) + O(N)\,.$$

Here we have used $a^k\le Q^k = q\le y$ to estimate from below each sum over b by $|h(a^k)|$.

Since $1+x\ge (1-x^2)e^x$ for $x\ge 0$, and in view of (2.6), (2.10), (2.11) and (3.11), we have

$$(3.16)\qquad \prod_{p\mid Q}(1+|h(p^k)|p^{-k}) \gg \exp\Big(\sum_{p\mid Q}|h(p^k)|p^{-k}\Big) \gg \exp(\Delta\cdot\Theta(t))\,.$$

The prime number theorem for arithmetic progressions gives

$$\log Q = \sum_{\substack{p\le t\\ p\equiv r_1,\dots,r_m \pmod M}} \log p + O(1) \sim \Delta t\,,$$

and therefore

$$\log\log Q = \log t + \log \Delta + o(1)\,. \tag{3.17}$$

Moreover, (2.9), (2.12) and (3.12) show that $q = Q^k \sim L(N)$, whence

$$\log\log Q = \log\log L(N) - \log k + o(1)\,. \tag{3.18}$$

Combining (3.17) and (3.18) we obtain

$$t \sim (k\Delta)^{-1} \log L(N)\,,$$

and thus by (3.15) and (3.16)

$$\sum_{n\le N} E(nq+\beta) \gg N \exp\left(\Delta\cdot \Theta\left(\frac{1+o(1)}{k\Delta}\log L(N)\right)\right). \tag{3.19}$$

The function $L^*(x)$ defined by

$$(L^*(x))^{-1} := (\log(x(\log x)^{-r}))^r\cdot\, \xi(x(\log x)^{-r})$$

is increasing for sufficiently large x and satisfies

$$\log L^*(x) = \log L(x) + o(1) \qquad (x\to\infty)\,.$$

Since $\Theta(x+O(1)) = \Theta(x)+o(1)$ it follows from (3.19) that

$$\sum_{n\le N} E(nq+\beta) \gg N \exp(\Delta\cdot\Theta((2\Delta k)^{-1}\log L^*(N)))\,. \tag{3.20}$$

As $nq+\beta \le N^2$ $(1\le n\le N)$ for large t, the relation

$$E(x) = o(\exp(\Delta\cdot\Theta((2\Delta k)^{-1}\log L^*(\sqrt{x}))))$$

or its equivalent

$$E(x) = o(\exp(\Delta\cdot\Theta((2\Delta k)^{-1}\log L(\sqrt{x}))))$$

would imply

$$\sum_{n\le N} E(nq+\beta) = o(N\exp(\Delta\cdot\Theta((2\Delta k)^{-1}\log L^*(N))))\,,$$

which contradicts (3.20). This proves the Ω_+-part of (2.15).

The same argument may be used to obtain the corresponding Ω_--result: one need only require t in (3.11) to run through values for which $\mu(Q) = -1$.

The estimate $E(x) \ll (\log x)^{r+1}$ follows immediately from (2.1), (3.1) and partial summation. This completes the proof of the theorem.

Proof of Corollary 2.3. From $f = I * h$ we infer

$$R(x) = -\frac{1}{2}x^2 \sum_{n>x} h(n)n^{-2} - x\sum_{n\le x}\frac{h(n)}{n}\left\{\frac{x}{n}\right\} + \frac{1}{2}x\sum_{n\le x}\frac{h(n)}{n} + \frac{1}{2}\sum_{n\le x} h(n)\left(\left\{\frac{x}{n}\right\}^2 - \left\{\frac{x}{n}\right\}\right).$$

Therefore (3.1) and the convergence of $\sum_{n\ge 1} h(n)\cdot n^{-1}$ yield

$$R(x) = xE(x) + O(x) + \frac{1}{2}\sum_{n\le x} h(n)\left(\left\{\frac{x}{n}\right\}^2 - \left\{\frac{x}{n}\right\}\right), \tag{3.21}$$

and consequently $R(x) \ll x(\log x)^{r+1}$ in view of (2.1) and (2.14).

Moreover, (2.1) and the assumption that $\xi(x) \ll (\log x)^{-1}$ yield

$$\begin{aligned}\Big|\sum_{n\le x} h(n)\left(\left\{\frac{x}{n}\right\}^2 - \left\{\frac{x}{n}\right\}\right)\Big| &\le \sum_{n\le\sqrt{x}} |h(n)| + \Big|x\int_{\sqrt{x}}^{x}\sum_{\sqrt{x}<n\le t} h(n)\left(2\left\{\frac{x}{t}\right\} - 1\right)t^{-2}\,dt\Big| \\ &\ll x^{3/4} + x\xi(\sqrt{x})\log x \ll x,\end{aligned}$$

since Abel's inequality gives

$$\Big|\sum_{\sqrt{x}<n\le t}\frac{h(n)}{n}n\Big| \le t\xi(\sqrt{x}).$$

Proof of Corollary 2.4. A comparison of formulas (3.1) and (3.21) shows that the assumption $\xi(x) = o(1/\log x)$ implies

$$R(x) = xE(x) + \tfrac{1}{2}b(h)x + o(x).$$

Therefore (2.20) follows from (2.19) by partial summation. To obtain (2.19) one may use the standard approach of Pillai and Chowla [14].

4. Applications. In some of the applications of Theorem 2.2 and its corollaries it is important to have estimates for sums involving iterates of the Möbius function.

LEMMA 4.1. *For $d \ge 2$ let $\mu_d := \mu_{d-1} * \mu$, where $\mu_1 := \mu$. Then for every $d \ge 1$ there is a positive constant c_d for which*

$$\sum_{n\le x}\mu_d(n)n^{-1} \ll_d \exp(-c_d\sqrt{\log x}).$$

Proof. By induction. The case $d = 1$ is the prime number theorem. Since $\mu_d(p^j) = (-1)^j\binom{d}{j}$, it follows that

$$\sum_{n\le x} |\mu_d(n)|n^{-1} \le \prod_{p\le x}\Big(\sum_{j\ge 0}|\mu_d(p^j)|p^{-j}\Big) \ll (\log x)^d\,.$$

The inductive step is therefore a consequence of the identity (cf. [1], Thm. 3.17),

$$\begin{aligned}\sum_{n\le x}\mu_d(n)n^{-1} = &\sum_{n\le\sqrt{x}}\mu_{d-1}(n)n^{-1}\sum_{m\le x/n}\mu(m)m^{-1}\\ &+\sum_{n\le\sqrt{x}}\mu(n)n^{-1}\sum_{m\le x/n}\mu_{d-1}(m)m^{-1}\\ &-\sum_{n\le\sqrt{x}}\mu_{d-1}(n)n^{-1}\sum_{n\le\sqrt{x}}\mu(n)n^{-1}\,.\end{aligned}$$

Our first application deals with Nagell's totient, which is defined for every natural j by

$$\theta(j,n) := n\prod_{p\mid n}\left(1-\frac{\varepsilon(j,p)}{p}\right)$$

where

$$\varepsilon(j,p) := \begin{cases}1 & \text{if } p\mid j,\\ 2 & \text{if } (p,j)=1.\end{cases}$$

Theorem 4.2. *For every positive integer j let*

$$\gamma(j) := \tfrac{1}{2}\prod_{p\mid j}(p^2-1)(p^2-2)^{-1}\prod_{p}(1-2p^{-2})\,.$$

Then

$$\sum_{n\le x}\theta(j,n) = \gamma(j)x^2 + R_j(x)$$

where

$$R_j(x) \ll x(\log x)^2$$

and

$$R_j(x) = \Omega_{\pm}(x\log\log x)\,.$$

Proof. Write $\theta(j,n) = I * h_j(n)$, where $h_j(p) := -\varepsilon(j,p)$ and $h_j(p^\alpha) := 0$ whenever $\alpha\ge 2$. A standard argument (cf. [5], Thm. 2) shows that

$$\sum_{n\le x}|h_j(n)| \ll \frac{x}{\log x}\prod_{p\le x}(1+|h_j(p)|p^{-1}) \ll x\log x\,,$$

whence $h_j \in \mathcal{C}(1,1)$.

In order to estimate $\sum_{x<n\le y} h_j(n)n^{-1}$, we factorize h_j as $h_j = \mu_2 * A_j$. The Euler product

$$\prod_p \Big(\sum_{\nu\ge 0} A_j(p^\nu)p^{-\nu s}\Big) = \prod_{p|j} \frac{1-p^{-s}}{1-2p^{-s}} \prod_p (1-(p^s-1)^{-2})$$

converges absolutely in $\mathrm{Re}\, s > 1/2$, and thus $\sum_{n\ge 1} A_j(n)n^{-1/2-\varepsilon}$ converges absolutely for every $\varepsilon > 0$.

Therefore by Lemma 4.1

$$\begin{aligned}\sum_{n\le x} h_j(n)n^{-1} &= \sum_{n\le\sqrt{x}} A_j(n)n^{-1} \sum_{m\le x/n} \mu_2(m)m^{-1} \\ &\quad + \sum_{\sqrt{x}<n\le x} A_j(n)n^{-1} \sum_{m\le x/n} \mu_2(m)m^{-1} \\ &\ll \exp(-c\sqrt{\log x})\end{aligned}$$

for some positive constant $c = c(j)$. Hence there exist constants $c_1 = c_1(j)$ and $c_2 = c_2(j)$ such that for $x > 0$ we have

$$\sup_{y>x} \Big| \sum_{x<n\le y} h_j(n)n^{-1} \Big| \le c_1 \exp(-c_2\sqrt{\log(1+x)}) =: \xi_j(x)\,.$$

Obviously $\xi_j(x)$ satisfies the assumptions of Corollary 2.3. Furthermore,

$$\Theta_j(x) = \sum_{p\le x} |h_j(p)|p^{-1} = 2\log\log x + O(1)\,,$$

and since $k = 1$ we may take $M = 3$ (which implies $\Delta = 1/2$), so (2.10) is fulfilled. As $\log L(\sqrt{x}) \gg \sqrt{\log x}$, we have

$$\Delta\cdot\Theta_j((2\Delta k)^{-1}\log L(\sqrt{x})) \ge \log\log\log x + O(1)$$

and Theorem 4.2 follows from Corollary 2.3.

In the same way we may also deal with Schemmel's totient, which is a multiplicative function defined for every natural j by

$$\Phi_j(p^\alpha) := \begin{cases} 0 & \text{if } p\le j, \\ p^\alpha(1-j/p) & \text{if } p > j. \end{cases}$$

Theorem 4.3. *For natural j let*

$$\lambda(j) := \tfrac{1}{2} \prod_{p\le j}(1-p^{-1}) \prod_{p>j}(1-jp^{-2})\,.$$

Then

$$\sum_{n\le x} \Phi_j(n) = \lambda(j)x^2 + R_j(x)$$

where

$$R_j(x) \ll x(\log x)^j$$

and

$$R_j(x) = \Omega_\pm(x(\log\log x)^{j/2})\,.$$

Proof. In this case we have $\Phi_j = I * h_j$, with

$$h_j(p^\alpha) := \begin{cases} 0 & \text{if } \alpha \ge 2, \\ -p & \text{if } \alpha = 1 \text{ and } p \le j, \\ -j & \text{if } \alpha = 1 \text{ and } p > j. \end{cases}$$

It is readily verified that $h_j \in \mathcal{C}(j-1,1)$. As before we factor h_j as $h_j = \mu_j * B_j$, where $\sum_{n\ge1} B_j(n)n^{-1/2-\varepsilon}$ converges absolutely for every $\varepsilon > 0$. In view of Lemma 4.1 we then obtain

$$\sup_{y>x}\Big|\sum_{x<n\le y} h_j(n)n^{-1}\Big| \ll \exp(-c\sqrt{\log x}) \tag{4.1}$$

for an appropriate constant $c = c(j) > 0$.

Again we may choose $M = 3$; since

$$\Delta\cdot\Theta_j(x) = \tfrac12\sum_{p\le x}|h_j(p)|p^{-1} = (j/2)\log\log x + O(1)$$

and $\log L(\sqrt{x}) \gg \sqrt{\log x}$, Corollary 2.3 yields the theorem.

As a further application of the results of Section 2 we will consider the multiplicative function φ_F defined with respect to an irreducible polynomial $F \in \mathbb{Z}[x]$ of degree $g \ge 1$ by

$$\varphi_F(n) := n\prod_{p\mid n}(1 - \varrho_F(p)/p)$$

where $\varrho_F(p)$ is the number of zeros of $F(x) \pmod p$. The verification of the premises of Theorem 2.2 and Corollary 2.3 is somewhat more arduous than in the first two examples and will be taken care of in a series of lemmas.

In the sequel $F(x) = a_g x^g + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ denotes a fixed irreducible polynomial of degree $g \ge 1$. Furthermore, let K be a splitting field of $F(x)/\mathbb{Q}$ and $\eta \in K$ a fixed zero of F. If we write $\varphi_F = I * h_F$, then

$$h_F(p^\alpha) = \begin{cases} -\varrho_F(p) & \text{if } \alpha = 1, \\ 0 & \text{if } \alpha \ge 2. \end{cases}$$

From Erdős ([3], Lemma 7) it follows that

$$\Theta_F(x) = \sum_{p\le x}|h_F(p)|p^{-1} = \sum_{p\le x}\varrho_F(p)p^{-1} = \log\log x + O(1)\,, \tag{4.2}$$

and thus (cf. [5], Thm. 2)

$$\sum_{n\le x}|h_F(n)| \ll \frac{x}{\log x}\prod_{p\le x}(1 + \varrho_F(p)/p) \ll x\,,$$

so that $h_F \in \mathcal{C}(0,1)$.

LEMMA 4.4. *For p unramified in $\mathbb{Q}(\eta)$, if a_g and the discriminant $\Delta(1,\eta,\ldots,\eta^{g-1})$ are p-adic units, then $\varrho_F(p)$ is the number of prime divisors of p of degree one in $\mathbb{Q}(\eta)$.*

Proof. For $a_g = 1$ the proof is well known (cf. [2], pp. 212–213). The general case is an immediate consequence of [7] (Thm. 7.6 and Prop. 7.7).

LEMMA 4.5. *There are positive constants $c_1 = c_1(F)$ and $c_2 = c_2(F)$ such that for $x > 0$*

$$\sup_{y>x}\Big|\sum_{x<n\le y} h_F(n)n^{-1}\Big| \le c_1\exp(-c_2(\log(1+x))^{1/12}).$$

Proof. By Lemma 4.4 there exists a positive integer D for which $\varrho_F(p)$ is the number of prime divisors of p of degree one in $\mathbb{Q}(\eta)$, whenever p does not divide D.

Let $\zeta_F(s) := \prod_{\mathfrak{p}}(1-N(\mathfrak{p})^{-s})^{-1}$ be the Dedekind zeta-function of $\mathbb{Q}(\eta)$, where $N(\mathfrak{p})$ denotes the norm of a prime ideal $\mathfrak{p}$ of $\mathbb{Q}(\eta)$. Then

$$H_F(s) := \sum_{n\ge1} h_F(n)n^{-s} = G_F(s)/\zeta_F(s),$$

where

$$\begin{aligned} G_F(s) &:= \sum_{n\ge1} b_F(n)n^{-s} \\ &= \prod_{p\mid D}(1-\varrho_F(p)p^{-s})\prod_{p\mid D}\prod_{\mathfrak{p}\mid p}(1-N(\mathfrak{p})^{-s})^{-1} \\ &\quad\times\prod_{p\nmid D}\prod_{\substack{\mathfrak{p}\mid p\\ f_{\mathfrak{p}}>1}}(1-N(\mathfrak{p})^{-s})^{-1}\prod_{p\nmid D}(1-\varrho_F(p)p^{-s})(1-p^{-s})^{-\varrho_F(p)} \end{aligned}$$

is absolutely convergent in $\operatorname{Re} s > 1/2$; here $f_{\mathfrak{p}}$ denotes the inertial degree of the prime ideal $\mathfrak{p}$. In particular, for every $\varepsilon > 0$

$$\sum_{\sqrt{x}<n\le x} |b_F(n)|n^{-1} \ll_\varepsilon x^{-1/4+\varepsilon}. \tag{4.3}$$

Writing $(\zeta_F(s))^{-1} = \sum_{n\ge1} a_F(n)n^{-s}$, we have (cf. Landau [10], pp. 80–89)

$$\sum_{n\ge1} a_F(n)n^{-1} = 0 \tag{4.4}$$

and

$$\sum_{n\le x} a_F(n) \ll x\exp(-c(\log x)^{1/12}) \tag{4.5}$$

for some positive constant $c = c(F)$.

Partial summation, (4.4) and (4.5) yield

$$\sum_{n\le x} a_F(n)n^{-1} \ll \exp(-c_1(\log x)^{1/12})\,. \tag{4.6}$$

The lemma now follows from (4.3), (4.6) and the identity

$$\sum_{n\le x} h_F(n)n^{-1} = \sum_{n\le\sqrt{x}} b_F(n)n^{-1} \sum_{m\le x/n} a_F(m)m^{-1} + \sum_{\sqrt{x}<n\le x} b_F(n)n^{-1} \sum_{m\le x/n} a_F(m)m^{-1}.$$

Lemma 4.6. *For a natural number M let ω_M be a primitive M-th root of unity and $\mathbb{Q}_M := \mathbb{Q}(\omega_M)$. If $\mathbb{Q}_M \cap K = \mathbb{Q}$, then for integers a relatively prime to M we have*

$$\sum_{\substack{p\le x\\ p\equiv a\,(\mathrm{mod}\,M)}} \varrho_F(p)p^{-1} = \frac{1}{\varphi(M)}\log\log x + O(1). \tag{4.7}$$

P r o o f. Denote by $\mathcal{G}al(K/\mathbb{Q})$ the Galois group of the extension $K/\mathbb{Q}$ and consider the decomposition $\mathcal{G}al(K/\mathbb{Q}) = \bigcup_{i=1}^r \Gamma_i$ into conjugation classes. For a rational prime p, unramified in K, let $\left[\dfrac{K/\mathbb{Q}}{(p)}\right]$ denote the conjugacy class of the Frobenius automorphism of any prime divisor $\mathfrak{p}$ of p. If D is defined as in the proof of Lemma 4.5, then for any p not dividing D, $\varrho_F(p)$ depends only upon $\left[\dfrac{K/\mathbb{Q}}{(p)}\right]$ (cf. [7], Ch. 3, Prop. 2.8), say $\varrho_F(p) = \gamma_i$ for $\left[\dfrac{K/\mathbb{Q}}{(p)}\right] = \Gamma_i$.

By assumption $\mathcal{G}al(K\mathbb{Q}_M/\mathbb{Q}) = \mathcal{G}al(K/\mathbb{Q}) \times \mathcal{G}al(\mathbb{Q}_M/\mathbb{Q})$. If τ_a is the element of $\mathcal{G}al(\mathbb{Q}_M/\mathbb{Q})$ such that $\tau_a(\omega_M) = \omega_M^a$, then we have the following decomposition into conjugation classes:

$$\mathcal{G}al(K\mathbb{Q}_M/\mathbb{Q}) = \bigcup_{i=1}^{r} \bigcup_{\substack{a\,(\mathrm{mod}\,M)\\ (a,M)=1}} \Gamma_i \times \{\tau_a\}.$$

Since $\left[\dfrac{K\mathbb{Q}_M/\mathbb{Q}}{(p)}\right] = \Gamma_i \times \{\tau_a\}$ implies $p \equiv a \pmod M$ and $\left[\dfrac{K/\mathbb{Q}}{(p)}\right] = \Gamma_i$, that is, $\varrho_F(p) = \gamma_i$, we have

$$\sum_{\substack{p\le x\\ p\equiv a\,(\mathrm{mod}\,M)}} \varrho_F(p) = \sum_{i=1}^{r} \gamma_i \cdot \pi_{(i,a)}(x) + O(1)\,, \tag{4.8}$$

where $\pi_{(i,a)}(x)$ is the number of primes p not exceeding x for which $\left[\frac{K\mathbb{Q}_M/\mathbb{Q}}{(p)}\right] = \Gamma_i \times \{\tau_a\}$.

By Chebotarev's density theorem with error term (cf. [9]), (4.8) implies that

$$\sum_{\substack{p\le x\\ p\equiv a\ (\mathrm{mod}\ M)}} \varrho_F(p) = \lambda\cdot \mathrm{li}(x) + O(x\exp(-c\sqrt{\log x})), \tag{4.9}$$

where the constant

$$\lambda := [K\mathbb{Q}_M:\mathbb{Q}]^{-1}\cdot\sum_{i=1}^{r}|\Gamma_i|\gamma_i$$

is independent of a. Partial summation in (4.9), gives

$$\sum_{\substack{p\le x\\ p\equiv a\ (\mathrm{mod}\ M)}} \varrho_F(p)p^{-1} = \lambda\log\log x + O(1)\,,$$

and a comparison with (4.2) yields $\lambda = 1/\varphi(M)$, which proves (4.7).

Using the previous two lemmas we can now easily prove

THEOREM 4.7. *For an irreducible nonconstant polynomial $F\in\mathbb{Z}[x]$ let*

$$\varphi_F(n) := n\prod_{p\mid n}(1-\varrho_F(p)/p),$$

where $\varrho_F(p)$ is the number of zeros of $F \pmod p$. If

$$c_F := \tfrac12\prod_{p}(1-\varrho_F(p)p^{-2})$$

and q denotes the smallest odd prime that is unramified in a splitting field K of $F(x)$, then

$$\sum_{n\le x}\varphi_F(n) = c_Fx^2 + R_F(x)$$

where

$$R_F(x) \ll x\log x$$

and

$$R_F(x) = \Omega_\pm(x(\log\log x)^{1/(q-1)}).$$

Proof. Recall that $\varphi_F = I * h_F$ with $h_F\in\mathcal{C}(0,1)$. By Lemma 4.5 there are positive constants c_1 and c_2 such that

$$\sup_{y>x}\Big|\sum_{x<n\le y} h_F(n)n^{-1}\Big| \le c_1\exp(-c_2(\log(1+x))^{1/12}) =: \xi_F(x)\,.$$

Obviously ξ_F satisfies the assumptions of Corollary 2.3.

Since q is totally ramified in $\mathbb{Q}_q$, we have $\mathbb{Q}_q \cap K = \mathbb{Q}$. Lemma 4.6 and formula (4.2) show that

$$\sum_{\substack{p \le x \\ p \equiv a \,(\mathrm{mod}\, q)}} \varrho_F(p) p^{-1} = \frac{1}{\varphi(q)} \Theta_F(x) + O(1) = \frac{1}{q-1} \log\log x + O(1)\,.$$

An application of Corollary 2.3 yields the proof.

Up to this point our examples have dealt with functions $I * h$, where $h \in \mathcal{C}(r,1)$ for some nonnegative r. In closing we will therefore bring an application of Corollary 2.3 which involves the class $\mathcal{C}(0,2)$. The relevant function f is defined by

$$f(n) := \sum_{\substack{d \mid n \\ (d,n/d)=1}} \varphi(d);$$

$f(n)$ is the number of integers possessing weak order $\pmod n$ (cf. [8]). In this case $f = I * h$ where

$$h(p^\alpha) := \begin{cases} 0 & \text{if } \alpha = 1, \\ 1-p & \text{if } \alpha \ge 2. \end{cases}$$

It can be seen without too much difficulty that $h \in \mathcal{C}(0,2)$ and it can be shown that

$$\sup_{y>x} \Big| \sum_{x<n\le y} h(n) n^{-1} \Big| \ll \exp(-c\sqrt{\log x})$$

(cf. [6]). Hence Corollary 2.3 gives

$$\sum_{n \le x} f(n) = \Big(\frac{1}{2} \sum_{n \ge 1} h(n) n^{-2}\Big) x^2 + R(x)$$

where $R(x) \ll x \log x$ and $R(x) = \Omega_\pm(x\sqrt{\log\log x})$.

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