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Uniform distribution preserving mappings

by

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1. Introduction. A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers $x_n \in [0,1)$ is called *uniformly distributed* $\bmod[0,1)$ if and only if every interval $[a,b) \subseteq [0,1)$ contains $x_1,\ldots,x_N$ with an asymptotic frequency corresponding to its length, that means,

$$\lim_{N\to\infty} \frac{1}{N} |\{n \le N \mid x_n \in [a,b)\}| = b-a \tag{1.1}$$

for all $0 \le a \le b \le 1$, or equivalently

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, d\lambda(x) \tag{1.2}$$

(λ denoting the Lebesgue measure) for every f of the form $f = \chi_{[a,b)}$, $0 \le a \le b \le 1$. (The characteristic function χ_A of a set A is defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.) It is well known (cf. [KN], [Hl]) that the following statements are equivalent:

(i) $(x_n)_{n\in\mathbb{N}}$ is uniformly distributed $\bmod[0,1)$, i.e. (1.2) holds for every $f = \chi_{[a,b)}$, $0 \le a \le b \le 1$.

(ii) (1.2) holds for every continuous f, i.e. $f \in C([0,1))$.

(iii) (1.2) holds for every Riemann-integrable f.

(iv) (1.2) holds for every f of the form $f(x) = e(kx)$ with $e(y) = e^{2\pi i y}$ and $k \in \mathbb{Z}$ (Weyl criterion).

This leads to a more general definition: A sequence $(x_n)_{n\in\mathbb{N}}$ in a compact space X is called *uniformly distributed with respect to the normalized Borel measure* μ if and only if (1.2) holds for every $f \in C(X)$ (with μ instead of λ). A more special property is well distribution: With the same notation as above a sequence is called *well distributed* if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_{n+s}) = \int_X f(x)\, d\mu(x) \tag{1.3}$$

uniformly in $s = 0, 1, 2, \ldots$ for every $f \in C(X)$. A further way of generalization is that of using limitation methods.

The infinite matrix $A = (a_{n,k})_{n,k\in\mathbb{N}}$ with real entries $a_{n,k}$ is called a *positive regular limitation method* if

(i) $a_{n,k} \ge 0$,
(ii) $\lim_{n\to\infty} a_{n,k} = 0$ for every $k \in \mathbb{N}$,
(iii) $\sum_{k=1}^{\infty} a_{n,k} < \infty$ for every $n \in \mathbb{N}$,
(iv) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$.

The sequence $(x_n)_{n\in\mathbb{N}}$ in X is called (A,μ)-*uniformly distributed* if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} f(x_k) = \int_X f(x)\, d\mu(x) \tag{1.4}$$

for every $f \in C(X)$. If we consider the case $a_{n,k} = 1/n$ for $k \le n$ and $a_{n,k} = 0$ for $k > n$ we obtain the standard concept of uniform distribution with respect to μ. If $a_{n,k} = p_k/P(n)$ with $p_k \ge 0$ and $P(n) = \sum_{k=1}^{n} p_k \to \infty$, then the $a_{n,k}$ are called *weighted means*.

In [PSS], [Bo] the authors have investigated uniform distribution preserving mappings (for short: u.d.p. mappings) f, i.e. maps generating uniformly distributed sequences $(f(x_n))_{n\in\mathbb{N}}$ for every uniformly distributed sequence $(x_n)_{n\in\mathbb{N}}$. They established some general results in case of uniform distribution $\bmod[0,1)$ and some special results for piecewise differentiable or piecewise linear f. In Section 2 we give generalizations of their results. We are concerned with compact spaces with regular Borel measures, limitation methods and well distribution. Section 3 contains some results demonstrating for instance that the restriction in [PSS] to piecewise differentiable functions is very restrictive in one sense, but in another sense even restriction to piecewise linear functions does not change very much: The set of continuous piecewise linear u.d.p. mappings is dense in the set of all continuous u.d.p. mappings on $[0,1]$ with respect to the topology of uniform convergence.

2. General results. In this section we establish some general criteria on u.d.p. transformations on compact metric spaces (X,d). For technical reasons we will assume two further conditions, one on the measure μ and the other one on the limitation method A:

$$\mu(B) = \mu(\overline{B}) \tag{M}$$

for every (open) ball B of positive radius, and

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k} - a_{n+1,k}| = 0\,. \tag{L}$$

Condition (L) is due to Lorentz [Lo]; it is sufficient for the existence of an (A,μ)-uniformly distributed sequence (cf. [De]). Furthermore we recall the definition of a Jordan-measurable function $g : X \to \mathbb{R}$: g is called (μ-)*Jordan-measurable* if it is continuous μ-almost everywhere.

THEOREM 2.1. *Let X be a compact metric space, μ a regular normalized Borel measure on X and A a positive regular limitation method such that the additional properties* (M) *and* (L) *are satisfied. Then a transformation $f : X \to X$ is μ-u.d.p. if and only if for every Jordan-measurable function $g : X \to \mathbb{R}$ the composition $g \circ f$ also is Jordan-measurable and*

$$\int\limits_X g(x)\,d\mu(x) = \int\limits_X g(f(x))\,d\mu(x)\,. \tag{2.1}$$

Proof. Let f be (A,μ)-u.d.p. and let (x_n) be an (A,μ)-uniformly distributed sequence on X. Then $(f(x_n))$ is also (A,μ)-uniformly distributed and for an arbitrary Jordan-measurable function $g : X \to \mathbb{R}$ we have

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}g(x_k) = \int\limits_X g(x)\,d\mu(x) = \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}g(f(x_k))\,.$$

By [Bi] $g \circ f$ is Jordan-measurable. Thus we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}g(f(x_k)) = \int\limits_X g(f(x))\,d\mu(x)\,,$$

and property (2.1) is proved.

Now assume that f is not u.d.p. Then there exists an (A,μ)-uniformly distributed sequence (x_n) such that $(f(x_n))$ is not (A,μ)-uniformly distributed. This means that there exists a Jordan-measurable function $g : X \to \mathbb{R}$ such that

$$\sum_{k=1}^{\infty} a_{n,k}g(f(x_k)) \tag{2.2}$$

does not converge to $\int_X g(x)\,d\mu(x)$ (for $n \to \infty$). Since we may suppose that $g \circ f$ is Jordan-measurable (2.2) necessarily converges to

$$\int\limits_X g(f(x))\,d\mu(x)$$

which contradicts (2.1) and the theorem is proved.

Remark. The composition of two Jordan-measurable functions is not automatically Jordan-measurable. In [Ma] it has been shown that every bounded function on $[0,1]$ is the composition of two Riemann-integrable, i.e. Jordan-measurable functions. In the following we will give a slightly different proof of this remarkable fact for functions on the s-dimensional unit cube $U = [0,1)^s$.

Let $e_1, \ldots, e_{2^s}$ be the vertex points of U and let $Q_1, \ldots, Q_{2^s}$ be the cubes $Q_j = [a_1, a_1 + 1/2) \times \ldots \times [a_s, a_s + 1/2)$ with $a_j = 0$ or $1/2$ such that e_j is a vertex point of Q_j. Then the cubes Q_j form a partition of U. We define a transformation $\tau_1 : U \to U$ by $\tau_1(x) = \frac{1}{2}(x + e_j)$ for $x \in Q_j$. The image $\operatorname{Im}\tau$ consists of 2^s cubes V_j of side length $1/4$. In a second step we consider a fixed cube V_j (instead of U) and define a transformation $\tau_{2,j} : V_j \to V_j$ in the same way as the transformation τ_1 above. Doing so for all cubes V_j we get an image consisting of 2^{2s} cubes. After n steps we have a transformation $\tau_n : U \to U$ such that the image $\operatorname{Im}\tau_n$ consists of 2^{ns} cubes of side length $1/4^n$. Now we set $\tau(x) = \lim_{n\to\infty} \tau_n(x)$. By construction, τ is injective. Furthermore, $\lambda(\operatorname{Im}\tau_n) = 2^{-ns}$, thus $\lambda(\operatorname{Im}\tau) = 0$. Clearly, the closure of $\operatorname{Im}\tau$ is a null-set, too. For a given bounded function f on U we define $\psi(y) = f(x)$ if $y = \tau(x) \in \operatorname{Im}\tau$ and $\psi(y) = 0$ if $y \notin \operatorname{Im}\tau$. Obviously, $f(x) = \psi(\tau(x))$. Thus f is the composition of two Jordan-measurable functions τ and ψ. Using coverings with open sets and coordinate mappings the above construction can be easily generalized to functions on manifolds.

In the following we extend Theorem 2.1 to mappings that preserve well distribution. One basic tool in the proof was Binder's generalization of De Bruijn and Post's [BP] classical result. We need an extension of this result to μ-well distributed (for short: μ-w.d.) sequences.

Lemma 1. *In every compact metric space X there exists a μ-w.d. sequence (a_n).*

This is a well-known result of Baayen and Hedrlin [BH].

Lemma 2. *Let (a_n) be μ-w.d. in (X, d) and let (x_n) be a sequence such that $\lim_{n\to\infty} d(a_n, x_n) = 0$. Then (x_n) is also μ-w.d.*

Proof. Let $f \in C(X)$ with $\int_X f\,d\mu = 0$; f is bounded: $|f| \le M$ and uniformly continuous on the compact space X. Thus for a given $\varepsilon > 0$ there exists n_0 such that $|f(x_n) - f(a_n)| < \varepsilon$ for all $n \ge n_0$. Then

$$\left|\frac{1}{N}\sum_{n=s+1}^{s+N} f(x_n)\right| \le \left|\frac{1}{N}\sum_{n=s+1}^{s+N} f(a_n)\right| + \frac{2Mn_0}{N} + \varepsilon\,.$$

Choosing N sufficiently large yields the desired result.

The following lemma is proved in [Bi], Hilfssatz 2.

Lemma 3. *Let (x_n) and (y_n) be two sequences of reals such that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n = a \quad \textit{and} \quad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} y_n = b$$

with $a \neq b$. Then there exists a sequence (z_n) with $z_n = x_n$ or $z_n = y_n$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} z_n$$

does not exist.

PROPOSITION 2.2. *Let $g : X \to \mathbb{R}$ be a function that is not Jordan-measurable. Then there exists a μ-w.d. sequence (z_n) on X such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(z_n)$$

does not exist.

Sketch of proof. We suppose that the lower integral $\underline{\mu}(g)$ is different from the upper integral $\overline{\mu}(g)$. By similar arguments to [Bi], Hilfssatz 5 (using the fixed μ-w.d. sequence (a_n) of Lemma 1) there exist two sequences (x_n) and (y_n) in X such that:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(x_n) = \underline{\mu}(g)\,, \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(y_n) = \overline{\mu}(g)\,,$$
$$\lim_{n\to\infty} d(x_n, a_n) = \lim_{n\to\infty} d(y_n, a_n) = 0\,.$$

Thus, by Lemma 2, (x_n) and (y_n) are μ-w.d. Applying Lemma 3 we obtain a μ-w.d. sequence (z_n) such that the above limit does not exist.

An immediate consequence of the proof of Theorem 2.1 and Proposition 2.2 is the following

THEOREM 2.3. *Let X be a compact metric space, and μ a regular normalized Borel measure on X with property* (M). *Then a transformation $f : X \to X$ preserves μ-well distribution if and only if for every Jordan-measurable function $g : X \to \mathbb{R}$ the composition $g \circ f$ is also Jordan-measurable and*

$$\int_X g(x)\, d\mu(x) = \int_X g(f(x))\, d\mu(x)\,.$$

Remark. Clearly the class of u.d.p. mappings coincides with the class of well distribution preserving mappings.

Remark. An analogue to Theorems 2.1 and 2.3 for completely uniform distribution (cf. [KN]) can be given by similar arguments.

Further generalizations of the results given in [PSS] can be given. As an example we state

PROPOSITION 2.4. *A mapping $f : X \to X$ is (A, μ)-u.d.p. if and only if*

(i) *f is Jordan-measurable and*
(ii) *$\mu(f^{-1}(M)) = \mu(M)$ for every μ-continuity set $M \subseteq X$.*

3. Further results on uniform distribution mod$[0, 1)$**.** In this section we restrict our investigations to the classical case of uniform distribution on the unit interval. In [PSS] a classification of piecewise differentiable u.d.p. mappings $f : [0, 1] \to [0, 1]$ is given. An answer to the question whether there are continuous u.d.p. mappings that are not piecewise differentiable is given by

THEOREM 3.1. *There exist continuous functions f on $[0, 1]$ that are nowhere differentiable and u.d.p.*

Proof. We introduce the following abbreviation:

$$\mathcal{A} = \Big\{ \bigcup_{k=1}^{n} [a_k, b_k] \;\Big|\; n \in \mathbb{N},\ I_k = [a_k, b_k] \subseteq [0, 1] \Big\}.$$

For an arbitrary subset A of a topological space (in our case the unit interval $[0, 1]$) let $\overline{A}$ be the topological closure, A° the interior. Furthermore let λ denote the Lebesgue measure and let

$$D = \{k/2^n \mid n \in \mathbb{N},\ k \in \{0, 1, \ldots, 2^n\}\}.$$

Without giving a formal proof we now note an obvious fact.

Claim 1. Let $A_1, A_2 \in \mathcal{A}$, $A_1 \subseteq A_2^\circ$, $\lambda(A_1) < \alpha < \lambda(A_2)$ and $\varepsilon > 0$. Then there exists an $A \in \mathcal{A}$ such that $A_1 \subset A^\circ$, $A \subset A_2^\circ$, $\lambda(A) = \alpha$ and for all $x \in A_2 \setminus A_1$ there are y with $|y - x| < \varepsilon$ contained in A as well as in the complement.

Construction of an f: Starting with $M_0 = \{0, 1\}$ and $M_1 = [0, 1]$ we define inductively M_α for every $\alpha \in D$. Let $\alpha = k/2^n$ with k odd. Then set $M_\alpha = A$ such that (with $A_1 = M_{\alpha - 1/2^n}$, $A_2 = M_{\alpha + 1/2^n}$ and $\varepsilon = 2^{-2n}$) the assertion of Claim 1 holds. Now we define

$$f(x) = \inf\{\alpha \in D \mid x \in M_\alpha\} = \sup\{\alpha \in D \mid x \notin M_\alpha\}.$$

Note that $M_\alpha = \{x \mid f(x) \le \alpha\}$ and f is continuous because

$$\begin{aligned}\{x \mid f(x) < \alpha_0\} &= \{x \mid \inf\{\alpha \in D \mid x \in M_\alpha\} < \alpha_0\} \\ &= \{x \mid \exists\, \alpha \in D : \alpha < \alpha_0,\ x \in M_\alpha\} \\ &= \bigcup \{M_\alpha \mid \alpha < \alpha_0,\ \alpha \in D\} = \bigcup \{M_\alpha^\circ \mid \alpha < \alpha_0,\ \alpha \in D\}\end{aligned}$$

is an open set for all $\alpha_0 \in [0, 1]$.

f is nowhere differentiable: Take $x \in [0,1]$ and $K \in \mathbb{R}$ arbitrarily. It suffices to find a $y \neq x$ such that

$$\left|\frac{f(y)-f(x)}{y-x}\right| > K. \tag{3.1}$$

To do this we take an n with $2^{n-1} > K$ and define α_1 and α_2 by

$$\alpha_1 = k/2^n < f(x) \leq (k+1)/2^n = \alpha_2, \quad k \in \{0,\ldots,2^n-1\}.$$

(For $f(x)=0$ we argue similarly to Case 1 below.) Suppose w.l.o.g. $f(x) \leq \alpha = \alpha_1 + l/2^{n+1} \in (\alpha_1,\alpha_2)$. (The other case is treated quite similarly.) With $\beta = \alpha_1 + m/2^{n+1}$ we have to distinguish two cases.

Case 1: $\alpha_1 < f(x) \leq \beta < \alpha < \alpha_2$. Since $x \in M_{\alpha_2} \setminus M_{\alpha_1}$ there is a $y \notin M_\alpha$ with $|y-x| < 2^{-2n-2}$. Therefore $f(y)-f(x) > \alpha-\beta = 2^{-n-2}$ and $|y-x| < \varepsilon = 2^{-2n-2}$, which implies (3.1).

Case 2: $\alpha_1 < \beta < f(x) \leq \alpha < \alpha_2$. Since $x \in M_\alpha \setminus M_{\alpha_1} \subseteq M_{\alpha_2} \setminus M_{\alpha_1}$ there exist $y_1 \in M_\beta$, $y_2 \notin M_\alpha$ with $|y_i - x| < 2^{-2(n+1)}$, $i=1,2$. Therefore $f(y_1) \leq \beta$ and $f(y_2) > \alpha$. With an easy geometric consideration we conclude

$$\sup\left\{\left|\frac{f(y)-f(x)}{y-x}\right| \,\middle|\, y \neq x\right\} \geq \left|\frac{f(y_2)-f(y_1)}{2^{-2n-1}}\right| > 2^{2n+1}(\alpha-\beta) > 2^{n-1} \geq K.$$

f preserves uniform distribution: For every interval I of the form $I=[0,\alpha]$, $\alpha \in D$, and every uniformly distributed sequence (x_n) we have

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\{n \leq N \mid f(x_n) \in I\}| &= \lim_{N\to\infty}\frac{1}{N}|\{n \leq N \mid x_n \in M_\alpha\}| \\ &= \lambda(M_\alpha) = \alpha = \lambda(I)\end{aligned}$$

(note $M_\alpha \in \mathcal{A}$). Since D is dense in $[0,1]$ this statement must hold for arbitrary $I=[a,b] \subseteq [0,1]$.

Thus the proof of Theorem 3.1 is complete.

Although Theorem 3.1 guarantees an abundance of continuous but nowhere differentiable u.d.p. mappings, in a topological sense the set of these mappings is not much larger than even the set of continuous piecewise linear u.d.p. mappings, described in a very satisfactory manner in [PSS].

THEOREM 3.2. *The set of all continuous u.d.p. mappings is the topological closure* (*with respect to uniform convergence*) *of the set of all continuous piecewise linear u.d.p. mappings.*

Proof. By [PSS], Proposition 2, the uniform limit of uniform distribution preserving mappings still has this property. This shows one set-theoretical inclusion.

For the other inclusion let f be continuous on [0,1] and u.d.p., $\varepsilon > 0$ an arbitrary positive real. We have to construct a mapping l of [0,1] onto itself which is continuous, piecewise linear, u.d.p. and satisfying $|l(x) - f(x)| < \varepsilon$ for all $x \in [0, 1]$.

Since $[0, 1]$ is compact f is uniformly continuous, hence for a fixed $n > 2/\varepsilon$, i.e. $1/n < \varepsilon/2$, there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1/n$ for all $x, y \in [0, 1]$. With $D = \{\alpha = k/n \mid k \in \{0, \ldots, n\}\}$ the fact that every open set of reals is a countable union of open intervals gives a representation of the form

$$f^{-1}([0,1] \setminus D) = [0, y_0) \cup \bigcup_{n \in \mathbb{N}} I_n \cup (x_0, 1]$$

with $I_n = (a_n, b_n)$, $a_n, b_n \in [0, 1]$. Write $K_i = (x_i, y_i)$ for those I_n for which $f(x_i) \neq f(y_i)$, thus $|f(x_i) - f(y_i)| = 1/n$. Therefore we have $y_i - x_i > \delta$ and the number of the K_i is bounded by $1/\delta$, hence it is finite. Now we consider the following partition of $[0, 1]$:

$$R_1 = [0, y_0) < A_1 = [y_0, x_1] < K_1 = (x_1, y_1) < \ldots$$
$$\ldots < A_m = [y_{m-1}, x_m] < R_2 = (x_m, 1]\,.$$

The following facts are obvious:

(i) $f(y_{i-1}) = f(x_i)$ for $i = 1, \ldots, m$,
(ii) $f(A_i) \subseteq (f(x_i) - 1/n, f(x_i) + 1/n)$, $i = 1, \ldots, m$,
(iii) $f(K_i) = (f(x_i), f(y_i))$ resp. $(f(y_i), f(x_i))$ for $i = 1, \ldots, m - 1$,
(iv) $f(R_1) \subseteq (f(y_0) - 1/n, f(y_0)]$ resp. $[f(y_0), f(y_0) + 1/n)$,
(v) $f(R_2) \subseteq (f(x_0) - 1/n, f(x_0)]$ resp. $[f(x_0), f(x_0) + 1/n)$.

Construction of l. On K_i: Linear connection of the points $(x_i, f(x_i))$ and $(y_i, f(y_i))$, $i = 1, \ldots, m - 1$.

On A_i: Linear connection of the points $(y_{i-1}, f(y_{i-1}))$, $(y_{i-1} + \varepsilon_1^{(i)}, f(y_{i-1}) + 1/n)$, $(y_{i-1} + \varepsilon^{(i)}, f(y_{i-1}))$, $(y_{i-1} + \varepsilon_2^{(i)}, f(y_{i-1}) - 1/n)$ and $(x_i, f(x_i))$ under the (possible) restrictions

$$0 < \varepsilon_1^{(i)} < \varepsilon^{(i)} < \varepsilon_2^{(i)} < x_i - y_{i-1} = \delta^{(i)}$$

and

$$\lambda(f^{-1}((f(y_{i-1}), f(y_{i-1}) + 1/n)) \cap A_i) = \varepsilon^{(i)} \lambda(A_i)\,,$$

$i = 1, \ldots, m$.

On R_1: Linear connection of the points

$$(0, f(y_0)), \quad (y_0/2, f(y_0) + 1/n) \quad \text{and} \quad (y_0, f(y_0))$$

in case $f(R_1) \geq f(y_0)$, and of the points

$$(0, f(y_0)), \quad (y_0/2, f(y_0) - 1/n) \quad \text{and} \quad (y_0, f(y_0))$$

in case $f(R_1) \leq f(y_0)$.

On R_2: As on R_1 *mutatis mutandis*.

P r o o f o f t h e p r o p e r t i e s o f l:

l is linear on each interval R_i, K_i and A_i and continuous at the points x_i and y_i, therefore it is continuous and piecewise linear on the unit interval.

l preserves uniform distribution: Let $J_j = ((j-1)/n, j/n)$, $j = 1, \ldots, n$. Then the inverse image of such an interval has the representation $l^{-1}(J_j) = \bigcup_{i=1}^{n_j} I_{j,i}$ with pairwise disjoint open intervals $I_{j,i}$ of the form $I_{j,i} = l^{-1}(J_j) \cap I^{(j,i)}$, where $I^{(j,i)}$ is one of the R_i, A_i or K_i. By Proposition 7 in [PSS] we have to show

$$\lambda(J_j) = \sum_{i=1}^{n_j} \lambda(I_{j,i}) \quad \text{for } j = 1, \ldots, n\,.$$

Indeed, we have—using that f preserves uniform distribution and applying Theorem 4 in [PSS]—

$$\lambda(f^{-1}(J_j)) = \lambda(J_j)\,.$$

Furthermore, by construction

$$\lambda(f^{-1}(J_j) \cap I) = \lambda(l^{-1}(J_j) \cap I)$$

for every $j = 1, \ldots, n$ and $I = R_i, A_i$ or K_i. Hence

$$\begin{aligned}\lambda(J_j) &= \lambda(f^{-1}(J_j)) = \lambda\Big(\bigcup_I (f^{-1}(J_j) \cap I)\Big) \\ &= \sum_I \lambda(f^{-1}(J_j) \cap I) = \sum_I \lambda(l^{-1}(J_j) \cap I) = \sum_{i=1}^{n_j} \lambda(I_{j,i})\,.\end{aligned}$$

Thus $|l(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$ by the construction of l and by observations (i)–(v) and the proof of Theorem 3.2 is complete.

THEOREM 3.3. *There are u.d.p. mappings which cannot be represented as a uniform limit of piecewise linear* (*not necessarily continuous*) *mappings.*

P r o o f. C l a i m 1: The uniform limit f of piecewise continuous maps l_n, $n \in \mathbb{N}$, is continuous with the exception of not more than countably many points.

P r o o f o f c l a i m 1. With

$U_f = \{x \mid f \text{ is not continuous in } x\}$ and $U_n = \{x \mid l_n \text{ is not continuous in } x\}$

by uniform convergence we have

$$U_f \subseteq \bigcup_{n \in \mathbb{N}} U_n\,,$$

which proves claim 1 because every U_n is even finite.

C l a i m 2: Let N be an arbitrary closed set of Lebesgue measure 0. Then the mapping $f : [0,1] \to [0,1]$ with $f(x) = 0$ for $x \in N$ and $f(x) = x$

otherwise preserves uniform distribution and satisfies $U_f = N$ (or $U_f = N \setminus \{0\}$).

Proof of claim 2. By [PSS], Theorem 2a, the relation

$$\int_0^1 g(f(x))\,dx = \int_N g(f(x))\,dx + \int_{[0,1]\setminus N} g(f(x))\,dx$$
$$= 0 + \int_{[0,1]\setminus N} g(x)\,dx = \int_0^1 g(x)\,dx$$

for every continuous $g : [0,1] \to \mathbb{R}$ proves claim 2 since the assertion on U_f is obvious.

It is well known that there exist closed sets N with $\lambda(N) = 0$ that are not countable. Hence claims 1 and 2 together give Theorem 3.3.

A very natural question is whether also pointwise convergence gives similar results in our context. The following two theorems show that pointwise convergence (even "convergence almost everywhere") preserves the property of measure preservation but not that of uniform distribution preservation, a further hint that uniform distribution is strongly connected with continuity and topology, and not only with measure theory. The following result is well known; we present a proof for completeness.

PROPOSITION 3.4. *Let the mappings $f_n : [0,1] \to [0,1]$, $n \in \mathbb{N}$, be measure preserving, i.e. $\lambda(f_n^{-1}(M)) = \lambda(M)$ for every measurable M, and let f_n converge to f almost everywhere. Then f is also measure preserving.*

Proof. By a standard measure-theoretical argument it suffices to show $\lambda(f^{-1}(a,b)) = b - a$ for arbitrary $0 \le a \le b \le 1$. Let $K = \{x \mid f_n(x) \to f(x)\}$ denote the set of convergence. By assumption we have $\lambda(K) = 1$. For $x \in K$ the statements "$x \in f^{-1}(a,b)$", "$f(x) \in (a,b)$" and therefore "there exists an $N \in \mathbb{N}$ such that for all $n \ge N$ $a < f_n(x) < b$" are equivalent. Hence

$$\lambda(f^{-1}(a,b)) \le \lambda\Big(\bigcup_{N\in\mathbb{N}} \bigcap_{n\ge N} f_N^{-1}((a,b))\Big)$$
$$= \lim_{N\to\infty} \lambda\Big(\bigcap_{n\ge N} f_n^{-1}((a,b))\Big) \le \lim_{N\to\infty} \lambda(f_N^{-1}((a,b))) = b - a\,.$$

For closed intervals this implies

$$\lambda(f^{-1}([a,b])) \le \lambda(f^{-1}((a-\varepsilon, b+\varepsilon))) \le b - a + 2\varepsilon$$

for every $\varepsilon > 0$, hence

$$\lambda(f^{-1}([a,b])) \le b - a$$

for all closed intervals. From these estimates the desired equality follows because

$$\lambda(f^{-1}([0,a])) + \lambda(f^{-1}((a,b))) + \lambda(f^{-1}([b,1])) = \lambda(f^{-1}([0,1])) = \lambda([0,1]) = 1.$$

THEOREM 3.5. *The pointwise limit of a sequence of u.d.p. mappings does not necessarily preserve uniform distribution.*

P r o o f. Let $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$ be an enumeration of the rationals. The sequence $(f_n)_{n\in\mathbb{N}}$ defined by $f_n(x) = 0$ for $x \in \{q_1, \ldots, q_n\}$, $f_n(x) = x$ otherwise, converges pointwise to $f(x) = 0$ for $x \in \mathbb{Q}$, $f(x) = x$ otherwise. Quite similarly to the proof of Theorem 3.3 one can show that every f_n preserves uniform distribution. But f does not because it is not Riemann-integrable (cf. Theorem 2.3 or [PSS], Theorem 1).

If one asks, on the other hand, whether there is a small class of u.d.p. mappings which produces all such mappings by taking limits almost everywhere one gets an affirmative answer.

THEOREM 3.6. *Let f be a u.d.p. mapping. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of u.d.p. mappings which are piecewise linear and bijective on the unit interval* $[0,1]$ *onto itself such that f_n converges to f almost everywhere.*

P r o o f. We construct f_n in such a way that there exists a set A_n with $\lambda(A_n) \le 1/2^n$ and $|f(x) - f_n(x)| < 1/2^n$ for all $x \notin A_n$. For the set N of points x where $f_n(x)$ does not converge to $f(x)$ we get

$$0 \le \lambda(N) \le \lambda\Big(\bigcap_{k\in\mathbb{N}} \bigcup_{n\ge k} A_n\Big) \le \lim_{k\to\infty} \sum_{n\ge k} \lambda(A_n) \le \lim_{k\to\infty} 2^{-k+1} = 0\,.$$

By [PSS], Theorem 1, f is Riemann-integrable, which means: For every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for every

$$P = \{0 = x_0 \le x_1 \le \ldots \le x_n = 1\}$$

satisfying $\|P\| = \max_{i=1,\ldots,n}(x_i - x_{i-1}) \le \delta_\varepsilon$ the difference between Riemann upper and lower sum fulfils $U_f(P) - L_f(P) < \varepsilon$. In our case we take

$$\varepsilon = \tfrac{1}{3}2^{-2n} \quad \text{and} \quad P = \{x_k = k/m \mid k = 0, \ldots, m\}$$

with

$$1/m < \min\{\delta_\varepsilon, d\} \quad \text{and} \quad d = \tfrac{1}{3}2^{-n}\,.$$

We obtain

$$\begin{aligned}\varepsilon > U_f(P) - L_f(P) &= \sum_{i=1}^{m} \frac{1}{m}\Big(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x)\Big)\\ &\ge \frac{d}{m}\big|\{i \mid \sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \ge d\}\big|\,,\end{aligned}$$

which implies for

$$A_n = \bigcup\{[x_{i-1}, x_i] \mid \sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \ge d\}$$

the desired upper bound

$$\lambda(A_n) \le \varepsilon/d = 2^{-n}\,.$$

We consider the following binary relation on the set $M = \{1, \ldots, m\}$:

$$R = \{(i,j) \in M^2 \mid f(x) = y \text{ for some } x \in [x_{i-1}, x_i],\ y \in [x_{j-1}, x_j]\}\,.$$

Since f preserves Lebesgue measure we conclude

$$\begin{aligned}\lambda(f^{-1}([x_{i_1} - x_{i_1-1}] \cup \ldots \cup [x_{i_k} - x_{i_k-1}]))\\ = \lambda([x_{i_1} - x_{i_1-1}] \cup \ldots \cup [x_{i_k} - x_{i_k-1}]) = k/m\,.\end{aligned}$$

Hence for every set $T = \{i_1, \ldots, i_k\} \subseteq M$ of indices we have $|R^{-1}(T)| \ge |T|$. Thus (marriage problem) there exists a bijection $\phi : M \to M$ such that for every $i \in M$ there are $x \in [(i-1)/m, i/m]$ and $y \in [(\phi(i)-1)/m, \phi(i)/m]$ with $f(x) = y$.

Construction of f_n: On every interval $((i-1)/m, i/m)$ we define f_n to be the linear connection of the points $((i-1)/m, (\phi(i)-1)/m)$ and $(i/m, \phi(i)/m)$. Of course it is possible to define the remaining values $f_n(i/m)$, $i = 0, \ldots, m$, in such a way that f_n is bijective. By [PSS], Proposition 7, it is clear that f_n preserves uniform distribution. It remains to prove

$$|f(x) - f_n(x)| < 2^{-n}$$

for $x \notin A_n$. By the definition of A_n, for $x_{i-1} \le x \le x_i$ we have

$$\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) < d = \tfrac{1}{3}2^{-n}\,.$$

Furthermore, there is an $x' \in [x_{i-1}, x_i]$ such that

$$y' = f(x') \in [x_{\phi(i)-1}, x_{\phi(i)}]\,.$$

Completing the proof of Theorem 3.6 we derive

$$\begin{aligned}|f(x) - f_n(x)| &\le |f(x) - f(x')| + |f(x') - f_n(x')| + |f_n(x') - f_n(x)|\\ &\le d + 1/m + 1/m < 2^{-n}\,.\end{aligned}$$

Remark. In the recent article [Ko] invariant measures for piecewise linear transformations were studied in detail.

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