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Hecke's functional equation and the average order of arithmetical functions

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1. Introduction. Let $r(n)$ denote the number of representations of n as a sum of two squares, and

(1) $$R(x) = \sum_{n=0}^{x} r(n) = \pi x + P(x),$$

so that $P(x)$ is the error term in the lattice-point problem for the circle. It has long been conjectured, but not proved, that

(2) $$P(x) = O(x^{\frac{1}{4}+\varepsilon}),$$

for every $\varepsilon > 0$, as $x \to \infty$ ([6]). Hardy ([7]) showed, however, that (2) is true on the average, that is to say

(3) $$\frac{1}{x}\int_{1}^{x} |P(y)|\,dy = O(x^{\frac{1}{4}+\varepsilon}).$$

He also proved ([7]) that

(4) $$\liminf_{x\to\infty} \frac{P(x)}{x^{\frac{1}{4}}\log^{\frac{1}{4}} x} < 0,$$

which, in the notation of Hardy and Littlewood and Ingham, is written as

(5) $$P(x) = \Omega_{-}(x^{\frac{1}{4}}\log^{\frac{1}{4}} x).$$

On the other side, Hardy also proved that

(6) $$\limsup_{x\to\infty} \frac{P(x)}{x^{\frac{1}{4}}} > 0,$$

that is

(7) $$P(x) = \Omega_{+}(x^{\frac{1}{4}}).$$

While a stronger result than (7), which would correspond to (5) with Ω_+ instead of Ω_-, is not known, Ingham proved in 1940 ([9]) that

(8) $$\limsup_{x\to\infty} \frac{P(x)}{x^{\frac{1}{4}}} = +\infty.$$

Ingham's method consists in making a skilful use of Kronecker's theorem on Diophantine approximation, and it applies also to the lattice-point problem for a rectangular hyperbola, in which the divisor function $d(n)$ takes the place of $r(n)$. A modification of this method, also due to Ingham, consists in avoiding the explicit use of Diophantine approximation, and in appealing, instead, to the argument used by Bohr and Jessen ([3]) in their proof of Kronecker's theorem. The modified method has been applied by Pennington ([15]) to Ramanujan's arithmetical function $\tau(n)$, to obtain the result

(9) $$\limsup_{x\to\infty} \frac{T(x)}{x^{23/4}} = +\infty, \qquad \liminf_{x\to\infty} \frac{T(x)}{x^{23/4}} = -\infty,$$

where $T(x) = \sum_{n\leqslant x} \tau(n)$.

In this paper we start with the functional equation

(10) $$(2\pi)^{-s}\Gamma(s)\varphi(s) = (2\pi)^{s-\delta}\Gamma(\delta-s)\psi(\delta-s),$$

where φ and ψ are representable by the Dirichlet series $\sum a_n\lambda_n^{-s}$ and $\sum b_n\mu_n^{-s}$ respectively, and consider the corresponding averages of their coefficients

(11) $$A_\lambda^\varrho(x) = \frac{1}{\Gamma(\varrho+1)}\sum_{\lambda_n\leqslant x} a_n(x-\lambda_n)^\varrho, \qquad B_\mu^\varrho(x) = \frac{1}{\Gamma(\varrho+1)}\sum_{\mu_n\leqslant x} b_n(x-\mu_n)^\varrho, \qquad \varrho \geqslant 0.$$

Here λ_n and μ_n are strictly increasing sequences of positive numbers, δ is real, and a_n, b_n are complex, and not all of them zero. We shall show that if $\operatorname{Re} a_n \neq 0$ for at least one value of n, then

(12) $$\operatorname{Re}[B_\mu^\varrho(x) - Q_\varrho(x)] = \Omega_\pm(x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}),$$

where the term Q_ϱ arises from the singularities of $\Gamma(s)\varphi(s)$. Similarly, if $\operatorname{Im} a_n \neq 0$ for at least one value of n, then

(12)′ $$\operatorname{Im}[B_\mu^\varrho(x) - Q_\varrho(x)] = \Omega_\pm(x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}).$$

We shall further show that if the sequence $\{a_n, \lambda_n\}$ satisfies certain additional arithmetical requirements, then

(13) $$\limsup_{x\to\infty} \frac{\operatorname{Re}[B_\mu^\varrho(x) - Q_\varrho(x)]}{x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}} = +\infty, \qquad \liminf_{x\to\infty} \frac{\operatorname{Re}[B_\mu^\varrho(x) - Q_\varrho(x)]}{x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}} = -\infty.$$

Corresponding results for the imaginary part also exist. By interchanging a_n with b_n, and λ_n with μ_n, one can write down similar results for A_λ^ϱ.

Since the Dirichlet series $\sum r(n)n^{-s}$ associated with the lattice point problem for the circle, and the series $\sum \tau(n)n^{-s}$ associated with Ramanujan's function, occur as solutions of functional equation (10), with $\delta = 1$ and $\delta = 12$ respectively, our results give, when $\varrho = 0$, the results of Ingham's, namely (8), and of Pennington's, namely (9). We also cover other arithmetical functions such as $r_k(n)$ which denotes the number of lattice points on a k-dimensional sphere, $\sigma_k(n)$ which denotes the sum of the kth powers of divisors of n when k is an odd integer, and Siegel's function $\mu(S, t)$, which is the measure of representation of the number t by S, where S is an indefinite quadratic form in four or more variables, with rational coefficients, and with positive determinant.

We prove these results by combining Ingham's method with a general arithmetical identity (Lemma 3) which is equivalent to functional equation (10).

2. Preliminary lemmas. We start with a precise formulation of the functional equation.

DEFINITION 1. Let $\{\lambda_n\}$, $\{\mu_n\}$, $n = 1, 2, \ldots$, be two strictly increasing sequences of positive numbers, and $\{a_n\}$, $\{b_n\}$ be two sequences of complex numbers not identically zero, and s a complex variable with $s = \sigma + it$. Let δ be a real number. Let the Dirichlet series

$$\varphi(s) = \sum_1^\infty a_n\lambda_n^{-s}, \qquad \psi(s) = \sum_1^\infty b_n\mu_n^{-s}$$

admit finite abscissae of absolute convergence. Then φ, ψ are said to satisfy the functional equation

(14) $$(2\pi)^{-s}\Gamma(s)\varphi(s) = (2\pi)^{s-\delta}\Gamma(\delta-s)\psi(\delta-s),$$

if there exists in the s-plane a domain D which is the exterior of a bounded closed set S, and in D there exists a holomorphic function $\chi(s)$ with the property

$$e^{-\varepsilon t}\chi(\sigma+it) = O(1), \quad 0 < \varepsilon < \tfrac{1}{2}\pi,$$

as $|t| \to \infty$, uniformly in each strip $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $-\infty < \sigma_1 < \sigma_2 < +\infty$, and

$$(15)\qquad \begin{aligned} \chi(s) &= (2\pi)^{-s}\Gamma(s)\varphi(s), \quad \text{for } \sigma > a,\\ \chi(s) &= (2\pi)^{s-\delta}\Gamma(\delta-s)\psi(\delta-s), \quad \text{for } \sigma < \beta, \end{aligned}$$

where a, β are some constants.

Amplifying the connexion between the Riemann zeta-function and the elliptic theta-function, Bochner ([2]) has shown that functional equation (14) is equivalent to a "modular relation". We state his result here as a lemma.

Lemma 1. (Bochner). *If the functional equation*

$$(16)\qquad (2\pi)^{-s}\Gamma(s)\varphi(s) = (2\pi)^{s-\delta}\Gamma(\delta-s)\psi(\delta-s)$$

holds in the sense of Definition 1, *then the following 'modular relation' holds*:

$$(17)\qquad \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = P(x) + \left(\frac{2\pi}{x}\right)^{\delta} \sum_{n=1}^{\infty} b_n e^{-4\pi^2\mu_n/x}, \qquad x > 0$$

where $P(x)$ is a 'residual function' given by

$$P(x) = \int_{\mathcal{C}} \chi(s)(2\pi)^s x^{-s}\, ds,$$

and $\mathcal{C}$ denotes any curve, or curves, in D encircling all of S.

Conversely, modular relation (17) *implies functional equation* (16).

The following lemma which will be required in the sequel has been proved by us in another paper ([4]).

Lemma 2. *Functional equation* (16) *implies the identity*

$$(18)\qquad \frac{1}{\Gamma(\varrho+1)} \sum_{\mu_n \leqslant x} b_n (x-\mu_n)^{\varrho} = Q_{\varrho}(x) + \frac{1}{(2\pi)^{\varrho}} \sum_{1}^{\infty} \left(\frac{x}{\lambda_n}\right)^{\frac{1}{2}(\delta+\varrho)} a_n J_{\delta+\varrho}\{4\pi\sqrt{(\lambda_n x)}\},$$

for $x > 0$, and $\varrho \geqslant 2a-\delta-\frac{1}{2}$, where a is a number for which $\sum_{n=1}^{\infty} |a_n|\lambda_n^{-a} < \infty$, and

$$Q_{\varrho}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\chi(s)(2\pi)^s x^{s+\varrho}}{\Gamma(\varrho+1+s)}\, ds.$$

Conversely, identity (18) *implies not only that $a > \frac{1}{2}(\delta+\frac{1}{2})$, but that functional equation* (16) *is satisfied.*

We shall now state and prove the principal lemma on which our results are based.

Lemma 3. *Functional equation* (16) *is equivalent to the relation*

$$(19)\qquad \sum_{n=1}^{\infty} a_n g(\lambda_n, t) = (2\pi)^{-\delta}\cdot\sqrt{\pi}\, 2^{2r+1} \int_0^{\infty} Q_{\varrho}(x^2) x^{2r} e^{-2tx}\, dx + {} + (2\pi)^{-\delta}\cdot 2^{\varrho+3/2} \left(\frac{d}{dt}\right)^{2r} \left[\sum_{n=1}^{\infty} \left(\frac{\sqrt{\mu_n}}{2t}\right)^{\varrho+\frac{1}{2}} b_n K_{\varrho+\frac{1}{2}}(2t\sqrt{\mu_n})\right],$$

where $\varrho \geqslant 0$, $\operatorname{Re} t > 0$, r is a sufficiently large integer, $K_{\varrho+\frac{1}{2}}(t)$ is the Bessel function of purely imaginary argument, and

$$g(\lambda_n, t) \equiv g(r, \varrho, \delta, \lambda_n, t) \equiv \sum_{\nu=0}^{r} c_{\nu} \frac{t^{2\nu}(-1)^{\nu+r}\Gamma(\nu+\gamma+r)}{(t^2+4\pi^2\lambda_n)^{\gamma+\nu+r}},$$

where c_{ν} is an absolute constant, and $\gamma = \delta+\varrho+\frac{1}{2}$.

If

$$B_{\mu}^{\varrho}(x) = \frac{1}{\Gamma(\varrho+1)} \sum_{\mu_n \leqslant x}' b_n (x-\mu_n)^{\varrho}, \quad \varrho \geqslant 0,$$

then relation (19) *can be written as*

$$(20)\qquad \sum_{n=1}^{\infty} a_n g(\lambda_n, \tfrac{1}{2}s) = (2\pi)^{-\delta}\sqrt{\pi}\, 2^{2r+1} \int_0^{\infty} [B_{\mu}^{\varrho}(x^2) - Q_{\varrho}(x^2)] x^{2r} e^{-sx}\, dx,$$

and if $\gamma \equiv \delta+\varrho+\frac{1}{2}$ is not a negative integer, relation (20) *can be written as*

$$(21)\qquad \sum_{n=1}^{\infty} a_n \left(\frac{d}{ds}\right)^{2r} \left[\frac{\Gamma(\delta+\varrho+\frac{1}{2})}{(s^2+16\pi^2\lambda_n)^{\delta+\varrho+\frac{1}{2}}}\right] = (2\pi)^{-\delta}\sqrt{\pi}\, 2^{-2(\delta+\varrho)} \int_0^{\infty} [B_{\mu}^{\varrho}(x^2) - Q_{\varrho}(x^2)] x^{2r} e^{-sx}\, dx.$$

Proof. Let

$$(22)\qquad h(t, x) = e^{-t^2 x}, \qquad h_r(t, x) = \left(\frac{\partial}{\partial t}\right)^{2r} h(t, x),$$

where r is a non-negative integer. By the ordinary rules of differentiation, we have, for any sufficiently smooth function F,

$$(23)\qquad \left(\frac{d}{dt}\right)^{2r} F(t^2) = \sum_{\nu=0}^{r} c_{\nu} t^{2\nu} F^{(r+\nu)}(t^2),$$

where c_{ν} is an absolute constant, and $c_r = 2^{2r}$.

Since we assume that the functional equation holds, we have, by Lemma 1, the relation

$$x^{\delta+\varrho}\sum_{n=1}^{\infty} a_n e^{-4\pi^2\lambda_n x} = x^{\delta+\varrho}P(4\pi^2 x)+(2\pi)^{-\delta}\sum_{n=1}^{\infty} b_n x^{\varrho} e^{-\mu_n/x}, \tag{24}$$

for $\varrho \geqslant 0$, and $\mathrm{Re}\, x > 0$. We shall show that if we multiply (24) throughout by $x^{-\frac{1}{2}}h_r(t,x)$, and integrate from 0 to ∞ with respect to x, we obtain (19). Taking first the left hand side, we note that

$$\int_0^\infty x^{\delta+\varrho-\frac{1}{2}}e^{-4\pi^2\lambda_n x}h_r(t,x)\,dx = \sum_{\nu=0}^{r} c_\nu t^{2\nu}(-1)^{\nu+r}\int_0^\infty x^{\delta+\varrho-\frac{1}{2}+\nu+r}e^{-(t^2+4\pi^2\lambda_n)x}\,dx$$

$$= \sum_{\nu=0}^{r} c_\nu t^{2\nu}(-1)^{\nu+r}\frac{\Gamma(\gamma+\nu+r)}{(t^2+4\pi^2\lambda_n)^{\gamma+\nu+r}} = g(\lambda_n, t),$$

so that, for sufficiently large r, we obtain

$$\int_0^\infty x^{\delta+\varrho-\frac{1}{2}}\Big(\sum_{n=1}^{\infty} a_n e^{-4\pi^2\lambda_n x}\Big)h_r(t,x)\,dx = \sum_{n=1}^{\infty} a_n g(\lambda_n, t). \tag{25}$$

The first member on the right-hand side of (24) gives

$$\int_0^\infty x^{\delta+\varrho-\frac{1}{2}}P(4\pi^2 x)h_r(t,x)\,dx = \sum_{\nu=0}^{r} c_\nu t^{2\nu}(-1)^{\nu+r}\int_0^\infty x^{\gamma-1+\nu+r}P(4\pi^2 x)e^{-t^2 x}dx. \tag{26}$$

Since

$$\int_0^\infty x^{\gamma-1+\nu+r}e^{-t^2x}\,dx\cdot\frac{1}{2\pi i}\int_{\mathcal{C}}\chi(s)(2\pi x)^{-s}ds$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}}\chi(s)(2\pi)^{-s}ds\cdot\int_0^\infty x^{\gamma-1+\nu+r-s}e^{-t^2x}dx$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}}\chi(s)(2\pi)^{-s}\Gamma(\gamma+\nu+r-s)\,e^{-2(\gamma+\nu+r-s)}\,ds,$$

we have

$$\int_0^\infty x^{\delta+\varrho-\frac{1}{2}}P(4\pi^2 x)h_r(t,x)\,dx = \sum_{\nu=0}^{r} c_\nu t^{2\nu}f^{(\nu+r)}(t^2), \tag{27}$$

where

$$f(t) = \frac{1}{2\pi i}\int_{\mathcal{C}}\chi(s)(2\pi)^{-s}\Gamma(\gamma-s)\,t^{-(\gamma-s)}ds,$$

the curve $\mathcal{C}$ (in Lemma 1) being so chosen that $\mathcal{C}$ as well as its transform $\mathcal{C}'$ by the substitution $s\to\delta-s$ are free from the poles of $\Gamma(\gamma-s)$. Using (23) in (27), we get

$$\int_0^\infty x^{\delta+\varrho-\frac{1}{2}}P(4\pi^2 x)h_r(t,x)\,dx = \left(\frac{d}{dt}\right)^{2r} f(t^2) \tag{28}$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}}\chi(s)(2\pi)^{-s}\Gamma(\gamma-s)\left(\frac{d}{dt}\right)^{2r} t^{-2(\gamma-s)}ds.$$

Since

$$\Gamma(\gamma-s)\cdot\left(\frac{d}{dt}\right)^{2r} t^{-2(\gamma-s)}$$

$$= \Gamma(\gamma-s)[2(\gamma-s)][2(\gamma-s)+1]\ldots[2(\gamma-s)+2r-1]t^{-2(\gamma-s+r)}$$

$$= 2^r\Gamma(\gamma-s+r)\cdot 2^r(\gamma+\tfrac{1}{2}-s)(\gamma+\tfrac{3}{2}-s)\ldots(\gamma+r-\tfrac{1}{2}-s)t^{-2(\gamma-s+r)}$$

$$= 2^{2r}\frac{\Gamma(\gamma-s+r)\cdot\Gamma(\gamma+r+\frac{1}{2}-s)}{\Gamma(\gamma+\frac{1}{2}-s)}\,t^{-2(\gamma-s+r)},$$

we may write (28) as

$$\int_0^\infty x^{\delta+\varrho-\frac{1}{2}}P(4\pi^2 x)h_r(t,x)\,dx$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}}\chi(s)(2\pi)^{-s}\cdot 2^{2r}\frac{\Gamma(\gamma-s+r)\Gamma(\gamma-s+r+\frac{1}{2})}{\Gamma(\gamma+\frac{1}{2}-s)}\cdot t^{-2(\gamma-s+r)}ds,$$

provided that r is chosen large enough. If we now change the variable s to $\delta-s$, and use the functional equation $\chi(s)=\chi(\delta-s)$, together with the duplication formula for the gamma-function, namely

$$\sqrt{\pi}\,\Gamma(x) = 2^{x-1}\Gamma(\tfrac{1}{2}x)\Gamma(\tfrac{1}{2}x+\tfrac{1}{2}),$$

we obtain

$$\int_0^\infty x^{\delta+\varrho-\frac{1}{2}}P(4\pi^2 x)h_r(t,x)\,dx \tag{29}$$

$$= -\frac{1}{2\pi i}\int_{\mathcal{C}'}\chi(s)(2\pi)^{s-\delta}2^{2r}2^{-2(\varrho+s+r)}\frac{\Gamma(2\varrho+2s+1+2r)}{\Gamma(\varrho+1+s)}\,t^{-2(s+\varrho+r+\frac{1}{2})}ds$$

$$= -\frac{\sqrt{\pi}(2\pi)^{-\delta}2^{2r+1}}{2\pi i}\int_{\mathcal{C}'}\chi(s)(2\pi)^{s}\frac{\Gamma(2\varrho+2r+1+2s)}{\Gamma(\varrho+1+s)}(2t)^{-2(s+\varrho+r+\frac{1}{2})}ds$$

$$= -\sqrt{\pi}(2\pi)^{-\delta}2^{2r+1}\int_0^\infty Q_\varrho(x^2)x^{2r}e^{-2tx}dx,$$

where

$$Q_\varrho(x) = \frac{1}{2\pi i}\int_{\mathscr{C}'} \frac{\chi(s)(2\pi)^s x^{s+\varrho}}{\Gamma(s+\varrho+1)}\,ds.$$

Let us now consider the second member of the right-hand side of (24). On using the relations

$$\int_0^\infty x^{\varrho-\frac{1}{2}} e^{-\mu_n/x} h_r(t,x)\,dx = \left(\frac{d}{dt}\right)^{2r}\int_0^\infty x^{\varrho-\frac{1}{2}} e^{-(\mu_n/x)-t^2x}\,dx,$$

and

$$K_\mu(z) = \tfrac{1}{2}(\tfrac{1}{2}z)^\mu \int_0^\infty \exp\left\{-\frac{1}{x}-\frac{z^2x}{4}\right\} x^{\mu-1}\,dx, \qquad \operatorname{Re} z^2 > 0,$$

where K_μ is the Bessel function of purely imaginary argument ([20], p. 78, p. 183), we obtain

$$(30)\qquad (2\pi)^{-\delta}\int_0^\infty x^{\varrho-\frac{1}{2}}\left(\sum_{n=1}^\infty b_n e^{-\mu_n/x}\right) h_r(t,x)\,dx$$

$$= (2\pi)^{-\delta}\cdot 2^{\varrho+\frac{3}{2}}\left(\frac{d}{dt}\right)^{2r}\left[\sum_{n=1}^\infty \left(\frac{\sqrt{\mu_n}}{2t}\right)^{\varrho+\frac{1}{2}} b_n K_{\varrho+\frac{1}{2}}(2t\sqrt{\mu_n})\right].$$

Combining (30) with (29) and (25), we obtain (19) as claimed in the lemma. Conversely, given (19) for a large integer r, we can, by repeated integration, obtain (19) with $r=0$, provided that ϱ is a sufficiently large integer. And since $t^{-\varrho-\frac{1}{2}}K_{\varrho+\frac{1}{2}}(t) = (-1)^\varrho(\frac{1}{2}\pi)^{\frac{1}{2}}\left(\frac{d}{t\,dt}\right)^\varrho\left[\frac{e^{-t}}{t}\right]$, the result follows from Lemma 6 of [4].

Since

$$K_\mu(z) = \frac{\Gamma(\frac{1}{2})(\frac{1}{2}z)^\mu}{\Gamma(\mu+\frac{1}{2})}\int_1^\infty e^{-zt}(t^2-1)^{\mu-\frac{1}{2}}\,dt,$$

for $\operatorname{Re}(\mu+\frac{1}{2})>0$, and $|\arg z|<\frac{1}{2}\pi$ ([20], p. 172), the right-hand side of (30) gives

$$(31)\qquad (2\pi)^{-\delta}\left(\frac{d}{dt}\right)^{2r}\left[\sum_{n=1}^\infty b_n\cdot\frac{2\sqrt{\pi}}{\Gamma(\varrho+1)}\int_{\mu_n}^\infty e^{-2ty}(y^2-\mu_n)^\varrho\,dy\right]$$

$$= \sqrt{\pi}(2\pi)^{-\delta}\left(\frac{d}{dt}\right)^{2r}\left[\frac{2}{\Gamma(\varrho+1)}\int_0 \sum_{\mu_n\le y} b_n(y^2-\mu_n)^\varrho e^{-2ty}\,dy\right]$$

$$= \sqrt{\pi}(2\pi)^{-\delta}\left(\frac{d}{dt}\right)^{2r}\left[2\int_0^\infty B_\mu^\varrho(y^2)e^{-2ty}\,dy\right]$$

$$= \sqrt{\pi}(2\pi)^{-\delta}2^{2r+1}\int_0^\infty B_\mu^\varrho(y^2)y^{2r}e^{-2ty}\,dy.$$

Substituting (31) for the second member on the right-hand side of (19), we obtain

$$\sum_{n=1}^\infty a_n g(\lambda_n,t) = (2\pi)^{-\delta}\sqrt{\pi}\,2^{2r+1}\int_0^\infty [B_\mu^\varrho(y^2)-Q_\varrho(y^2)]y^{2r}e^{-2ty}\,dy.$$

Writing $t=s/2$, we obtain (20), from which (21) obviously follows.

3. The results. We are now in a position to prove the following results.

THEOREM I. *Suppose that functional equation (14) is satisfied, and that the sequence $\{\lambda_n\}$ contains a subset $\{\lambda_{n_k}\}$ such that no number $\lambda_n^{\frac{1}{2}}$ is representable as a linear combination of the numbers $\lambda_{n_k}^{\frac{1}{2}}$ with coefficients ± 1, unless $\lambda_n^{\frac{1}{2}} = \pm\lambda_{n_r}^{\frac{1}{2}}$ for some r, in which case $\lambda_n^{\frac{1}{2}}$ has no other representation. Suppose, in addition, that*

$$(32)\qquad \sum_{n=1}^\infty \frac{|\operatorname{Re} a_{n_k}|}{\lambda_{n_k}^{\frac{1}{2}\delta+\frac{1}{2}\varrho+\frac{1}{4}}} = +\infty, \qquad \varrho\geqslant 0.$$

Then

$$(33)\qquad \limsup_{x\to\infty}\frac{\operatorname{Re}[B_\mu^\varrho(x)-Q_\varrho(x)]}{x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}} = +\infty,$$

$$(34)\qquad \liminf_{x\to\infty}\frac{\operatorname{Re}[B_\mu^\varrho(x)-Q_\varrho(x)]}{x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}} = -\infty,$$

where $Q_\varrho(x)$ is defined as in Lemma 2.

If in assumption (32) we replace $\operatorname{Re} a_{n_k}$ by $\operatorname{Im} a_{n_k}$, then in conclusions (33) and (34) we should have $\operatorname{Im}[B_\mu^\varrho(x)-Q_\varrho(x)]$ in place of $\operatorname{Re}[B_\mu^\varrho(x)-Q_\varrho(x)]$.

THEOREM II. *If $\operatorname{Re} a_n\neq 0$ for at least one value of n, then*

$$(35)\qquad \operatorname{Re}[B_\mu^\varrho(x)-Q_\varrho(x)] = \Omega_\pm(x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}), \qquad \varrho\geqslant 0.$$

If $\operatorname{Im} a_n\neq 0$ for at least one value of n, then

$$(36)\qquad \operatorname{Im}[B_\mu^\varrho(x)-Q_\varrho(x)] = \Omega_\pm(x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}), \qquad \varrho\geqslant 0.$$

The exponent $\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}$ *in* (35) *and* (36) *is the best possible for large* ϱ, *in the sense that if* α *is such that* $\sum_{n=1}^{\infty}|a_n|\lambda_n^{-\alpha}<\infty$, *and* $\varrho\geqslant 2\alpha-\delta-\frac{1}{2}$, *then*

$$B_\mu^\varrho(x)-Q_\varrho(x)=O(x^{\frac{1}{2}\delta+\frac{1}{2}\varrho-\frac{1}{4}}). \tag{37}$$

Remark. It is obvious that if a_n is interchanged with b_n, and λ_n with μ_n, we have conclusions about $A_\lambda^\varrho(x)$.

Proof. Consider identity (20) in Lemma 3. It is easy to see that if

$$\operatorname{Im} s\neq\pm 4\pi\lambda_n^{\frac{1}{2}},$$

then

$$\lim_{\sigma\to+0}\sigma^{\gamma+2r}g(\lambda_n,s/2)=0, \tag{38}$$

where $\gamma=\delta+\varrho+\frac{1}{2}$, and r is an integer with the property $\gamma+2r>0$. On the other hand, if $\operatorname{Im} s=\pm 4\pi\lambda_n^{\frac{1}{2}}$, then $(s^2+16\pi^2\lambda_n)^{\gamma+2r}=(\sigma^2+2\sigma\cdot 4\pi\lambda_n^{\frac{1}{2}}e^{\pm\pi i/2})^{\gamma+2r}$, so that

$$\lim_{\sigma\to+0}\sigma^{\gamma+2r}g(\lambda_n,s/2)=\Gamma(\gamma+2r)\cdot 2^{-\gamma}(4\pi\lambda_n^{\frac{1}{2}})^{-\gamma}e^{\mp i\frac{\pi}{2}(\gamma+2r)}.$$

Hence

$$\lim_{\sigma\to+0}\sigma^{\gamma+2r}\sum_{n=1}^{\infty}a_n g(\lambda_n,s/2) \tag{39}$$

$$=\begin{cases}\Gamma(\gamma+2r)2^{-3\gamma}\pi^{-\gamma}\cdot a_n\cdot\lambda_n^{-\gamma/2}e^{\mp\pi i\gamma/2}, & \text{if}\quad \operatorname{Im} s=\pm 4\pi\lambda_n^{\frac{1}{2}},\\ 0, & \text{otherwise.}\end{cases}$$

Using (39) in identity (20), we obtain

$$\lim_{\sigma\to+0}\sigma^{\gamma+2r}\int_0^\infty[B_\mu^\varrho(x^2)-Q_\varrho(x^2)]x^{2r}e^{-sx}dx \tag{40}$$

$$=\begin{cases}\Gamma(\gamma+2r)\pi^{\delta-\gamma-\frac{1}{2}}2^{-3\gamma+\delta-2r-1}a_n\cdot\lambda_n^{-\gamma/2}e^{\mp\pi i\gamma/2}, & \text{if}\quad \operatorname{Im} s=\pm 4\pi\lambda_n^{\frac{1}{2}},\\ 0, & \text{otherwise.}\end{cases}$$

Now let

$$W(x)=\prod_{k=1}^{N}V(4\pi\lambda_{n_k}^{\frac{1}{2}}x-\theta_k), \tag{41}$$

where N is a positive integer, θ_k is a real number, and

$$V(x)=2\cos^2(\tfrac{1}{2}x)=\tfrac{1}{2}(2+e^{ix}+e^{-ix}). \tag{42}$$

We note that $V(x)$ is real and non-negative for real x. Using (42) in (41), we can write

$$W(x)=1+\tfrac{1}{2}\sum_{k=1}^{N}[e^{+4\pi i\lambda_{n_k}^{\frac{1}{2}}x-i\theta_k}+e^{-4\pi i\lambda_{n_k}^{\frac{1}{2}}x+i\theta_k}]+U(x),$$

where $U(x)$ is a trigonometrical polynomial with exponents

$$4\pi(r_1\lambda_{n_1}^{\frac{1}{2}}+r_2\lambda_{n_2}^{\frac{1}{2}}+\ldots+r_N\lambda_{n_N}^{\frac{1}{2}}), \tag{43}$$

where r_ν takes the values 0, $+1$, or -1, and no combination $(r_1,r_2,\ldots,r_N)$ is such that all the component r's are zero, nor such that all but one are zero and the remaining one equals ±1. By our hypothesis it follows that none of the exponents (43) can be equal to $\lambda_n^{\frac{1}{2}}$ for any n.

If we now consider the limit

$$\lim_{\sigma\to+0}\sigma^{\gamma+2r}\int_0^\infty[B_\mu^\varrho(x^2)-Q_\varrho(x^2)]x^{2r}e^{-\sigma x}W(x)\,dx \tag{44}$$

it follows that because of relation (40), $U(x)$ contributes nothing to the integral, and we have

$$\lim_{\sigma\to+0}\frac{\sigma^{\gamma+2r}}{\Gamma(\gamma+2r)}\int_0^\infty[B_\mu^\varrho(x^2)-Q_\varrho(x^2)]x^{2r}e^{-\sigma x}W(x)\,dx \tag{45}$$

$$=c\sum_{k=1}^{N}\frac{a_{n_k}\cos(\theta_k-\pi\gamma/2)}{\lambda_{n_k}^{\gamma/2}},$$

where $c=\pi^{-\varrho-1}2^{-2\delta-3\varrho-2r-5/2}$. Since

$$\lim_{\sigma\to+0}\frac{\sigma^{\gamma+2r}}{\Gamma(\gamma+2r)}\int_0^\infty\operatorname{Re}[B_\mu^\varrho(x^2)-Q_\varrho(x^2)]x^{2r}e^{-\sigma x}W(x)\,dx$$

$$\leqslant\limsup_{x\to\infty}\frac{\operatorname{Re}[B_\mu^\varrho(x^2)-Q_\varrho(x^2)]}{x^{\gamma-1}},$$

we obtain from (45) the inequality

$$\limsup_{x\to\infty}\frac{\operatorname{Re}[B_\mu^\varrho(x^2)-Q_\varrho(x^2)]}{x^{\gamma-1}}\geqslant c\sum_{k=1}^{\infty}\frac{\operatorname{Re}a_{n_k}\cos(\theta_k-\pi\gamma/2)}{\lambda_{n_k}^{\gamma/2}}. \tag{46}$$

Choose

$$\theta_k=\begin{cases}\pi\gamma/2, & \text{if}\quad \operatorname{Re}a_{n_k}\geqslant 0,\\ \pi\gamma/2-\pi, & \text{if}\quad \operatorname{Re}a_{n_k}<0.\end{cases}$$

Then (46) leads to the inequality

$$\limsup_{x\to\infty} \frac{\operatorname{Re}[B_\mu^\varrho(x^2)-Q_\varrho(x^2)]}{x^{\gamma-1}} \geqslant c \sum_{k=1}^{\infty} \frac{|\operatorname{Re} a_{n_k}|}{\lambda_{n_k}^{\gamma/2}}, \tag{47}$$

with $\gamma = \delta+\varrho+\frac{1}{2}$.

Similarly we obtain also

$$\liminf_{x\to\infty} \frac{\operatorname{Re}[B_\mu^\varrho(x^2)-Q_\varrho(x^2)]}{x^{\gamma-1}} \leqslant -c \sum_{k=1}^{\infty} \frac{|\operatorname{Re} a_{n_k}|}{\lambda_{n_k}^{\gamma/2}}. \tag{48}$$

Hence, if $\sum_{k=1}^{\infty} \frac{|\operatorname{Re} a_{n_k}|}{\lambda_{n_k}^{\gamma/2}} = +\infty$, then we have

$$\overline{\underline{\lim}}_{x\to\infty} \frac{\operatorname{Re}[B_\mu^\varrho(x)-Q_\varrho(x)]}{x^{\frac{1}{2}(\gamma-1)}} = \pm\infty, \tag{49}$$

which proves (33) and (34) as claimed in Theorem 1. It is obvious that the corresponding result holds for the imaginary part. (In consequence of Lemma 2, we note that the functional equation implies that $\sum_{n=1}^{\infty} |\operatorname{Re} a_n|\,\lambda_n^{-\gamma/2} = +\infty$, in case $\varrho = 0$.)

Since we can always pick just one term of the sequence $\{\lambda_n\}$ to serve as the stipulated subsequence λ_{n_k}, inequalities (47) and (48) lead to Theorem II.

If we use Lemma 2, and observe that $J_\mu(x) = O(x^{-\frac{1}{2}})$ as $x\to\infty$, it follows easily that $B_\mu^\varrho(x)-Q_\varrho(x) = O(x^{\frac{1}{2}(\delta+\varrho)-\frac{1}{4}})$ for $\varrho \geqslant 2a-\delta-\frac{1}{2}$.

4. Examples. We shall now work out the implications of the theorems in a few important cases. It will appear that even for the special arithmetical functions considered, our theorems yield new results.

Example 1. If $\tau(n)$ is Ramanujan's arithmetical function, defined by the relation

$$\sum_{n=1}^{\infty} \tau(n) z^n = z\,\{(1-z)(1-z^2)(1-z^3)\ldots\}^{24}, \qquad |z|<1,$$

then it is well known ([8]) that its generating function defined by

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$$

satisfies functional equation (14), with $\delta = 12$, $\lambda_n = \mu_n = n$, and $a_n = b_n = \tau(n)$. Further, the function $Q_\varrho(x)$ of Lemma 2 is zero in this case.

For the subset $\{\lambda_{n_k}\}$ whose existence is stipulated in the hypothesis of Theorem I, we can choose the set of positive square-free integers. That this set satisfies the requirements is a consequence of Besicovitch's theorem that the fractional powers of certain classes of integers are linearly independent over the field of rational numbers ([1]). To verify condition (32) of Theorem I, we observe that the proof of a lemma of Pennington's ([15]) carries the following implication:

$$\sum_{1}^{\infty} \frac{|\tau(n)|}{n^\sigma} = +\infty \quad \text{implies} \quad \sum_{q} \frac{|\tau(q)|}{q^\sigma} = +\infty,$$

where q runs over all positive, square-free integers. Thus if

$$\sum_{n=1}^{\infty} \frac{|\tau(n)|}{n^{\gamma/2}} = +\infty, \quad \text{where} \quad \gamma = \delta+\varrho+\tfrac{1}{2}, \tag{50}$$

then we obtain, by Theorem I,

$$\limsup_{x\to\infty} \frac{T^\varrho(x)}{x^{\frac{1}{2}(\gamma-1)}} = +\infty, \qquad \liminf_{x\to\infty} \frac{T^\varrho(x)}{x^{\frac{1}{2}(\gamma-1)}} = -\infty, \tag{51}$$

where

$$\{\Gamma(\varrho+1)\}\,T^\varrho(x) = \sum_{n\leqslant x} \tau(n)(x-n)^\varrho, \quad \varrho \geqslant 0.$$

If

$$\sum_{n=1}^{\infty} \frac{|\tau(n)|}{n^{\gamma/2}} < +\infty, \quad \text{where} \quad \gamma = \delta+\varrho+\tfrac{1}{2}, \tag{52}$$

then, since $\tau(n)$ is not identically zero, by Theorem II, we have

$$T^\varrho(x) = O(x^{\frac{1}{2}(\gamma-1)}), \qquad T^\varrho(x) = \Omega_\pm(x^{\frac{1}{2}(\gamma-1)}). \tag{53}$$

Since $\delta = 12$, and we know, for instance, by Lemma 2, that $\sum_{1}^{\infty} |\tau(n)|\,n^{-25/4} = +\infty$, condition (50) is satisfied for $\varrho = 0$. Hence Theorem I yields the conclusions

$$\limsup_{x\to\infty} \frac{T(x)}{x^{23/4}} = +\infty, \qquad \liminf_{x\to\infty} \frac{T(x)}{x^{23/4}} = -\infty. \tag{54}$$

This result is due to Pennington ([15]). On the other hand, since $\sum_{1}^{\infty} |\tau(n)|\,n^{-13/2-\varepsilon} < \infty$ for $\varepsilon > 0$ ([8], p. 173), condition (52) is satisfied for $\varrho > \frac{1}{2}$, and Theorem 2 yields the conclusions

$$T^\varrho(x) = O(x^{23/4+\frac{1}{2}\varrho}), \qquad T^\varrho(x) = \Omega_\pm(x^{23/4+\frac{1}{2}\varrho}) \tag{55}$$

for $\varrho > \frac{1}{2}$.

EXAMPLE 2. Let $\sigma_k(n)$ denote the sum of the kth powers of the divisors of n, where k is an *odd* integer, $k \neq 0$. Then $\sum_{n=1}^{\infty} \sigma_k(n) n^{-s} = \zeta(s)\zeta(s-k)$, and functional equation (14) is satisfied by $\zeta(s)\zeta(s-k)$ with $\delta = k+1$, $\lambda_n = \mu_n = n$, $a_n = \sigma_k(n)$, and $b_n = (-1)^{(k+1)/2}\sigma_k(n)$. Further the function $Q_\varrho(x)$ of Lemma 2 in this case is the sum of the residues of the function

$$\frac{\zeta(s)\zeta(s-k)x^{s+\varrho}\Gamma(s)}{\Gamma(\varrho+1+s)} \tag{56}$$

at its poles ([4]). For the subset $\{\lambda_{n_k}\}$ we can again take the positive, square-free integers.

Case (i). First consider $\sigma_k(n)$, where k is a positive, odd integer. We have $n^k \leqslant \sigma_k(n) < cn^{k+\varepsilon}$, $\varepsilon > 0$, and $\sum_{n=1}^{\infty} \sigma_k(n) n^{-a} < \infty$, for $a > k+1$, while $\sum_{n=1}^{\infty} \sigma_k(n) n^{-a} = +\infty$ for $a \leqslant k+1$. Further, if q stands for any square-free integer, then $\sum_q \sigma_k(q) q^{-a} < +\infty$ for $a > k+1$, while $\sum_q \sigma_k(q) q^{-a} = +\infty$ for $a \leqslant k+1$ since $\sum_q q^{-a} < +\infty$ for $a > 1$, while $\sum_q q^{-a} = +\infty$ for $a \leqslant 1$. Hence by Theorems I and II we obtain the following results: If $\Gamma(\varrho+1) S_k^\varrho(x) = \sum_{n\leqslant x} \sigma_k(n)(x-n)^\varrho$, and k is a positive, odd integer, then

$$S_k^\varrho(x) - Q_\varrho(x) = \begin{cases} O(x^{\frac{1}{2}(k+\varrho+\frac{1}{2})}), \\ \Omega_\pm(x^{\frac{1}{2}(k+\varrho+\frac{1}{2})}), \end{cases} \tag{57}$$

for $\varrho > k+\frac{1}{2}$, while

$$\limsup_{x\to\infty} \frac{S_k^\varrho(x)-Q_\varrho(x)}{x^{\frac{1}{2}(k+\varrho+\frac{1}{2})}} = +\infty, \quad \liminf_{x\to\infty} \frac{S_k^\varrho(x)-Q_\varrho(x)}{x^{\frac{1}{2}(k+\varrho+\frac{1}{2})}} = -\infty \tag{58}$$

for $\varrho \leqslant k+\frac{1}{2}$.

Case (ii). Now consider $\sigma_k(n)$, where k is a negative odd integer. We have $\sum \sigma_k(n) n^{-a} < \infty$ if $a > 1$, while $\sum \sigma_k(n) n^{-a} = +\infty$ if $a \leqslant 1$. Arguing as in Case (i), we obtain the following:

$$S_k^\varrho(x) - Q_\varrho(x) = \begin{cases} O(x^{\frac{1}{2}(k+\varrho+\frac{1}{2})}), \\ \Omega_\pm(x^{\frac{1}{2}(k+\varrho+\frac{1}{2})}), \end{cases} \tag{59}$$

if $\varrho > |k|+\frac{1}{2}$, while

$$\limsup_{x\to\infty} \frac{S_k^\varrho(x)-Q_\varrho(x)}{x^{\frac{1}{2}(k+\varrho+\frac{1}{2})}} = +\infty, \quad \liminf_{x\to\infty} \frac{S_k^\varrho(x)-Q_\varrho(x)}{x^{\frac{1}{2}(k+\varrho+\frac{1}{2})}} = -\infty \tag{60}$$

if $\varrho \leqslant |k|+\frac{1}{2}$.

EXAMPLE 3. Let $r_k(n)$ denote the number of lattice points on a k-dimensional sphere of radius $\sqrt{n}$. The generating function $\zeta_k(s) = \sum_{n=1}^{\infty} r_k(n) n^{-s}$, $k \geqslant 2$, is Epstein's zeta-function ([5]), and functional equation (14) is therefore satisfied with $\varphi(s) = 2^s \zeta_k(s)$, $\delta = k/2$, $a_n = b_n = r_k(n)$, and $\lambda_n = \mu_n = \frac{1}{2}n$. Further, the function $Q_\varrho(x)$ of Lemma 2 in this case is given by

$$Q_\varrho(x) = \frac{\pi^{k/2} x^{k/2+\varrho}}{\Gamma(\varrho+1+k/2)} - \frac{x^\varrho}{\Gamma(\varrho+1)}. \tag{61}$$

It is easy to see that $\sum r_k(n) n^{-k/2} = +\infty$, and $\sum r_k(n) n^{-k/2-\varepsilon} < \infty$, for $\varepsilon > 0$. In order to invoke Theorem I we choose the subset $\{\lambda_{n_k}\}$ to be the numbers $\frac{1}{2}q$, where q is a positive, square-free integer.

Case (i). First consider $r_k(n)$ for $k \geqslant 4$. We then have $r_k(n) \geqslant cn^{k/2-1}$ for n odd, and $\sum r_k(q) q^{-k/2} = +\infty$. The inequality is well known ([13], p. 113, Satz 172) for $k = 4$, and can be derived for $k > 4$ by observing that $r_k(n) = \sum_{m^2\leqslant n} r_{k-1}(n-m^2)$. Hence, by Theorems I and II, we have the following:

If $$P_k^\varrho(x) = \frac{1}{\Gamma(\varrho+1)} \sum_{0\leqslant n\leqslant x} r_k(n)(x-n)^\varrho - \frac{\pi^{k/2} x^{k/2+\varrho}}{\Gamma(\varrho+1+k/2)},$$

with $r_k(0) = 1$, and $k \geqslant 4$, then

$$P_k^{\varrho}(x) = \begin{cases} O(x^{\frac{1}{2}(k/2+\varrho-\frac{1}{2})}), \\ \Omega_\pm(x^{\frac{1}{2}(k/2+\varrho-\frac{1}{2})}), \end{cases} \tag{62}$$

if $\varrho > \frac{1}{2}(k-1)$, while

$$\limsup_{x\to\infty} \frac{P_k^\varrho(x)}{x^{\frac{1}{2}(k/2+\varrho-\frac{1}{2})}} = +\infty, \quad \liminf_{x\to\infty} \frac{P_k^\varrho(x)}{x^{\frac{1}{2}(k/2+\varrho-\frac{1}{2})}} = -\infty, \tag{63}$$

if $\varrho \leqslant \frac{1}{2}(k-1)$.

Case (ii). Let $k = 3$. Since $\sum_{n=1}^{\infty} \frac{r_3(n)}{n^{3/2}} = +\infty$, and $r_3(4n) = r_3(n)$, we have $\sum_{4\nmid n} \frac{r_3(n)}{n^{3/2}} = +\infty$, and $r_3(n) = 0$ if and only if $n = 4^a(8m+7)$. If $n = g^2 q$, where q is square-free, and $4\nmid n$, then $g^2 \equiv 1 \bmod 8$, so that $q \equiv 7 \bmod 8$ if and only if $n \equiv 7 \bmod 8$, i.e. $r_3(n) \neq 0$ if and only if

$r_3(q) \neq 0$. From this and the explicit formula for $r_3(n)$, originating in the work of Gauss ([11], p. 253; [14]), we obtain

$$r_3(n) \leqslant c_\varepsilon g^{1+\varepsilon} r_3(q), \quad 0 < \varepsilon < 1$$

if $4 \nmid n$. Hence

$$\sum_{4\nmid n} \frac{r_3(n)}{n^{3/2}} \leqslant c_\varepsilon \sum_q \frac{r_3(q)}{q^{3/2}} \sum_{g \text{ odd}} \frac{g^{1+\varepsilon}}{g^3} = c \sum_q \frac{r_3(q)}{q^{3/2}}$$

and since the series on the left diverges, so does $\sum \frac{r_3(q)}{q^{3/2}}$. Hence (62) and (63) hold for $k = 3$ as well.

Case (iii). Let $k = 2$. Since any prime p of the form $4n+1$ is expressible as a sum of two squares, we have, if q stands for a square-free integer,

$$\sum \frac{r_2(q)}{q} \geqslant \sum_{p \equiv 1 \bmod 4} \frac{r_2(p)}{p} \geqslant \sum_{p \equiv 1 \bmod 4} \frac{1}{p} = +\infty .$$

(See [11], p. 155). Hence (62) and (63) hold for $k = 2$ as well. Thus, if $\varrho > \frac{1}{2}$, we have

$$P_2^\varrho(x) = \begin{cases} O(x^{\frac{1}{2}(\varrho+\frac{1}{2})}), \\ \Omega_\pm(x^{\frac{1}{2}(\varrho+\frac{1}{2})}), \end{cases}$$

whereas if $\varrho \leqslant \frac{1}{2}$,

$$\limsup_{x\to\infty} \frac{P_2^\varrho(x)}{x^{\frac{1}{2}(\varrho+\frac{1}{2})}} = +\infty, \qquad \liminf_{x\to\infty} \frac{P_2^\varrho(x)}{x^{\frac{1}{2}(\varrho+\frac{1}{2})}} = -\infty .$$

If $\varrho = 0$, we get Ingham's result (8).

Remark. For $k \geqslant 4$, better estimates are known in the case $\varrho = 0$. See Petersson [16].

Example 4. Case (i). Let S be an indefinite quadratic form in more than four variables, with rational coefficients, and positive determinant. Then the generating function is Siegel's zeta-function ([18], [19]) which satisfies functional equation (14) and the same conclusions as in Example 3 are valid. We have only to note that $\mu(S, t) > c \cdot t^{k/2-1}$ if it is non-zero, and that all sufficiently large integers of certain arithmetical progressions are representable ([17]).

Case (ii). The same remarks as above apply to the case of positive definite forms, with rational coefficients, in which case the generating function is Epstein's zeta-function, which satisfies equation (14). If the coefficients are real, then the second part of estimate (62) holds for $\varrho > 0$.

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