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Distributions of the values of some arithmetical functions

by

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§ 1. Y. Wang and A. Schinzel proved, by Brun's method, the following theorem ([3]):

For any given sequence of h non-negative numbers $a_1, a_2, \ldots, a_h$ and $\varepsilon > 0$, there exist positive constants $c = c(a, \varepsilon)$ and $x_0 = x_0(a, \varepsilon)$ such that the number of positive integers $n \leqslant x$ satisfying

$$\left|\frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i\right| < \varepsilon \quad (1 \leqslant i \leqslant h)$$

is greater than $cx/\log^{h+1}x$, whenever $x > x_0$.

They also proved the analogous theorem for the function σ.

Shao Pin Tsung, also using Brun's method, extended this result to all multiplicative positive functions $f_s(n)$ satisfying the following conditions ([4]):

I. *For any positive integer l and prime number p:*

$$\lim_{p\to\infty} (f_s(p^l)/p^{ls}) = 1 \quad (p \textit{ denotes primes}).$$

II. *There exists an interval $\langle a, b\rangle$, $a = 0$ or $b = \infty$, such that for any integer $M > 0$ the set of numbers $f_s(N)/N^s$, where $(N, M) = 1$, is dense in $\langle a, b\rangle$.*

(This formulation is not the same but equivalent to the original one.)

In this paper we shall show without using Brun's method that *if we replace the condition* I *by the condition*

$$\sum \frac{(f_s(p)-p^s)^2}{p^{2s+1}} < \infty$$

(*but preserving condition* II) *then there exist more than $C(a, \varepsilon)x$ posi-*

tive integers $n \leqslant x$ for which

$$\left|\frac{f_s(n+i)}{f_s(n+i-1)} - a_i\right| < \varepsilon \quad (i = 1, 2, \ldots, h).$$

This theorem follows easily from the following stronger theorem.

THEOREM 1. *Let $f(n)$ be an additive function, satisfying the following conditions*

1. $\sum_p (\|f(p)\|^2/p)$ *is convergent, where $\|f\|$ denotes $f(p)$ for $|f(p)| \leqslant 1$ and 1 for $|f(p)| > 1$.*

2. *There exists a number c_1 such that, for any integer $M > 0$, the set of numbers $f(N)$, where $(N, M) = 1$ is dense in (c_1, ∞).*

Then, for any given sequence of h real numbers $a_1, a_2, \ldots, a_h$ and $\varepsilon > 0$, there exist more than $C(a, \varepsilon)x$ positive integers $n \leqslant x$ for which

$$(1) \qquad |f(n+i) - f(n+i-1) - a_i| < \varepsilon \quad (i = 1, 2, \ldots, h);$$

$C(a, \varepsilon)$ is a positive constant, depending on ε and a_i.

LEMMA. *There exists an absolute constant c such that the number of the integers of the form $pq > x$ for which one can find $n \leqslant x$ satisfying $n \equiv b \pmod a$, $n \equiv 0 \pmod p$ and $n+1 \equiv 0 \pmod q$ is for $x > x_0(a)$ less than cx/a.*

Proof. Let $c_1, c_2, \ldots$ denote absolute constants. Assume $p > x^{1/2}$ ($q > x^{1/2}$ can be dealt similarly). Denote by $A_l(x)$ the number of integers of the form pq satisfying

$$pq > x, \quad x^{1-1/2^l} \leqslant p < x^{1-1/2^{l+1}}, \quad n \equiv b \pmod a, \quad p|n, \; q|n+1, \quad \text{for some } n,\; 1 \leqslant n \leqslant y,$$

and by $A_l'(x)$ the number of integers pq for which

$$x^{1-1/2^l} \leqslant p < x^{1-1/2^{l+1}}, \quad q > x^{1/2^{l+1}}, \quad n \equiv b \pmod a, \quad p|n, \quad q|n+1, \quad \text{for some } n, \quad 1 \leqslant n \leqslant x.$$

Clearly $A_l'(x) \geqslant A_l(x)$ and it will suffice to prove that for $x > x_0(a)$, $\sum_{l=1}^{\infty} A_l'(x) < cx/a$.

Define positive integer l_x by the inequality

$$2^{l_x} \geqslant \frac{1}{a}\log x > 2^{l_x - 1}.$$

The number k of integers n satisfying

$$(2) \quad n \leqslant x, \quad n \equiv b \pmod a, \quad n \equiv 0 \pmod p, \quad x^{1-1/2^l} < p < x^{1-1/2^{l+1}}$$

for an $l \geqslant l_x$ does not exceed $\sum_{x \geqslant p > x^{1-2^{-l_x}}} \left(\left[\frac{x}{pa}\right] + 1\right)$, thus by theorems of Mertens and Chebyshev

$$k < \frac{c_1 x}{a 2^{l_x}} + \frac{c_2 x}{\log x}$$

and by the definition of l_x

$$k < \frac{c_3 x}{\log x}.$$

Denote the numbers satisfying (2) for an $l \geqslant l_x$ by $a_1 < a_2 < \ldots < a_k \leqslant x$. Since for all $y \leqslant x$, $\nu(y) < c_4 \log x / \log\log x$ (from the prime number theorem or from more elementary results), we have

$$(3) \qquad \sum_{l \geqslant l_x} A_l'(x) \leqslant \sum_{i=1}^{k} \nu(a_i) < \frac{c_3 x}{\log x} \cdot \frac{c_4 \log x}{\log\log x} < \frac{c_5 x}{a}$$

for $x > x_1(a)$.

For $l < l_x$ denote numbers satisfying (2) by $a_1^{(l)} < a_2^{(l)} < \ldots < a_{k_l}^{(l)}$. Similarly as for k we have for k_l the inequality

$$k_l < \frac{c_6 x}{a 2^{l+1}} + \frac{c_2 x}{\log x}$$

hence by $l < l_x$

$$(4) \qquad k_l < \frac{c_7 x}{a \cdot 2^l}.$$

We shall prove that for $l < l_x$ and sufficiently large x

$$(5) \qquad A_l'(x) = \sum_{i=1}^{k_l} \nu_l(a_i^{(l)} + 1) < \frac{c_8 x}{a \cdot l^2}$$

where $\nu_l(m)$ denotes the number of prime factors $> x^{1/2^{l+1}}$ of m.

For this purpose, we split the summands of the sum (5) into two classes. In the first class are the integers $a_i^{(l)}$ for which $\nu_l(a_i^{(l)}+1) \leqslant 2^l/l^2$. From (4) it follows that the contribution of these integers $a_i^{(l)}$ to (5) is less than $c_7 x/al^2$. The integer in the second class satisfy $\nu_l(a_i^{(l)}+1) > 2^l/l^2$. Thus these integers are divisible by more than $2^l/l^2$ primes $q > x^{1/2^{l+1}}$. Thus the number of integers of the second class is less than

$$\frac{x\left(\sum_{x^{1/2^{l+1}} < p \leqslant x} \frac{1}{q}\right)^{[2^l/l^2]}}{[2^l/l^2]!} + \left[\frac{2^l}{l^2}\right]! < \frac{x(c_9 l)^{[2^l/}}{a[2^l/l^2]!} + \left[\frac{2l^{l_x}}{l_x^2}\right]! < \frac{x}{a \cdot 5^l} + \left[\frac{2\log x}{a \log\log x}\right]! < \frac{x}{a \cdot 4^l}$$

for $l > c_{10}$, $x > x_2(a)$. By definition, $\nu_l(a_i^{(l)}+1) < 2^{l+1}$. Thus, for $l > c_{10}$, the contribution of the numbers of the second class to (5) is $< x/a\cdot 2^{l-1}$; for $l \leqslant c_{10}$ the contribution is clearly $< 2^{c_{10}+1}x$. Thus, for $l < l_x$, $x > x_2(a)$,

$$A'_l(x) < c_8 x/al^2$$

and in view of (3) we have for $x > x_0(a)$

$$\sum_{l=1}^{\infty} A'_l(x) < \frac{c_5 x}{a} + \sum_{l<l_x} \frac{c_8 x}{al^2} < \frac{cx}{a}$$

which proves the Lemma.

Proof of the theorem. Let ε be a positive number and let a sequence a_i $(i = 1, 2, \ldots, h)$ be given.

By condition 2 we can find positive integers $N_0, N_1, \ldots, N_h$ such that

(6) $\big(N_i, (h+1)!\big) = 1 \quad (i = 0, 1, \ldots, h), \quad (N_i, N_j) = 1 \quad (0 \leqslant i < j \leqslant h),$

$$f(N_0) > c_1 + \max_{1\leqslant i\leqslant h}\Big\{f(i+1) - \sum_{j=1}^{i} a_j\Big\}$$

and

$$\Big|f(N_i) - \Big\{f(N_0) - f(i+1) + \sum_{j=1}^{i} a_j\Big\}\Big| < \tfrac{1}{4}\varepsilon \quad (1 \leqslant i \leqslant h);$$

hence

(7) $$\big|f\big((i+1)N_i\big) - f(iN_{i-1}) - a_i\big| < \tfrac{1}{2}\varepsilon \quad (1 \leqslant i \leqslant h).$$

Let k_1 be the greatest prime factor of $N_0 N_1 \ldots N_h$. Put $\mu = \varepsilon/\sqrt{96hc}$ (c is the constant of the Lemma). By condition 1, $\sum_{|f(p)|\geqslant\mu} (1/p)$ is convergent. Since $\sum_p (1/p^2)$ is also convergent, there exists a k_2 such that

(8) $$\sum_{\substack{|f(p)|\geqslant\mu \\ p>k_2}} \frac{1}{p} + \sum_{p>k_2} \frac{1}{p^2} < \frac{1}{3(h+1)}.$$

Finally by condition 1 there exists a k_3 such that

(9) $$\sum_{\substack{|f(p)|<\mu \\ p>k_3}} \frac{f(p)^2}{p} < \frac{\varepsilon^2}{48h}.$$

Let us put

$$k = \max(k_1, k_2, k_3), \quad N = N_1 N_2 \ldots N_h, \quad P = \prod_{\substack{p\leqslant k \\ p\nmid N}} p, \quad Q = (h+1)!\,N^2 P$$

and let us consider the following system of congruences

$$n \equiv 1 \,(\mathrm{mod}\,(h+1)!P), \quad n \equiv -i + N_i \,(\mathrm{mod}\, N_i^2), \quad 0 \leqslant i \leqslant h.$$

By (6) and the Chinese Remainder Theorem there exists a number n_0 satisfying these congruences.

It is easy to see that

(10) *for every integer t the numbers $(Qt+n_0+i)/(i+1)N_i$ $(i = 1, 2, \ldots, h)$ are integers which are not divisible by any prime $\leqslant k$;*

(11) *the number of terms not exceeding x of the arithmetical progression $Qt+n_0$ is $x/Q+O(1)$.*

In order to prove Theorem 1 we shall estimate the number of integers n of the progression $Qt+n_0$ which satisfy the inequalities

(12) $$n \leqslant x, \quad \sum_{i=1}^{h}\Big(f(n+i) - f(n+i-1) - f\big((i+1)N_i\big) + f(iN_{i-1})\Big)^2 > \tfrac{1}{4}\varepsilon^2.$$

We divide the set of integers $n \equiv n_0 \,(\mathrm{mod}\, Q)$ for which the inequalities (12) hold into two classes. Integers n such that $n(n+1)\ldots(n+h)$ is divisible by a prime $p > k$ with $|f(p)| \geqslant \mu$, or by p^2, $p > k$, are in the first class and all other integers are in the second class.

(13) *The number of integers $n \leqslant x$, $n \equiv r \,(\mathrm{mod}\, Q)$ which are divisible by a given integer $d > 0$ is equal to $x/dQ+O(1)$ for $(d, Q) = 1$,*

hence the number of integers $n \leqslant x$, $n \equiv n_0 \,(\mathrm{mod}\, Q)$ of the first class is less than

$$(h+1)\frac{x}{Q}\Big(\sum_{\substack{p>k \\ |f(p)|\geqslant\mu}} \frac{1}{p} + \sum_{p>k} \frac{1}{p^2}\Big) + O\Big(\sum_{p\leqslant x+h} 1 + \sum_{p^2\leqslant x+h} 1\Big).$$

By the inequality (8) and the definition of k this number is less than $\frac{1}{3}x/Q + o(x)$.

For the integers of the second class, by remark (10) we have

$$\sum_n{}'' \sum_{i=1}^{h}\Big(f(n+i) - f(n+i-1) - f\big((i+1)N_i\big) + f(iN_{i-1})\Big)^2$$

$$= S = \sum_n{}'' \sum_{i=1}^{h}\Big\{\sum_{\substack{p|n+i \\ p>k}} f(p) - \sum_{\substack{p|n+i-1 \\ p>k}} f(p)\Big\}^2,$$

where $\sum''$ means that the summation runs through the integers of the second class. In view of remark (13), since $(Q,p)=1$ we have

$$S \leqslant \sum_{\substack{n\equiv n_0 (\mathrm{mod}\, Q)\\ n\leqslant x}} \sum_{i=1}^{h} \Big\{ \sum_{\substack{p|n+i\\ p>k,\, |f(p)|<\mu}} f(p) - \sum_{\substack{p|n+i-1\\ p>k,\, |f(p)|<\mu}} f(p) \Big\}^2$$

$$= \sum_{\substack{x+h\geqslant p>k\\ |f(p)|<\mu}} f^2(p)\Big(\frac{2hx}{Qp}+O(1)\Big)+$$

$$+\sum_{\substack{n\equiv n_0 (\mathrm{mod}\, Q)\\ n\leqslant x}} \sum_{i=1}^{h} \Big\{ 2 \sum_{\substack{pq|n+i,\, q>p>k,\\ |f(p)|<\mu,\, |f(q)|<\mu}} f(p)f(q) +$$

$$+2 \sum_{\substack{pq|n+i-1,\, q>p>k\\ |f(p)|<\mu,\, |f(q)|<\mu}} f(p)f(q) - 2 \sum_{\substack{p|n+i,\, q|n+i-1,\, q>k\\ p>k,\, |f(p)|<\mu,\, |f(q)|<\mu}} f(p)f(q) \Big\}$$

$$\leqslant \frac{2hx}{Q} \sum_{p>k,\, |f(p)|<\mu} \frac{f^2(p)}{p} + \sum_{\substack{n\equiv n_0 (\mathrm{mod}\, Q)\\ n\leqslant x}} \sum_{i=1}^{h} 2 \sum_{\substack{p|n+i,\, q|n+i-1\\ pq\geqslant x,\, p>k,\, q>k\\ |f(p)|<\mu,\, |f(q)|<\mu}} |f(p)f(q)| +$$

$$+O\Big(\sum_{\substack{p\leqslant x+h\\ |f(p)|<\mu}} f^2(p) + \sum_{\substack{p>q>k,\, pq\leqslant x+h\\ |f(p)|<\mu,\, |f(q)|<\mu}} |f(p)f(q)| \Big).$$

Thus finally from (9), Lemma, the equality $\mu^2 = \varepsilon^2/96hc$ and from the fact that the number of integers of the form pq not exceeding $x+h$ is $o(x)$, we get

$$S < \frac{\varepsilon^2}{12}\cdot\frac{x}{Q}+o(x).$$

Thus the number of integers of the second class is less than $\frac{1}{3}x/Q+o(x)$.

Hence there exist less than $\frac{2}{3}x/Q+o(x)$ positive integers $n\leqslant x$, $n\equiv n_0 (\mathrm{mod}\, Q)$ for which

$$\sum_{i=1}^{h}\Big(f(n+i)-f(n+i-1)-f\big((i+1)N_i\big)+f(iN_{i-1})\Big)^2 > \tfrac{1}{4}\varepsilon^2.$$

Therefore by (11) there exist more than $\frac{1}{3}x/Q+o(x)$ positive integers $n\leqslant x$, for which

$$\sum_{i=1}^{h}\Big(f(n+i)-f(n+i-1)-f\big((i+1)N_i\big)+f(iN_{i-1})\Big)^2 \leqslant \tfrac{1}{4}\varepsilon^2$$

and then

$$\big|f(n+i)-f(n+i-1)-f\big((i+1)N_i\big)+f(iN_{i-1})\big| \leqslant \tfrac{1}{2}\varepsilon \quad (i=1,2,\ldots,h).$$

In view of (7), the proof is complete.

Theorem 2. *Let $f(n)$ be an additive function satisfying the conditions of Theorem 1 and such that partial sums of $\sum(\|f(p)\|/p)$ are bounded:*

$$A > |S_k|,\quad S_k = \sum_{p\leqslant k}\frac{\|f(p)\|}{p}. \tag{14}$$

Then for any given natural number h there exists a number c_h such that for any $\varepsilon>0$ and every sequence of h numbers: $a_1, a_2, \ldots, a_h \geqslant c_h$, there exist more than $C(a,\varepsilon)x$ positive integers $n\leqslant x$, for which

$$|f(n+i)-a_i| < \varepsilon \quad (i=1,2,\ldots,h). \tag{15}$$

$C(a,\varepsilon)$ is a positive constant, depending on ε and a_i.

Proof. Let ε be a positive number, $c_h = c_1+\max f(i)$ and let a sequence $a_i\geqslant c_h$ $(i=1,2,\ldots,h)$ be given.

By condition 2 we can find positive integers $N_1, N_2, \ldots, N_h$ such that

$$(N_i, h!)=1 \quad (i=1,2,\ldots,h), \qquad (N_i,N_j)=1 \quad (1\leqslant i<j\leqslant h) \tag{16}$$

and

$$|f(N_i)-a_i+f(i)| < \tfrac{1}{2}\varepsilon \quad (i=1,2,\ldots,h). \tag{17}$$

Let k_1 be the greatest prime factor of $N_1N_2\ldots N_h$. Let C be an absolute constant such that

$$\sum_{y\leqslant p<z}\frac{1}{p} < C\log\frac{\log z}{\log y} \quad \text{for all} \quad z>y>1.$$

Put $\mu = \varepsilon/7C\sqrt{h}$. By condition 1, $\sum_{|f(p)|\geqslant\mu}(1/p)$ is convergent. Since $\sum(1/p^2)$ is also convergent, there exists a k_2 such that

$$\sum_{|f(p)|\geqslant\mu,\, p>k_2}\frac{1}{p}+\sum_{p>k_2}\frac{1}{p^2} < \frac{1}{3h}. \tag{18}$$

By condition 1 there exists also a k_3 such that

$$\sum_{p>k_3,\, |f(p)|<\mu}\frac{f(p)^2}{p} < \frac{\varepsilon^2}{24h}. \tag{19}$$

Put $\eta = \varepsilon/\sqrt{96h}$, $B = A+1/3h$ and denote by I_ν the interval

$$[\nu\eta-\tfrac{1}{2}\eta,\ \nu\eta+\tfrac{1}{2}\eta], \quad \nu = 0, \pm1, \pm2, \ldots, \pm[B/\eta+1]$$

and let k_ν be the least integer $k > \max(k_1, k_2, k_3)$ such that $\sum_{p\leqslant k, |f(p)|k\mu} (f(p)/p) \in I_\nu$ if such integers k exist, otherwise let $k_\nu = 1$.

Now if $\sum_{p\leqslant x+h, |f(p)|<\mu} (f(p)/p) \in I_{\nu_x}$—by the condition (14) and by (18) such ν_x certainly exists—we put $k_{\nu_x} = k$ and then we get

$$\left|\sum_{\substack{x+h\geqslant p>k\\ |f(p)|<\mu}} \frac{f(p)}{p}\right| < \eta, \quad k \leqslant \max_{|\nu|\leqslant[B/\eta]+1} k_\nu = \bar{k}. \tag{20}$$

Let $\sum'$ denote that the summation runs through all primes p, q satisfying conditions $p > q > k$, $pq \leqslant x+h$, $|f(p)| < \mu$, $|f(q)| < \mu$. From (20) we get

$$2\sum{}' \frac{f(p)f(q)}{pq} \leqslant \Bigg(\sum_{\substack{x+h\geqslant p+k\\ |f(p)|<\mu}} \frac{f(p)}{p}\Bigg)^2 + \sum_{x+h\geqslant p>\sqrt{x+h}} \frac{\mu}{p} \sum_{x+h\geqslant q>\frac{x+h}{p}} \frac{\mu}{q} \tag{21}$$

$$\leqslant \frac{\varepsilon^2}{96h} + \sum_{l=2}^{\infty} \sum_{(x+h+1)^{1-1/2^l}>p\geqslant(x+h+1)^{1-1/2^{l-1}}} \frac{\mu}{p} \sum_{x+h\geqslant q>\frac{x+h}{p}} \frac{\mu}{q}$$

$$\leqslant \frac{\varepsilon^2}{96h} + \mu^2 C^2 \sum_{l=2}^{\infty} \frac{l}{2^l} = \frac{\varepsilon^2}{96h} + \mu^2 C^2 \frac{3}{2} < \frac{\varepsilon^2}{24h}.$$

Let us put $N = N_1 N_2 \ldots N_h$, $\quad P = \prod_{p\leqslant k, p\nmid N} p$,

$$Q = h!\,N^2 P \leqslant h!\,N^2 \prod_{p\leqslant \bar{k}, p\nmid N} p = \bar{Q} \tag{22}$$

and let us consider the following system of congruences:

$$n \equiv 0 \;(\mathrm{mod}\, h!P), \quad n \equiv -i + N_i \;(\mathrm{mod}\, N_i^2).$$

By (16) and the Chinese Remainder Theorem there exists a number n_0 satisfying these congruences.

It is easy to see, that

(23) *for every integer t the numbers* $\dfrac{Qt+n_0+i}{iN_i}$ $(i = 1, 2, \ldots, h)$ *are integers, which are not divisible by any prime* $\leqslant k$.

Analogously, as in the proof of Theorem 1, we shall estimate the number of integers n of the progression $Qt+n_0$, which satisfy the inequalities

$$n \leqslant x, \quad \sum_{i=1}^{h} \big(f(n+i) - f(iN_i)\big)^2 > \tfrac{1}{4}\varepsilon^2. \tag{24}$$

We divide the set of integers $n \equiv n_0 \,(\mathrm{mod}\, Q)$, for which the inequalities (24) hold, into two classes. Integers n such that $(n+1)(n+2)\ldots(n+h)$ is divisible by a prime $p > k$ with $|f(p)| \geqslant \mu$ or by p^2, $p > k$, are in the first class and all others integers are in the second class.

By remark (13) the number of integers $n \leqslant x$, $n \equiv n_0 \,(\mathrm{mod}\, Q)$ of the first class is less than

$$h\frac{x}{Q}\Bigg(\sum_{p>k, |f(p)|\geqslant\mu} \frac{1}{p} + \sum_{p>k} \frac{1}{p^2}\Bigg) + O\Bigg(\sum_{p\leqslant x+h} 1 + \sum_{p^2\leqslant x+h} 1\Bigg).$$

By the inequality (18) and the definition of k this number is less than $\frac{1}{3}x/Q + o(x)$.

For the integers of the second class, by remark (23), we have

$$\sum_{i=1}^{h} \big(f(n+i) - f(iN_i)\big)^2 = \sum_{i=1}^{h}\Bigg(\sum_{p|n+i, p>k} f(p)\Bigg)^2$$

and

$$\sum_n{}'' \sum_{i=1}^{h} \big(f(n+i) - f(iN_i)\big)^2 = \sum_n{}'' \sum_{i=1}^{h}\Bigg(\sum_{p|n+i, p>k} f(p)\Bigg),$$

where $\sum_n''$ means that the summation runs through the integers of the second class. In view of remark (13), we have

$$\sum_n{}'' \sum_{i=1}^{h} \big(f(n+i) - f(iN_i)\big)^2 \leqslant \sum_{\substack{n\equiv n_0 (\mathrm{mod} Q)\\ n\leqslant x}} \sum_{i=1}^{h}\Bigg(\sum_{p|n+i, p>k} f(p)\Bigg)^2$$

$$= \sum_{\substack{x+h\geqslant p>k\\ |f(p)|<\mu}} f^2(p)\left(\frac{hx}{Qp} + O(1)\right) + 2\sum{}' f(p)f(q)\left(\frac{hx}{Qpq} + O(1)\right)$$

$$\leqslant \frac{hx}{Q}\Bigg(\sum_{p>k, |f(p)|<\mu} \frac{f^2(p)}{p} + 2\sum{}' \frac{f(p)f(q)}{pq}\Bigg) +$$

$$+ O\Bigg(\sum_{\substack{p\leqslant x+h\\ |f(p)|<\mu}} f^2(p) + \sum{}' |f(p)f(q)|\Bigg).$$

Thus, finally from (19), (21) and from the fact that the number of integers of the form pq not exceeding $x+h$ is $o(x)$ we get

$$\sum{}'' \sum_{i=1}^{h} \big(f(n+i) - f(iN_i)\big)^2 < \frac{\varepsilon^2}{12} \cdot \frac{x}{Q} + o(x).$$

Thus the number of integers of the second class is less than $\frac{1}{3}x/Q+o(x)$.

Hence, there exist less than $\frac{2}{3}x/Q+o(x)$ positive integers $n \leqslant x$, $n \equiv n_0 \pmod{Q}$ for which

$$\sum_{i=1}^{h}\big(f(bn+i)-f(iN_i)\big)^2 > \tfrac{1}{4}\varepsilon^2.$$

By (11) and (22) there exist, therefore, more than $\frac{1}{3}x/\bar{Q}+o(x)$ positive integers $n \leqslant x$, for which

$$\sum_{i=1}^{h}\big(f(n+i)-f(iN_i)\big)^2 \leqslant \tfrac{1}{4}\varepsilon^2,$$

and then

$$|f(n+i)-f(iN_i)| \leqslant \tfrac{1}{2}\varepsilon \quad (i = 1, 2, \ldots, h).$$

In view of (16) and (17), this completes the proof.

Theorem 2 is best possible. Assume only that there exists an a and a $c>0$ so that the number of integers $n \leqslant x$ satisfyng $|f(n)|<a$ is greater than cx.

Then $\sum \frac{\|f(p)\|^2}{p}$ *converges and* $\sum \frac{\|f(p)\|}{p}$ *has bounded partial sums.*

In the paper [2], P. Erdös proved([1]) the following theorem:

If there exist two constants c_1 and c_2 and an infinite sequence $x_k \to \infty$ so that for every x_k there are at least $c_1 x_k$ integers:

$$1 \leqslant a_1 < a_2 < \ldots < a_l \leqslant x_k, \quad l \geqslant c_1 x_k,$$

for which

$$|f(a_i)-f(a_j)| < c_2, \quad 1 \leqslant i < j \leqslant l,$$

then

$$f(n) = \alpha \log n + g(n), \quad \textit{where} \quad \sum \frac{\|g(p)\|^2}{p} < \infty.$$

In our case the conditions of this theorem are clearly satisfied and, in fact, we clearly must have $\alpha = 0$. This implies that

$$\sum \frac{\|f(p)\|^2}{p} < \infty.$$

([1]) The proof of Lemma 8 [2] is not clear and on p. 15 needs more details similar to these given above.

Assume now that $\sum(\|f(p)\|/p)$ does not have bounded partial sums. Let e.g. $\sum_{p<x}(\|f(p)\|/p) = A$, A large. Then by the method of Turán ([5], cf. also [2]) we obtain

$$\sum_{n=1}^{x}\big(f(n)-A\big)^2 < c_3 x$$

which implies that $|f(n)-A| < A-a$ for all but ηx integers $n \leqslant x$, where $\eta = c_3/(A-a)^2$. For sufficiently large A, it contradicts the assumption that $|f(n)|<a$ has cx solutions $n \leqslant x$, thus the proof is complete.

In Theorem 1 one can replace $\sum(\|f(p)\|^2/p) < \infty$ by: *there is an α so that if we put $f(n)-\alpha\log n = g(n)$ then $\sum(\|g(p)\|^2/p) < \infty$.* We think that here we again have a necessary and sufficient condition, but we cannot prove this. In fact, we conjecture that if there exist an a and an $c>0$ such that the number of integers $n \leqslant x$ satisfying $|f(n+1)-f(n)|<a$ is $>cx$, then

$$f(n) = \alpha\log n + g(n) \quad \text{with} \quad \sum \frac{\|g(p)\|^2}{p} < \infty.$$

§ 2. The proof of Theorem 2 is very similar to the proof of Lemma 1 of P. Erdös' paper [1]. Using ideas and results from that paper we can prove the following theorem.

THEOREM 3. *Let $f(n)$ be an additive function satisfying condition 1 of Theorem 1 and let $\sum_{f(p)\neq 0}(1/p)$ be divergent, $\sum(\|f(p)\|/p)$ convergent, then the distribution function of h-tuples $\{f(m+1), f(m+2), \ldots, f(m+h)\}$ exists, and it is a continuous function.*

Proof. We denote by $N(f; c_1, c_2, \ldots, c_h)$ the number of positive integers m not exceeding n, for which

$$f(m+i) \geqslant c_i, \quad i = 1, 2, \ldots, h,$$

where c_i are given constants.

It is sufficient to consider, as in [1], the special case in which, for any α, $f(p^\alpha) = f(p)$, so that

$$f(m) = \sum_{p|m} f(p).$$

Let us also consider the function $f_k(m) = \sum_{p|m,\, p\leqslant k} f(p)$. We are going to show that the sequence $N(f_k; c_1, c_2, \ldots, c_h)/n$ is convergent. For, if we denote by $A_{i,j}$ $(j \leqslant j_{0,i})$ the squarefree integers whose prime factors are not greater than k, and for which $f_k(A_{i,j}) \geqslant c_i$, we can see that the integers m for which

$$f_k(m+i) \geqslant c_i \quad (i = 1, 2, \ldots, h)$$

are distributed periodically with the period $\prod_{\substack{1\leqslant i\leqslant h\\ 1\leqslant j\leqslant j_{0,i}}} A_{i,j}$. Hence $N(f_k; c_1, c_2, \ldots, c_h)/n$ has a limit.

To prove the existence of a limit of $N(f; c_1, c_2, \ldots, c_h)/n$ it is sufficient to show that for arbitrary $\varepsilon > 0$ there exists k_0 such that for every $k > k_0$ and $n > n(\varepsilon)$

$$|N(f; c_1, c_2, \ldots, c_h) - N(f_k; c_1, c_2, \ldots, c_h)|/n < \varepsilon.$$

To show this, it is enough to prove that the number of integers $m \leqslant n$ for which there exists $i \leqslant h$ such that $f_k(m+i) < c_i$ and $f(m+i) \geqslant c_i$ or $f_k(m+i) \geqslant c_i$ and $f(m+i) < c_i$ is less than εhn. But it is an immediate consequence of the analogous theorem for $h = 1$ proved in [1], p. 123.

In order to prove that the distribution function is continuous we must show that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\Delta = N(f; c_1-\delta, c_2-\delta, \ldots, c_h-\delta) - N(f; c_1+\delta, c_2+\delta, \ldots, c_h+\delta) < \varepsilon.$$

Now

$$\Delta = \sum_{i=1}^{h} \{N(f; c_1+\delta, \ldots, c_{i-1}+\delta, c_i-\delta, \ldots, c_h-\delta) - N(f; c_1+\delta, \ldots, c_i+\delta, c_{i+1}-\delta, \ldots, c_h-\delta)\}$$

and by Lemma 2 of [1] each term of this sum is less than ε/h for suitably chosen δ. This completes the proof.

We conclude from Theorems 2 and 3 that if an additive function f satisfies conditions 1, 2, $\sum_{f(p)\neq 0} (1/p)$ is divergent and $\sum (\|f(p)\|/p)$ convergent, then the distribution function of $\{f(m+1), \ldots, f(m+h)\}$ exists, is continuous and strictly decreasing on some half straight-line, thus the sequence of integers n for which inequality (15) holds has a positive density. Similarly we can prove the following:

THEOREM 4. *Assume that* $\sum_{f(p)\neq 0} \frac{1}{p} = \infty$ *and that* $\sum \frac{\|f(p)\|^2}{p} < \infty$ *then* $\{f(n+1)-f(n), f(n+2)-f(n+1), \ldots, f(n+k)-f(n+k-1)\}$ *has a continuous distribution function.*

It is easy to see that condition 2 can be replaced by the conditions

$$\lim_{p\to\infty} f(p) = 0 \quad \text{and} \quad \sum_p |f(p)| = \infty.$$

§ 3. Y. Wang proved in [6] that the number N of primes $p < x$ satisfying

$$\left|\frac{\varphi(p+\nu+1)}{\varphi(p+\nu)} - a_\nu\right| < \varepsilon, \quad 1 \leqslant \nu \leqslant k$$

is greater than

$$c(a, \varepsilon)\frac{x}{(\log x)^{k+2}\log\log x}.$$

By our methods we can obtain in that case

$$N > c_1(a, \varepsilon)\frac{x}{\log x}.$$

After having passed to the additive function $\log(\varphi(n)/n)$ the proof is similar to the proof of Theorem 1. We use the fact, that $\log(\varphi(n)/n)$ is always negative, and apply the asymptotic formula for the number of primes in arithmetical progression instead of (11) and the Brun-Titchmarsh theorem instead of (13).

We can also prove that there exists distribution function $N(c_1, c_2, \ldots, c_k)$ defined as

$$\lim_{x\to\infty} \frac{1}{\pi(x)} N\left(p < x; \frac{\varphi(p+\nu)}{p+\nu} \geqslant c_\nu, \quad \nu = 1, 2, \ldots, k\right).$$

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