ACTA ARITHMETICA
VI (1961)

On Weyl's inequality and Waring's problem for cubes

by

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1. The most important single result used in the treatment of Waring's Problem and similar problems is Weyl's inequality. We first recall this, limiting ourselves to cubic exponential sums of the type

$$T = \sum_{x=1}^{P} e(\alpha x^3 + \alpha_2 x^2 + \alpha_1 x), \tag{1}$$

where α, α_2, α_1 are real and $e(\theta)$ denotes $e^{2\pi i\theta}$. Let h/q be any rational approximation to α satisfying

$$(h, q) = 1, \qquad |\alpha - h/q| \leqslant q^{-2}; \tag{2}$$

then Weyl's inequality (¹) asserts that

$$|T| \ll P^{\varepsilon}(P^{\frac{3}{4}} + P^{\frac{1}{4}}q^{\frac{1}{4}} + Pq^{-\frac{1}{4}}) \tag{3}$$

for any fixed $\varepsilon > 0$. In particular, if q lies between fixed multiples of P and P^2, we get the estimate $P^{3/4+\varepsilon}$, and these conditions on q correspond (roughly) to α being on the 'minor arcs' in the application to Waring's Problem.

The main object of the present paper is to show that Weyl's inequality can be extended, without loss of precision, to an exponential sum in two variables, in which αx^3 is replaced by $\alpha f(x, y)$, where

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \tag{4}$$

is any fixed binary cubic form with integral coefficients and non-zero discriminant (²). The inequality does not extend in the same way to

(¹) There are many slightly different forms of Weyl's inequality in the literature. That given here seems to be due to Vinogradov, Izvestiya Akad. Nauk SSSR 21 (1927), pp. 567-578.

(²) The discriminant of the form (4) is the invariant $18abcd + b^2c^2 - 4ac^3 - 4db^3 - 27a^2d^2$.

cubic forms in more than two variables or to forms of higher degree in two variables (3). Stated formally, our result is:

THEOREM 1. *Let $f(x, y)$ be any fixed binary cubic form with integral coefficients and non-zero discriminant, and let $\varphi(x, y)$ be any real polynomial of degree at most 2. Let*

$$S = \sum_{x=1}^{P}\sum_{y=1}^{Q} e\big(af(x, y)+\varphi(x, y)\big), \tag{5}$$

where $1 \leqslant Q \leqslant P$. Then, subject to (2), we have

$$|S| \ll P^{\varepsilon}(P^{\frac{3}{2}}+P^{\frac{1}{2}}q^{\frac{1}{2}}+P^{2}q^{-\frac{1}{2}}) \tag{6}$$

for any fixed $\varepsilon > 0$.

This theorem enables us to generalize some of the results related to Waring's Problem for cubes which are known already, either explicitly or implicitly. If $a_1, \ldots, a_8$ are any non-zero integers and N is any integer, it can be proved by a method of Davenport (4) that the equation

$$a_1x_1^3+\ldots+a_8x_8^3 = N \tag{7}$$

has infinitely many solutions in integers, provided that the corresponding congruence is soluble to every prime-power modulus. If $N = 0$, the latter condition is always satisfied and can be omitted. Theorem 1 enables us to replace the pair of terms $a_1x_1^3+a_2x_2^3$ in (7) by any binary cubic form $f(x_1, x_2)$, and to make similar replacements with other pairs of terms provided one pair of cubes is retained.

We shall prove one result of this kind in moderate detail:

THEOREM 2. *Let a_1, a_2 be non-zero integers and let f_3, f_4, f_5 be binary cubic forms with non-zero discriminants. Then the equation*

$$a_1x_1^3+a_2x_2^3+f_3(x_3, y_3)+f_4(x_4, y_4)+f_5(x_5, y_5) = 0 \tag{8}$$

has infinitely many primitive solutions in integers, with all the integers arbitrarily large.

We base our proof of the solubility of (8) in every p-adic field, which is a necessary part of the proof of Theorem 2, on a theorem of D. J. Lewis (see § 9).

(3) More recently Dr Birch has shown that our result extends to those forms of degree k in n variables that are expressible as a linear combination of n kth powers of real or complex linear forms. If $k > 3$, or if $k = 3$ and $n > 2$, such a representation is not generally possible.

(4) *On Waring's Problem for cubes*, Acta Mathematica 71 (1939), pp. 123-143. This paper will be referred to as W.P.C.

Two other results, which can be proved on similar lines, but for which the p-adic condition cannot be omitted, are as follows:

(a) *With the hypotheses of Theorem 1, the equation (8) with an integer $N \neq 0$ on the right has infinitely many solutions provided N is such that the corresponding congruence is soluble to every prime-power modulus.*

(b) *With similar hypotheses, the equation*

$$a_1x_1^3+a_2x_2^3+f(x_3, y_3) = N \tag{9}$$

is soluble for almost all those positive integers N for which the corresponding congruence is soluble to every prime-power modulus; that is, the number of exceptional N between 0 and X is $o(X)$ as $X \to \infty$. In particular, the positive integers represented by the left hand side of (9) have positive density.

We are grateful to Dr. B. J. Birch and Professor D. J. Lewis for helpful comments.

2. In this section we prove Theorem 1 under the additional hypothesis that the discriminant D of $f(x, y)$ is such that $-3D$ is not a square; the excluded case will be dealt with in § 3.

LEMMA 1. *Let*

$$\begin{aligned} B_1(x_1, y_1;\ x_2, y_2) &= 3ax_1x_2+b(x_1y_2+x_2y_1)+cy_1y_2, \\ B_2(x_1, y_1;\ x_2, y_2) &= bx_1x_2+c(x_1y_2+x_2y_1)+3dy_1y_2. \end{aligned} \tag{10}$$

Then for the sum S in (5) we have (5)

$$|S|^4 \ll P^2 \sum_{x_1, y_1}\sum_{x_2, y_2} \min(P, \|2\alpha B_1\|^{-1})\min(P, \|2\alpha B_2\|^{-1}), \tag{11}$$

where the summations are over integers of absolute value less than P.

Proof. See Lemma 3.1 of Davenport, *Cubic forms in 32 variables*, Phil. Trans. Royal Soc. A, 251 (1959), pp. 193-232 (6), where a similar result is proved for cubic forms in n variables. The proof is a natural extension of that used for the original Weyl's inequality; the bilinear forms B_1, B_2 arise as the coefficients of x and y in the second difference

$$f(x+x_1+x_2, y+y_1+y_2)-f(x+x_1, y+y_1) \\ -f(x+x_2, y+y_2)+f(x, y).$$

LEMMA 2. *Suppose that $-3D$ is not a square. Then for any integers m_1, m_2 the number of solutions of*

$$B_1(x_1, y_1;\ x_2, y_2) = m_1, \quad B_2(x_1, y_1;\ x_2, y_2) = m_2 \tag{12}$$

(5) We use $\|\theta\|$ to denote the difference between θ and the nearest integer.

(6) This paper will be referred to as C.F.

in integers x_1, y_1, not both 0, and x_2, y_2, not both 0, all of absolute value less than P, is $\ll P^\varepsilon$ for any fixed $\varepsilon > 0$.

Proof. Let

(13) $$\Delta(x, y) = (bx+cy)^2-(3ax+by)(cx+3dy) = Ax^2+Bxy+Cy^2,$$

say. The discriminant of this form is

(14) $$B^2-4AC = (bc-9ad)^2-4(b^2-3ac)(c^2-3bd) = -3D,$$

and this is not a perfect square. Hence $\Delta(x, y) \neq 0$ for integers x, y, not both 0.

We shall prove that the equations (12) imply that

(15) $$\Delta(x_1, y_1)\Delta(x_2, y_2) = \Delta(m_2, -m_1)$$

identically in x_1, y_1, x_2, y_2. Write (12) as linear equations in x_2, y_2:

(16) $$(3ax_1+by_1)x_2+(bx_1+cy_1)y_2 = m_1,$$

(17) $$(bx_1+cy_1)x_2+(cx_1+3dy_1)y_2 = m_2.$$

Their determinant is $-\Delta(x_1, y_1)$, and their solution is

$$\Delta(x_1, y_1)x_2 = -(cx_1+3dy_1)m_1+(bx_1+cy_1)m_2,$$
$$\Delta(x_1, y_1)y_2 = (bx_1+cy_1)m_1-(3ax_1+by_1)m_2.$$

Put

$$\lambda = 3am_2-bm_1, \quad \mu = bm_2-cm_1, \quad \nu = cm_2-3dm_1,$$

so that

(18) $$\mu^2-\lambda\nu = \Delta(m_2, -m_1).$$

Then

(19) $$\Delta(x_1, y_1)x_2 = \mu x_1+\nu y_1,$$

(20) $$\Delta(x_1, y_1)y_2 = -\lambda x_1-\mu y_1.$$

Hence

(21) $$\Delta(x_1, y_1)(\mu x_2+\nu y_2) = \Delta(m_2, -m_1)x_1.$$

On the other hand, the interchange of x_1, y_1 and x_2, y_2 in (19) gives

(22) $$\Delta(x_2, y_2)x_1 = \mu x_2+\nu y_2.$$

Hence, provided $x_1 \neq 0$, the last two equations imply (15). But in view of the identical nature of (15), it must hold independently of this condition.

In (15), $\Delta(m_2, -m_1)$ is a non-zero integer, since $\Delta(x_1, y_1) \neq 0$ and $\Delta(x_2, y_2) \neq 0$. Also $|\Delta(m_2, -m_1)| \ll P^4$. Hence the number of possibilities for $\Delta(x_1, y_1)$ and $\Delta(x_2, y_2)$ as factors of $\Delta(m_2, -m_1)$ is $\ll P^\varepsilon$.

If $\Delta(x, y)$ is a definite form, the number of pairs x_1, y_1 for which it assumes a particular value is $\ll P^\varepsilon$, and similarly for x_2, y_2, whence the desired result. If $\Delta(x, y)$ is indefinite, the same conclusion can still be drawn, for although the numbers of pairs x_1, y_1 may be infinite, they fall into $\ll P^\varepsilon$ classes, and pairs in the same class are related by a power of the fundamental solution of the Pellian equation $X^2-(B^2-4AC)Y^2 = -4$. Thus the number in each class with $|x_1| < P$, $|y_1| < P$ is $\ll \log P$, and the result follows.

Lemma 3. *Suppose α satisfies (2). Then for any integer H we have*

$$\sum_{m=H+1}^{H+q} \min(P, \|\alpha m\|^{-1}) \ll P+q\log q.$$

Proof. See Vinogradov, *The method of trigonometrical sums in the theory of numbers* (trans. Roth and Davenport), Lemma 8a of Chapter I, where a slightly more general result is proved.

Proof of Theorem 1 (first case). In the estimate (11) for $|S|^4$, we remove from the summation all terms with $x_1 = y_1 = 0$ and all terms with $x_2 = y_2 = 0$. This gives

$$|S|^4 \ll P^6+P^2 \sum_{x_1, y_1}{}' \sum_{x_2, y_2}{}' \min(P, \|2\alpha B_1\|^{-1})\min(P, \|2\alpha B_2\|^{-1}).$$

Putting $2B_1 = m_1$ and $2B_2 = m_2$ and using the result of Lemma 2, we obtain

(23) $$|S|^4 \ll P^6+P^{2+\varepsilon} \sum_{m_1} \sum_{m_2} \min(P, \|\alpha m_1\|^{-1})\min(P, \|\alpha m_2\|^{-1}),$$

the summations being over $|m_1| \ll P^2$, $|m_2| \ll P^2$. The last double sum factorizes, giving

$$|S|^4 \ll P^6+P^{2+\varepsilon}\Big\{\sum_m \min(P, \|\alpha m\|^{-1})\Big\}^2.$$

Dividing the inner sum into blocks of at most q consecutive terms, and applying Lemma 3 to each, we infer that this sum is

$$\ll (P^2q^{-1}+1)(P+q\log q),$$

and this gives (6). (Note that we can suppose $\log q < 3\log P$, since otherwise (6) is trivial.)

3. We turn to the case in which $-3D$ is a perfect square, but not 0. It is necessary to modify Lemma 2.

Lemma 4. *Suppose that $-3D$ is a square, not 0. Then the conclusion of Lemma 2 holds with the following exceptions:* (i) *if m_1/m_2 has one of two* *special values, the number of solutions of* (12) *is* $\ll P^{1+\varepsilon}$, (ii) *if* $m_1 = m_2 = 0$ *the number of solutions is* $\ll P^2$.

Proof. The quadratic form $\Delta(x, y)$ now factorizes rationally, but the factors are distinct (i.e. not proportional). The proof of Lemma 2 remains valid if m_1, m_2 are such that $\Delta(m_2, -m_1) \neq 0$.

Suppose that $\Delta(m_2, -m_1) = 0$ but that m_1, m_2 are not both 0; this means that m_1/m_2 has one of two special values. By (15) we can suppose without loss of generality that $\Delta(x_1, y_1) = 0$. Suppose first that $m_1 \neq 0$. Then $3ax_1 + by_1 \neq 0$, for if $3ax_1 + by_1 = 0$ then $bx_1 + cy_1 \neq 0$ by (16), contrary to the supposition that $\Delta(x_1, y_1) = 0$. Since the determinant of (16), (17) is 0, these equations are inconsistent unless

$$\frac{m_2}{m_1} = \frac{bx_1 + cy_1}{3ax_1 + by_1}.$$

Now (16) can be written

$$(3ax_1 + by_1)(m_1x_2 + m_2y_2) = m_1^2.$$

This gives $\ll P^{\varepsilon}$ possibilities for each factor. Since x_1/y_1 is limited to two values from $\Delta(x_1, y_1) = 0$, we obtain $\ll P^{\varepsilon}$ possibilities for x_1, y_1 and $\ll P^{1+\varepsilon}$ possibilities for x_2, y_2. There is a similar argument if $m_2 \neq 0$.

Suppose finally that $m_1 = m_2 = 0$. Then $\Delta(x_1, y_1) = 0$ by (19), (20), and similarly $\Delta(x_2, y_2) = 0$. Each of x_1/y_1, x_2/y_2 has only two possibilities, giving $\ll P^2$ solutions.

Proof of Theorem 1 (second case). We have to add two additional terms on the right of (23), corresponding to the two additional cases in Lemma 4. The additional term arising from $m_1 = m_2 = 0$ is $\ll P^6$. The other term is

$$\ll P^2 \cdot P^{1+\varepsilon} \sum_{m_1, m_2} \min(P, \|\alpha m_1\|^{-1}) \min(P, \|\alpha m_2\|^{-1}),$$

where the summation is over m_1, m_2 for which m_2/m_1 has one of two special values. We can take $m_1 \neq 0$. Replacing the second minimum by P, we obtain

$$\ll P^{4+\varepsilon} \sum_{m_1} \min(P, \|\alpha m_1\|^{-1})$$
$$\ll P^{4+\varepsilon}(P^2q^{-1}+1)(P+q\log q).$$

Since $(P^2q^{-1}+1)(P+q\log q) \gg P^2$, this is

$$\ll P^{2+\varepsilon}\{(P^2q^{-1}+1)(P+q\log q)\}^2$$

giving the same result as before. This completes the proof.

It does not seem to be possible to extend the proof of Theorem 1 to cover the case $D = 0$. A form with $D = 0$, if non-degenerate, is transformable rationally into X^2Y. The corresponding exponential sum is easily estimated if the summation is over a rectangle in the X, Y plane, but that is not the hypothesis of Theorem 1.

4. Preliminaries to the proof of Theorem 2. We are concerned with the integral solutions of (8). The first step is to choose a real solution of the equation without the last term:

$$(24) \qquad a_1\xi_1^3 + a_2\xi_2^3 + f_3(\xi_3, \eta_3) + f_4(\xi_4, \eta_4) = 0$$

with all the unknowns different from 0. By changing signs of variables, we can suppose that $\xi_1, \ldots, \eta_4$ are all positive. We next choose a 6-dimensional box B:

$$(25) \qquad \xi_j' < x_j < \xi_j'' \quad (j = 1, \ldots, 4), \qquad \eta_j' < y_j < \eta_j'' \quad (j = 3, 4)$$

containing the point $(\xi_1, \ldots, \eta_4)$ and having its sides sufficiently small to satisfy a later condition (Lemma 15).

Let P be a large positive number and let $\mathcal{N}(P)$ denote the number of integer solutions of (8) satisfying

$$(26) \qquad \xi_j'P < x_j < \xi_j''P \quad (j = 1, \ldots, 4),$$

$$(27) \qquad \eta_j'P < y_j < \eta_j''P \quad (j = 3, 4),$$

$$(28) \qquad P^{\frac{4}{5}} < x_5 < 2P^{\frac{4}{5}}, \quad P^{\frac{4}{5}} < y_5 < 2P^{\frac{4}{5}}.$$

We establish ultimately an asymptotic formula for $\mathcal{N}(P)$ as $P \to \infty$, namely $\mathcal{N}(P) \sim CP^{23/5}$, where $C > 0$ depends only on the equation (8) and on the choice of the box B. This proves Theorem 2.

Corresponding to each term in (8) we define an exponential sum, as follows:

$$(29) \qquad T_j(\alpha) = \sum_{(26)} e(\alpha a_j x_j^3) \quad (j = 1, 2),$$

$$(30) \qquad S_j(\alpha) = \sum_{(26),(27)} e\big(\alpha f_j(x_j, y_j)\big) \quad (j = 3, 4),$$

$$(31) \qquad U(\alpha) = \sum_{(28)} e\big(\alpha f_5(x_5, y_5)\big).$$

Then

$$(32) \qquad \mathcal{N}(P) = \int_0^1 T_1(\alpha)T_2(\alpha)S_3(\alpha)S_4(\alpha)U(\alpha)\,d\alpha.$$

We adopt substantially the same subdivision of the range of integration as in W. P. C.

Let δ be a fixed small positive number. For each pair of integers h, q with

$$(33) \qquad 1 \leqslant h < q, \quad (h, q) = 1, \quad 1 \leqslant q \leqslant P^{\frac{4}{5}(1-\delta)},$$

the *major arc* $\mathfrak{M}_{h,q}$ consists of the interval

$$(34) \qquad |\alpha - h/q| < q^{-1}P^{-2-\delta}.$$

If α is not in any major arc, there exists h, q satisfying (34) and

$$P^{\frac{4}{5}(1-\delta)} < q \leqslant P^{2+\delta}.$$

We define the *proper minor arcs* $\mathfrak{m}'$ to consist of those α for which

$$(35) \qquad P^{1-\delta} < q \leqslant P^{2+\delta}$$

and the *improper minor arcs* $\mathfrak{m}''$ to consist of those α for which

$$(36) \qquad P^{\frac{4}{5}(1-\delta)} < q \leqslant P^{1-\delta}.$$

5. The proper minor arcs.

Lemma 5. *Let $f(x, y)$ be a binary cubic form with integer coefficients, of discriminant $D \neq 0$. Then the number of solutions of*

$$(37) \qquad f(x, y) = f(x', y')$$

in integers, all of absolute value less than P, is $\ll P^{2+\varepsilon}$ for any fixed $\varepsilon > 0$.

Proof [7]. It suffices to prove the result for the equation $f(x+x_1, y+y_1) = f(x-x_1, y-y_1)$, and this can be written

$$(38) \qquad (3ax_1+by_1)x^2+2(bx_1+cy_1)xy+(cx_1+3dy_1)y^2 = -f(x_1, y_1).$$

The quadratic form on the left has discriminant $4\Delta(x_1, y_1)$ by (13), and the discriminant of $\Delta(x_1, y_1)$ itself is $-3D$ by (14).

Suppose first that $-3D$ is not a square, so that $\Delta(x_1, y_1) \neq 0$ for integers x_1, y_1, not both 0. If $f(x_1, y_1) \neq 0$, the equation (38) determines x, y with $\ll P^{\varepsilon}$ possibilities, by the argument used in the proof of Lemma 2. If $f(x_1, y_1) = 0$ then x_1, y_1 are restricted to $\ll P$ possibilities and for each of these (38) gives $\ll P$ possibilities for x, y. Hence the result.

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[7] We could deduce the result from Mahler's theorem (Math. Ann. 107 (1933), pp. 691-730 and 108 (1934), pp. 37-55; Folgerung 2 on pp. 52-53) that the number of representations of m by an irreducible binary cubic is $\ll m^{\varepsilon}$. But this theorem is of a much deeper character than Lemma 5.

Suppose now that $-3D$ is a square, not 0, so that $\Delta(x_1, y_1)$ factorizes rationally. The preceding argument still applies if $\Delta(x_1, y_1) \neq 0$. If $\Delta(x_1, y_1) = 0$, then x_1, y_1 are restricted to $\ll P$ possibilities, and for each of these (38) gives $\ll P$ possibilities for x, y (independently of whether $f(x_1, y_1) = 0$ or not). This completes the proof.

Lemma 6. *We have*

$$\int_0^1 |T_1(\alpha)U(\alpha)|^2 d\alpha \ll P^{\frac{13}{5}+\varepsilon}.$$

Proof. The integral represents the number of solutions of

$$a_1x_1^3+f_5(x_5, y_5) = a_1x_1'^3+f_5(x_5', y_5')$$

subject to (26) and (28) for both sets of variables. More simply, we write the equation as

$$(39) \qquad ax^3+f(y, z) = ax'^3+f(y', z')$$

with

$$\xi'P < x < \xi''P, \quad |y| \ll P^{\frac{4}{5}}, \quad |z| \ll P^{\frac{4}{5}},$$

and the same for x', y', z'. We follow the lines of Lemma 1 of W. P. C.

The number of solutions with $x = x'$ is $\ll P\cdot(P^{4/5})^{2+\varepsilon}$ by Lemma 5. Hence we can suppose $x' > x$. With $x' = x+t$, (39) becomes

$$(40) \qquad at(3x^2+3xt+t^2)+f(y', z') = f(y, z).$$

Plainly $3P^2t \ll |f(y, z)|+|f(y', z')| \ll P^{12/5}$, so that $0 < t \ll P^{2/5}$.

Let $\varrho(m)$ denote the number of representations of m by $f(y, z)$, and let $R(t, m)$ denote the number of representations of m by the left hand side of (40). Then the number of solutions of (40) is

$$\sum_{t,m} \varrho(m)R(t, m) \leqslant \Big\{\sum_{t,m} \varrho^2(m)\Big\}^{1/2}\Big\{\sum_{t,m} R^2(t, m)\Big\}^{1/2}.$$

We have $\sum_m \varrho^2(m) \ll P^{8/5+\varepsilon}$ by Lemma 5, so the above is

$$(41) \qquad \ll \{P^{\frac{2}{5}}P^{\frac{8}{5}+\varepsilon}\}^{\frac{1}{2}}\Big\{\sum_{t,m} R^2(t, m)\Big\}^{\frac{1}{2}},$$

on using the bound for t.

The last double sum is the number of solutions of

$$at(3x_1^2+3x_1t)+f(y_1, z_1) = at(3x_2^2+3x_2t)+f(y_2, z_2),$$

in the same ranges as before. If $x_1 = x_2$, there are $\ll P$ possibilities for x_1, and $\ll P^{2/5}$ possibilities for t, and $\ll (P^{4/5})^{2+\varepsilon}$ possibilities for

y_1, z_1, y_2, z_2 by Lemma 5, making altogether $P^{3+\varepsilon}$. If $x_1 \neq x_2$, we observe that y_1, z_1, y_2, z_2 can be chosen in $\ll (P^{4/5})^4$ ways, and that they determine the factors of

$$3at(x_1-x_2)(x_1+x_2+t)$$

in $\ll P^{\varepsilon}$ ways. Hence

$$\sum_{t,m} R^2(t,m) \ll P^{3+\varepsilon}+(P^{\frac{4}{5}})^4 P^{\varepsilon} \ll P^{\frac{16}{5}+\varepsilon}.$$

On substitution in (41), this gives the result stated.

Lemma 7. *We have*

$$\int_{\mathfrak{m}'} |T_1(\alpha)T_2(\alpha)S_3(\alpha)S_4(\alpha)U(\alpha)|\,d\alpha \ll P^{\frac{91}{20}+3\delta}.$$

Proof. The sum $T_2(\alpha)$ in (29) can be estimated by the original Weyl's inequality (3), and on using (34) and (35), which hold in $\mathfrak{m}'$, we obtain

$$|T_2(\alpha)| \ll P^{\frac{3}{4}+\delta}.$$

The double sum $S_3(\alpha)$ can be estimated by Theorem 1, and in the same way we obtain

$$|S_3(\alpha)| \ll P^{\frac{3}{2}+\delta}.$$

Hence the integral of the enunciation is

$$\ll P^{\frac{9}{4}+2\delta}\int_0^1 |T_1(\alpha)S_4(\alpha)U(\alpha)|\,d\alpha$$

$$\ll P^{\frac{9}{4}+2\delta}\Big\{\int_0^1 |S_4(\alpha)|^2\,d\alpha\Big\}^{\frac{1}{2}}\Big\{\int_0^1 |T_1(\alpha)U(\alpha)|^2\,d\alpha\Big\}^{\frac{1}{2}}.$$

The first integral here is $\ll P^{2+\varepsilon}$ by Lemma 5, and the second is $\ll P^{13/5+\varepsilon}$ by Lemma 6. Hence the result.

6. The improper minor arcs. We require some lemmas from W. P. C. and from C. F. Those quoted from the former were proved there for the sums $T_1(\alpha)$, $T_2(\alpha)$ without the coefficients a_1, a_2, but remain valid since a_1, a_2 are constants. The lemmas are not restricted to α in $\mathfrak{m}''$, and indeed they find their main application on the major arcs (§ 7).

Lemma 8. *Suppose that*

$$\alpha = \frac{h}{q}+\beta, \quad |\beta| < q^{-1}P^{-2-\delta}, \quad (h,q)=1, \quad q \leqslant P^{1-\delta}. \tag{42}$$

Then, for $j=1,2$,

$$T_j(\alpha) = q^{-1}S_j(h,q)I_j(\beta)+O(q^{\frac{2}{3}+\varepsilon}), \tag{43}$$

where

$$S_j(h,q) = \sum_{x=1}^{q} e(ha_jx^3/q) \quad (j=1,2), \tag{44}$$

$$I_j(\beta) = \int_{\xi_j'P}^{\xi_j''P} e(\beta a_j x^3)\,dx \quad (j=1,2). \tag{45}$$

Proof. This is Lemma 7 of W. P. C., except that there a sum was used in place of the integral $I_j(\beta)$. The difference between the sum and the integral is $O(1)$.

Lemma 9. *We have*

$$|S_j(h,q)| \ll q^{\frac{2}{3}} \quad (j=1,2), \tag{46}$$

$$|I_j(\beta)| \ll \min(P, P^{-2}|\beta|^{-1}) \quad (j=1,2). \tag{47}$$

Proof. (46) is Lemma 2 of W. P. C. The first part of (47) is trivial, the second is easily found on integrating by parts.

Lemma 10. *With the hypotheses* (42) *we have, for* $j=3,4$,

$$S_j(\alpha) = q^{-2}S_j(h,q)I_j(\beta)+O(S_0P^{1+\varepsilon}q^{-1})+O(1), \tag{48}$$

where

$$S_j(h,q) = \sum_{x=1}^{q}\sum_{y=1}^{q} e\big(hf_j(x,y)/q\big) \quad (j=3,4), \tag{49}$$

$$I_j(\beta) = \int_{\xi_j'P}^{\xi_j''P}\int_{\eta_j'P}^{\eta_j''P} e\big(\beta f_j(x,y)\big)\,dx\,dy \quad (j=3,4), \tag{50}$$

and where S_0 *is an upper bound for the absolute value of any sum of the type*

$$S(h,q,l,l') = \sum_{x=1}^{q}\sum_{y=1}^{q} e\Big(\frac{hf(x,y)+lx+l'y}{q}\Big). \tag{51}$$

Proof. This is the case $n=2$ of Lemma 5.5 of C. F., except that the definition of S_0 differs from that of the number T used there. The new definition (which costs a factor P^{ε}) is justified by the argument of Lemma 5.6 of C. F.

LEMMA 11. *We have*

$$S_0 \ll q^{\frac{3}{2}+\varepsilon}, \tag{52}$$

and in particular (taking $l = l' = 0$ in (51)*),*

$$|S_j(h, q)| \ll q^{\frac{3}{2}+\varepsilon} \quad (j = 3, 4). \tag{53}$$

Proof. Apply Theorem 1 to the sum (51) with $P = Q = q$ and with $\alpha = h/q$.

LEMMA 12. *We have*

$$\int_{\mathfrak{m}''} |T_1(\alpha) T_2(\alpha) S_3(\alpha) S_4(\alpha) U(\alpha)|\, d\alpha \ll P^{\frac{68}{15}+\delta}.$$

Proof. We recall that $\mathfrak{m}''$ is defined by (34), (36). By Lemmas 8, 9, for α in $\mathfrak{m}''$ we have

$$|T_j(\alpha)| \ll q^{-\frac{1}{3}} \min(P, P^{-2}|\beta|^{-1}) + q^{\frac{2}{3}+\varepsilon}$$
$$\ll q^{-\frac{1}{3}+\varepsilon} \min(P, P^{-2}|\beta|^{-1}).$$

By Lemmas 10, 11 we have

$$|S_j(\alpha)| \ll q^{-\frac{1}{2}} P^2 + P^{1+\varepsilon} q^{\frac{1}{2}+\varepsilon} + 1$$
$$\ll q^{-\frac{1}{2}} P^2.$$

As for $U(\alpha)$, we note that Theorem 1 is applicable to this sum, with $P^{4/5}$ as the range of each variable, and gives

$$|U(\alpha)| \ll (P^{\frac{4}{5}})^{\frac{3}{2}+\delta}.$$

Thus the integrand in the enunciation is

$$\ll q^{-\frac{2}{3}+2\varepsilon} \min(P^2, P^{-4}\beta^{-2})\, q^{-1} P^4 P^{\frac{6}{5}+\delta}.$$

Integration over $|\beta| < q^{-1}P^{-2-\delta}$ gives

$$q^{-\frac{5}{3}+2\varepsilon} P^{4+\frac{1}{5}+\delta},$$

and then summation for h and for q with $q \leqslant P^{1-\delta}$ gives the result stated.

7. The major arcs. Lemma 8 gives approximations to $T_1(\alpha)$, $T_2(\alpha)$ on an individual major arc $\mathfrak{M}_{h,q}$, and Lemma 10 gives approximations to $S_3(\alpha)$, $S_4(\alpha)$. As regards $U(\alpha)$ we can apply Lemma 10 with $P^{4/5}$ in place of P, since $q \leqslant P^{\frac{4}{5}(1-\delta)}$ and

$$|\beta| < q^{-1}P^{-2-\delta} < q^{-1}(P^{\frac{4}{5}})^{-2-\delta}.$$

Thus on $\mathfrak{M}_{h,q}$

$$U(\alpha) = q^{-2} S_5(h, q) I_5(\beta) + O(P^{\frac{4}{5}+\varepsilon} q^{\frac{1}{2}}), \tag{54}$$

where $S_5(h, q)$ is defined as in (49), but

$$I_5(\beta) = \int_{P^{4/5}}^{2P^{4/5}} \int_{P^{4/5}}^{2P^{4/5}} e(\beta f_5(x, y))\, dx\, dy. \tag{55}$$

We denote by $V(\alpha)$ the integrand in (32) and by $V^*(\alpha, h, q)$ the product of the main terms in the approximations, valid on $\mathfrak{M}_{h,q}$, already found for the factors of $V(\alpha)$. Thus

$$V^*(\alpha, h, q) = q^{-8} \prod_{j=1}^{5} S_j(h, q) \prod_{j=1}^{5} I_j(\beta). \tag{56}$$

LEMMA 13. *Let $\mathfrak{M}$ denote the aggregate of the major arcs. Then*

$$\int_{\mathfrak{M}} V(\alpha)\, d\alpha = \sum_{q \leqslant P^{4(1-\delta)/5}} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \int_{\mathfrak{M}_{h,q}} V^*(\alpha, h, q)\, d\alpha + O(P^{\frac{67}{15}}).$$

Proof. The main terms in the approximations (43), (48), (54) are respectively estimated by

$$q^{-\frac{1}{3}} \min(P, P^{-2}|\beta|^{-1}), \quad q^{-\frac{1}{2}} P^2, \quad q^{-\frac{1}{2}} (P^{\frac{4}{5}})^2,$$

and the error terms by

$$q^{\frac{2}{3}+\varepsilon}, \quad P^{1+\varepsilon} q^{\frac{1}{2}}, \quad P^{\frac{4}{5}+\varepsilon} q^{\frac{1}{2}}.$$

It is easily deduced that, on $\mathfrak{M}_{h,q}$,

$$|V(\alpha) - V^*(\alpha, h, q)| \ll P^{4+\frac{4}{5}+\varepsilon} q^{-\frac{7}{6}} \min(P^2, P^{-4}\beta^{-2})$$
$$+ P^{5+\frac{3}{5}+\varepsilon} q^{-\frac{7}{6}} \min(P, P^{-2}|\beta|^{-1}).$$

Integration for β over $|\beta| < q^{-1}P^{-2-\delta}$, then summation over h and over $q \leqslant P^{\frac{4}{5}(1-\delta)}$ lead to the estimate stated.

8. Completion of the analytical argument. It follows from (32) and Lemmas 7, 12, 13 that

$$\mathfrak{N}(P) = \sum_{q\leqslant P^{4(1-\delta)/5}} A(q) \int_{|\beta|<q^{-1}P^{-2-\delta}} I(\beta)\,d\beta + O(P^{\frac{23}{5}-\delta}), \tag{57}$$

where

$$A(q) = q^{-8}\sum_{\substack{h=1\\(h,q)=1}}^{q}\prod_{j=1}^{5} S_j(h,q) \tag{58}$$

and

$$I(\beta) = \prod_{j=1}^{5} I_j(\beta). \tag{59}$$

By Lemmas 9, 11 we have

$$|A(q)| \ll q^{-8} q\,(q^{\frac{2}{3}+\varepsilon})^2 (q^{\frac{3}{2}+\varepsilon})^3 \ll q^{-\frac{7}{6}}. \tag{60}$$

Thus the 'singular series'

$$\mathfrak{S} = \sum_{q=1}^{\infty} A(q) \tag{61}$$

is absolutely convergent. Its sum depends only on the coefficients in equation (8).

Lemma 14. *We have*

$$\mathfrak{N}(P) = P^{\frac{23}{5}}\mathfrak{S}\int_{|\gamma|<P^{\delta}} J(\gamma)\,d\gamma + O(P^{\frac{23}{5}-\delta}), \tag{62}$$

where

$$J(\gamma) = \int_B e\big(\gamma F(x_1,\ldots,y_4)\big)\,dx_1\ldots dy_4 \tag{63}$$

and

$$F(x_1,\ldots,y_4) = a_1x_1^3 + a_2x_2^3 + f_3(x_3,y_3) + f_4(x_4,y_4), \tag{64}$$

and B is the 6-dimensional box (25).

Proof. We begin by contracting the range of integration in (57) to $|\beta| < P^{-3+\delta}$. Since

$$|I(\beta)| \ll \min(P^2, P^{-4}\beta^{-2})P^4P^{\frac{8}{5}} \tag{65}$$

by (59), (47), (50), (55), the change in the integral is $\ll P^{\frac{22}{5}-\delta}$ and is negligible.

The range for β is now independent of q. By (60), (65) we can extend

the summation over q to ∞ and neglect the error introduced. We now have

$$\mathfrak{N}(P) = \mathfrak{S}\int_{|\beta|<P^{-3+\delta}} I(\beta)\,d\beta + O(P^{\frac{23}{5}-\delta}).$$

We put $x_j = Px_j'$, $y_j = Py_j'$ for $j\leqslant 4$ and $x_5 = P^{\frac{4}{5}}x_5'$, $y_5 = P^{\frac{4}{5}}y_5'$ in the integrals $I_1(\beta),\ldots,I_5(\beta)$. Putting also $\beta = P^{-3}\gamma$, we obtain

$$\int_{|\beta|<P^{-3+\delta}} I(\beta)\,d\beta = P^{\frac{23}{5}}\int_{|\gamma|<P^{\delta}} J_1(\gamma)\,d\gamma,$$

where

$$J_1(\gamma) = \int_B dx_1\ldots dy_4 \int_1^2\int_1^2 e\Big(\gamma\big(F+P^{-\frac{3}{5}}f_5(x_5,y_5)\big)\Big)\,dx_5\,dy_5$$

with F as in (64). Plainly we can omit the integration over x_5, y_5 with a negligible error, thus obtaining the result stated.

Lemma 15. *If the box B in* (25) *is chosen sufficiently small, but independent of P, then*

$$\lim_{P\to\infty}\int_{|\gamma|<P^{\delta}} J(\gamma)\,d\gamma = J_0$$

exists, and $J_0 > 0$.

Proof. See Lemma 6.2 of C.F., where the result is proved for a general cubic form. The point $(\xi_1,\ldots,\eta_4)$ inside B must be a non-singular solution of (24), but this condition is satisfied because $\xi_1\neq 0$.

Lemma 16. *We have*

$$\mathfrak{N}(P) = J_0P^{\frac{23}{5}}\mathfrak{S} + o(P^{\frac{23}{5}}). \tag{66}$$

Proof. Lemmas 14, 15.

9. The singular series. Plainly $\mathfrak{S}\geqslant 0$ from (66); it remains to be proved that $\mathfrak{S}>0$. Once this is established we have $\mathfrak{N}(P)\sim J_0\mathfrak{S}P^{23/5}$, as asserted in § 4.

It follows from well-known general arguments [8] that the following condition is sufficient to ensure that $\mathfrak{S}>0$: *for every prime p the equation* (8) *has a non-singular solution in the p-adic field.* It remains to prove this result. Since the two first partial derivatives of a binary cubic form (of non-zero discriminant) cannot vanish unless both variables vanish, it will be enough to prove the existence of a non-trivial solution of (8) in every p-adic field.

[8] See § 7 of C. F.

We base the proof on a theorem of D. J. Lewis [9], and it will appear incidentally that the result holds for any equation like (8) which contains 7 or more variables. The theorem of Lewis is as follows.

LEMMA 17. *If K is a complete field under a non-Archimedean valuation, then every equation of the form*

$$c_1 z_1^3 + \ldots + c_n z_n^3 = 0 \quad (n \geqslant 7), \tag{67}$$

with the c_i in K, has a non-trivial solution in K.

We need also the following classical result.

LEMMA 18. *A binary cubic form with coefficients in a field Ω of characteristic 0 and with discriminant $D \neq 0$ can be expressed identically (after possibly a preliminary unimodular change of variables) as*

$$R(x+ay)^3 + S(x+\beta y)^3,$$

where a, β and R, S are conjugate pairs of elements of the field $\Omega(\sqrt{-3D})$.

Proof. With the usual notation, $f(x, y)$ has the quadratic covariant $\Delta(x, y)$ of (13), and by a preliminary change of variables we can ensure that $A \neq 0$.

The desired identity is equivalent to

$$\begin{aligned} R+S &= a, \\ Ra+S\beta &= \tfrac{1}{3}b, \\ Ra^2+S\beta^2 &= \tfrac{1}{3}c, \\ Ra^3+S\beta^3 &= d. \end{aligned} \tag{68}$$

These equations imply

$$\begin{aligned} RS(a-\beta)^2 &= -\tfrac{1}{9}A, \\ RS(a+\beta)(a-\beta)^2 &= -\tfrac{1}{9}B, \\ RSa\beta(a-\beta)^2 &= -\tfrac{1}{9}C, \end{aligned}$$

whence

$$a+\beta = B/A, \quad a\beta = C/A.$$

Thus a, β are conjugate elements of $\Omega(\sqrt{-3D})$, since $-3D = B^2-4AC$, and the same is true of R, S by the first two equations in (68). It is easily verified that the values of a, β, R, S determined in this way satisfy all the equations (68).

LEMMA 19. *With the hypotheses of Theorem 2, the equation (8) has a non-trivial solution in every p-adic field.*

(9) *Cubic congruences*, Michigan Math. Journ. 4 (1957), pp. 85-95, Theorem 2.

Proof. Let Ω_p denote the p-adic field. By Lemma 18, the equation (8) is equivalent to (67) with $n = 8$,

$$c_1 = a_1,\ c_2 = a_2,\ c_3 = R_3,\ c_4 = S_3,\ \text{etc.},$$

and

$$z_1 = x_1,\ z_2 = x_2,\ z_3 = x_3+ay_3,\ z_4 = x_3+\beta y_3,\ \text{etc.}$$

The coefficients and variables are now in the field Ω_p' obtained from Ω_p by adjoining [10] $\sqrt{-3D_1}$, $\sqrt{-3D_2}$, $\sqrt{-3D_3}$, where D_1, D_2, D_3 are the discriminants of f_3, f_4, f_5.

It may be of interest to note (though the fact is not important for our argument) that Ω_p' can be derived from Ω_p by at most *two* quadratic extensions. For there are only three distinct quadratic extensions of Ω_p, namely by adjoining $\sqrt{p}$ or $\sqrt{a}$ or $\sqrt{ap}$, where a is a particular p-adic unit which is not a square, and any two of these extensions include the third.

The field Ω_p' satisfies the requirements of Lemma 17, and hence (8) has a solution with $x_1, \ldots, y_5$ in Ω_p'. From such a solution we can derive one in Ω_p by the following device [11]. Suppose first that Ω_p' is simply a quadratic extension of Ω_p and that $x_1, \ldots, y_5$ are in Ω_p' but not in Ω_p. The conjugates $x_1', \ldots, y_5'$ in Ω_p' relative to Ω_p provide another solution of (8). The linear combination

$$ux_1+vx_1', \ldots, uy_5+vy_5'$$

will satisfy (8) if u, v satisfy an equation of the form $Hu^2v+H'uv^2 = 0$, where H, H' are conjugates in Ω_p'. Taking $u = H'$, $v = -H$ (or $u = 1$, $v = 1$ if $H = H' = 0$), we obtain a solution of (8) which, apart from a factor of proportionality, lies in Ω_p.

A similar argument applies if Ω_p' is a double quadratic extension of Ω_p; we first derive a solution in the intermediate field and then repeat the argument.

This proves Lemma 19, and so completes the proof of Theorem 2.

(10) p may be 3, but that does not affect the argument.

(11) The argument is not new; see D. J. Lewis, *Cubic forms over algebraic number fields*, Mathematika 4 (1957), pp. 97-101.

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Reçu par la Rédaction le 2. 11. 1960