Remarks on the number of factors of an odd perfect number

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I. Introduction. It is still unknown whether or not odd perfect numbers exist, but many necessary conditions for their existence have been established. Among the most interesting of these are a theorem of Kühnel ([1]) that an odd perfect number $N$ must have at least six different prime factors, and a result published by Kanold ([9]) stating that $N > 10^{16}$.

In 1888, G. Servais ([3]) showed that if $N = p_1^a p_2^{a_2} \cdots p_k^{a_k}$ is an odd perfect number with $p_1 < p_2 < \cdots < p_k$, then $k > 2$. This result was refined by Grün ([4]), who gave the inequality $k > \frac{1}{2} p_1 - 2$. The object of this paper is to make a considerable improvement in Grün’s inequality for large $p_1$ and to give some general theorems concerning the number of prime factors of $N$ as a function of its smallest factor. Other theorems of interest (including two on the size of $N$) will be given, and a table of numerical results will be included.

We shall let $p_n$ denote the $n$th prime number, where $p_1 = 2$. The symbol $N$ always denotes an odd perfect number. The function $li(x)$ is the familiar logarithmic integral:

$$\text{li}(x) = \int_2^x \frac{dt}{\log t} = \lim_{a \to 0} \left[ \int_2^x \frac{dt}{\log t} + \int_1^a \frac{dt}{\log t} \right].$$

We shall commonly write $\log^a f(x)$ in place of $[\log f(x)]^a$. We let $\Theta(x) = \sum_{p \leq x} \log p$, where $p$ runs through the primes not exceeding $x$, and $\pi(x)$ will denote the number of primes not exceeding $x$.

Now suppose that $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is perfect, where $3 < p_1 < \cdots < p_k$, so

$$2 = \prod_{r=1}^k p_r^{a_r} - 1 < \prod_{r=1}^k p_r - 1.$$
If \( p_1 = P_n \), then it is plain that (1) implies
\[
2 < \prod_{r = 1}^{s = n+1} \frac{P_r}{P_r - 1}.
\]

Let the function \( a(n) \) be defined for \( n \geq 2 \) by the following double inequality:
\[
\prod_{r = n+1}^{s = n+2} \frac{P_r}{P_r - 1} < 2 < \prod_{r = n+1}^{s = n+2} \frac{P_r}{P_r - 1}.
\]

From (1), (2), and (3), it follows that if \( P_s \) is the smallest prime factor of \( N \), then \( N \) has at least \( a(n) \) different prime factors. Also, \( N \) must have a prime factor at least as large as \( P_{a(n) - 1} = P_s \). (Throughout the remainder of this paper, the letter \( s = s(n) \) will represent \( n + a(n) - 1 \). The calculation of the exact values of \( a(n) \) and \( P_s \) for \( 2 \leq n \leq 100 \) was performed by ILLIAC (the University of Illinois Automatic Digital Computer). These values are found in the table at the end of this paper. The table has four columns, giving the numbers \( n, P_n, a(n), \) and \( P_s \), respectively.

I am deeply indebted to my father, Professor H. W. Norton, who spent many hours of careful work preparing the program for ILLIAC. I am also especially grateful to Professor Paul T. Bateman, who read and criticized the manuscript, and whose suggestions (particularly for the proof of Theorem 4) were most helpful.

II. Auxiliary theorems. Some lemmas due to Rosser ([5], [9]) are indispensable in obtaining our results.

**Lemma 1.** \( P_n > n \log n \) for all \( n \geq 1 \).

**Lemma 2.** (a) If \( 1 < x \leq e^{10} \), or if \( x \geq e^{100} \), then \( x/\log x < \pi(x) \).
(b) If \( 2 < n < e^2 \), then \( P_n < n \log n + n \log \log n \).
(c) For \( x \geq 2 \), we have
\[
\left( 1 - \frac{2.35}{\log x} \right) x < \theta(x) < \left( 1 + \frac{2.35}{\log x} \right) x.
\]

For \( 1 < x \leq e^{10} \), \( \theta(x) < \left( 1 + \frac{1}{\log x} \right) x \), and if \( 41 < x \leq e^{100} \), then
\[
\left( 1 - \frac{1}{\log x} \right) x < \theta(x) < \left( 1 + \frac{1}{\log x} \right) x.
\]

III. Main results. We now wish to study the function \( a(n) \) in some detail.

**Theorem 1.** \( a(n) > n^2 - 2n - \frac{n+1}{\log n} - \frac{5}{4} - \frac{1}{2n} - \frac{1}{4n \log n} \).

**Proof.** Using (3) and Lemma 1, we have
\[
\log 2 < \log \left( \frac{P_n}{P_{n-1}} \right) + \sum_{r=n+1}^{s} \left( \frac{1}{P_r} - \frac{1}{P_{r+1}} \right) + \frac{1}{P_{s+1}} + \frac{1}{3P_{s+1}} + \ldots
\]
\[
< \log \left( \frac{P_s}{P_{n-1}} \right) + \sum_{r=n+2}^{s} \frac{1}{P_r} + \frac{1}{2} \sum_{r=n+1}^{s} \left( \frac{1}{P_r^2} + \frac{1}{P_r} \right)
\]
\[
< \log \left( \frac{P_s}{P_{n-1}} \right) + \frac{1}{2} \int_{n+1}^{s} \frac{ds}{\log^2(n+1)} + \frac{1}{2} \int_{n+1}^{s+1} \frac{1}{2 \log^2(n+1)} \frac{ds}{\log n} < \frac{1}{2}\frac{1}{\log(n+1)} \frac{1}{4n \log^2(n+1)}
\]
\[
\log(n+1) > 2 \log(n+1) \left( 1 - \frac{1}{\log n} \right) \left( 1 - \frac{1}{2 \log n \log(n+1)} \right)
\]
\[
= \frac{1}{4n \log^4(n+1)}
\]
\[
> 2 \log(n+1) - \frac{2}{n} - \frac{1}{n \log n} - \frac{1}{n \log^2 n}.
\]

It follows that
\[
x \in (n+1)^2 \left( 1 - \frac{2}{n \log n} \right) = n^2 - n + \frac{n+1}{\log n} - \frac{7}{4} - \frac{1}{2n} - \frac{1}{4n \log n}.
\]

This proves the theorem.

**Theorem 2.** Suppose that \( t \left( 1 - \frac{a}{\log t} \right) < \theta(t) < t \left( 1 + \frac{b}{\log t} \right) \) for \( P_n < t < P_s \), where \( a \) and \( b \) are constants and \( 0 < a \leq 3 \), \( 0 < b \leq 3 \).

Then
\[
P_s > e^{-P_s} \left( 1 - \frac{4a + 3b + b^2}{2\log P_n} - \frac{4\log P_n}{547} \right).
\]

In particular,
\[
P_s > e^{-P_n} \left( 1 - \frac{4a + 3b + b^2}{2\log 547} - \frac{4\log 547}{547} \right)
\]
for \( n \geq 4 \).
Proof. The second inequality follows from the first, since $P_r > P_n$, for $r \leq n \leq 100$ (see the table of numerical results), and $P_{100} = 547$.

Now, for $r \gg 2$,

$$\log \left( \frac{P_r}{P_{r-1}} \right) < \frac{1}{P_r} + \frac{1}{P_r^2},$$

so

$$\log 2 < \log \prod_{r=2}^{\infty} \frac{P_r}{P_{r-1}} < \sum_{r=2}^{\infty} \frac{1}{P_r} + \sum_{r=2}^{\infty} \frac{1}{P_r^2} < \int_1^{\theta(n)} \frac{dt}{\log t} + \frac{2}{P_n} \frac{\theta(P_n)}{P_n \log P_n - \frac{\theta(P_n)}{P_n}} + K(n),$$

where

$$K(n) = \int_{P_n}^{\theta(n)} \frac{\theta(t) (\log t + 1) dt}{f \log t} < \log \log P_n - \log \log P_n + \frac{1+b}{\log P_n} + \frac{1+b}{2 \log^2 P_n} + \frac{b}{2 \log^2 P_n} - \frac{b}{2 \log^2 P_n}.$$

Hence,

$$\log \log P_n - \frac{b}{2 \log^2 P_n} > \log \log P_n - \frac{b}{2 \log^2 P_n} - \frac{2}{\log^2 P_n} - \frac{b}{2 \log^2 P_n} - \frac{b}{2 \log^2 P_n}.$$

Now, $e^{-x} < 1 - x - x^2/2$ for $0 < x$. Since $0 < b < 3$ and $P_r > 7$, it follows that

$$0 < \frac{b}{\log P_n} < \frac{b}{2 \log^2 P_n},$$

so we have

$$\log P_n - \frac{b}{2 \log^2 P_n} > \log \left( \frac{P_n}{P_{n-1}} \right) \exp \left( - \frac{b}{\log P_n} - \frac{b}{2 \log^2 P_n} \right)$$

$$> (2 \log P_n) \left( 1 - \frac{b}{2 \log^2 P_n} - \frac{2a+b}{2 \log^2 P_n} \right)$$

and since

$$2b \log P_n < 2b \log P_n,$$

it follows that

$$P_n > e^{-b} \exp \left( - \frac{4a+3b}{2 \log P_n} - \frac{4 \log P_n}{P_n} \right)$$

$$> e^{-b} \left( 1 - \frac{4a+3b}{2 \log P_n} - \frac{4 \log P_n}{P_n} \right).$$

Corollary. For $2 \leq n < e^{a_3}$, and for $n > e^{a_{100}}$, we have

$$P_r > P_n \left( 1 - \frac{4 \log P_n}{2 \log P_n} - \frac{4 \log P_n}{P_n} \right).$$

Proof. Using Lemma 2(b), it is hard to show that $n < e^{a_3}$ implies $s < e^{a_3}$ and $P_r < e^{a_{100}}$. Also, for $n > e^{a_{100}}$, $P_n > 4 \log^2 P_n > e^{a_{100}}$. We now apply Lemma 2(e) and Theorem 2, noting that the result is obvious for $2 \leq n < 12$.

Recent unpublished theorems of Rosser and Schoenfeld (with better inequalities for $\theta(x)$) indicate that the inequality of the Corollary can be improved in various ways. For example,

$$P_r > P_n \left( 1 - \frac{15}{2 \log P_n} - \frac{4 \log P_n}{P_n} \right)$$

for all $r > 2$. Using these results and Lemma 2(a), various inequalities for $a(n)$ can easily be deduced.

We wish to make some remarks on the size of $N$. It was proved by Euler that any odd perfect number $N$ can be written in the form

$$N = p^a q^{2b} q^{2b} \ldots q^{2b},$$

where $p \equiv a \equiv 1 \mod 4$. By two theorems of Kanold (27, 28), $N$ is not perfect if $2b < 10$ and $b_1 = b_2 = \ldots = b_n = 1$, or if $b_1 = b_2 = 2$ and $b_3 = b_4 = \ldots = b_n = 1$. From this, we obtain without difficulty the following lemma:

Lemma 3. If $P_n$ is the smallest prime factor of $N$, then

$$N > P_n^{2(1+P_n+P_{n+1}) + \ldots + P_{n+m}).}$$

Taking logarithms, we get $\log N > \theta(P_n) - 2(\theta(P_n) + 6 \log P_n + + 2 \log P_{n+1} - \log P_n).$ Using Lemma 2(e), the proof of the Corollary, the table of numerical results, and Kanold's theorem that $N > 10^4$, we have

Theorem 3. If $P_n$ is the smallest factor of $N$, where $2 \leq n < e^{a_3}$ or $n > e^{a_{100}}$, then

$$\log N > 2P_n \left( 1 - \frac{1}{2 \log P_n} \right) - 2P_n \left( 1 + \frac{1}{2 \log P_n} \right) + + 6 \log P_n - 2 \log P_{n+1} - \log P_n.$$

Using the recent results of Rosser and Schoenfeld, this inequality can be improved to read

$$\log N > 2P_n \left( 1 - \frac{1}{2 \log P_n} \right) - 2P_n \left( 1 + \frac{1}{2 \log P_n} \right) + + 6 \log P_n - 2 \log P_{n+1} - \log P_n$$

We now obtain a result giving the true order of magnitude of \(a(n)\) and \(P_s\). From this we can easily get a function which is a lower bound for \(\log N\). The statement and proof of the following theorem are due to Professor Paul T. Bateman, who communicated to the author his improvements of the author's proof of a similar theorem.

**Theorem 4.** Let \(N\) be an odd perfect number with smallest prime factor \(P_n\), and let \(b\) be any number less than \(\frac{1}{4}\). Then:

1. \(N\) has at least \(a(n)\) different prime factors, where
   \[a(n) = \text{li}(P_n^2) + O(n^\varepsilon e^{-\varepsilon n^h}).\]

2. \(N\) has a prime factor at least as large as
   \[P_n = P_n^2 + O(n^\varepsilon e^{-\varepsilon n^h}).\]

3. \(\log N > 2P_n + O(n^\varepsilon e^{-\varepsilon n^h}).\)

**Proof.** It is known that \(\theta(x) = x + A(x), \) where \(A(x) = O(x e^{-\varepsilon x})\) and \(a\) is any number less than \(\frac{1}{4}\) (see Prachar [10]). Hence,

\[
\sum_{p < x} \frac{1}{p} = \frac{1}{2} + \int_1^x \frac{\theta(t)}{\log t} dt = \frac{1}{2} + \int_1^x \frac{dt + dA(t)}{\log t} dt
\]

\[= \log \log x + \frac{A(x)}{x \log x} + B - \int_1^x \frac{dt}{\log t + \frac{1}{x \log x}} dt + O\left(\int_1^x \frac{t}{\log t} dt\right),
\]

where \(B\) is a constant. Let \(b < a < \frac{1}{4}\). Then

\[
\int_1^x \frac{e^{-\varepsilon n^h}}{\log t} dt = O(e^{-\varepsilon n^h}).
\]

Therefore,

\[
\sum_{p < x} \frac{1}{p} = \log \log x + B + O(e^{-\varepsilon n^h}) \quad \text{for all} \quad a < \frac{1}{4}.
\]

It follows that

\[
\sum_{p < x} - \log \left(1 - \frac{1}{p}\right) = \sum_{p < x} \left(- \log \left(1 - \frac{1}{p}\right) - \frac{1}{p}\right) + O\left(\frac{1}{x}\right)
\]

\[= \log \log x + c + O(e^{-\varepsilon n^h}),
\]

where \(c\) is a constant. Thus, for \(h = o(x^2),\)

\[
\sum_{p < x} - \log \left(1 - \frac{1}{p}\right) = \log \log (x^3 + h) - \log \log x + O(e^{-\varepsilon n^h}).
\]

Let \(h = a^2 e^{-\varepsilon n^h}, \) where \(b < a < \frac{1}{4}\). Then

\[
\sum_{p < x} - \log \left(1 - \frac{1}{p}\right) = \log x^{3} + O\left(\frac{e^{-\varepsilon n^h}}{\log x}\right) + O\left(\frac{e^{-\varepsilon n^h}}{\log x}\right) > \log 2
\]

for all sufficiently large \(x\). Also, if \(h = -a^2 e^{-\varepsilon n^h}, \) where \(b < a < \frac{1}{4}, \) then

\[
\sum_{p < x} - \log \left(1 - \frac{1}{p}\right) = \log x^{2} + O\left(\frac{e^{-\varepsilon n^h}}{\log x}\right) + O\left(\frac{e^{-\varepsilon n^h}}{\log x}\right) < \log 2
\]

for all sufficiently large \(x\). Hence, if \(g(x)\) is the smallest number such that

\[
\sum_{p < x} - \log \left(1 - \frac{1}{p}\right) > \log 2,
\]

then \(g(x) = a^2 + O(a^2 e^{-\varepsilon n^h})\) for all \(a < \frac{1}{4}\). Therefore, if \(x = n + a(n) - 1\) is the smallest integer such that

\[
\prod_{p < x} P_p = \prod_{p < x} P_p > 2^x
\]

then

\[
\sum_{p < x} - \log \left(1 - \frac{1}{p}\right) > \log 2, \quad \text{so} \quad P_s = P_n + O\left(P_n e^{-\varepsilon n^h}\right), \quad \text{or} \quad P_s = P_n + O\left(n^2 e^{-\varepsilon n^h}\right)
\]

for any \(b < a\). This proves statement (2) of the theorem.

Now, it is known that \(\pi(x) = \text{li}(x) + O(x e^{-\varepsilon x})\) for all \(a < \frac{1}{4}\) (see Prachar [10]). Hence,

\[
a(n) = a - n + 1 = \pi\left(P_n^2 + O\left(n^2 e^{-\varepsilon n^h}\right)\right) - n + 1
\]

\[
= \pi\left(P_n^2\right) + O\left(n^2 e^{-\varepsilon n^h}\right)
\]

\[
= \text{li}\left(P_n^2\right) + O\left(P_n^2 e^{-\varepsilon n^h}\right) + O\left(n^2 e^{-\varepsilon n^h}\right)
\]

\[
= \text{li}\left(P_n^2\right) + O\left(n^2 e^{-\varepsilon n^h}\right)
\]

where we have assumed as before that \(b < a < \frac{1}{4}\). This proves part (1) of the theorem.
Finally, if the odd perfect number \( N \) has smallest prime factor \( P_n \), then (cf. Lemma 3)

\[
\log N > \log \left( \prod_{i=1}^{n-1} P_i^2 \right) = 2\phi(P_n) + O(P_n) = 2P_n + O(\log N).
\]

The proof is complete.

Using known results (see Landau [8]) for \( P_n \) as a function of \( n \), we can rewrite parts (1) and (2) in the (weaker) form

\[
\begin{align*}
(5) \quad a(n) &= \frac{1}{2} n \log n \log 2 + \frac{1}{2} n \log n - \frac{3}{4} n^2 + \frac{n}{2} \log n + O(1), \\
(6) \quad P_n &= n \log n + 2n \log \log n - 2n \log n + n \log 2 + O(n^2).
\end{align*}
\]

The inequality \( N > \exp\left( 2P_n + O(\log n) \right) \), while rather crude, serves to show the rapidity with which \( N \) increases as a function of \( n \) (or \( P_n \)).

### IV. Numerical results

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#### References

Criteria for Kummer’s congruences

by

L. Carlitz (Durham, N. C.)

1. Kummer ([2]) obtained the well-known congruence for the Euler numbers
\[
\sum_{r=0}^n (-1)^r \binom{n}{r} E_{n+np-1} = 0 \pmod{p^r} \quad (n \geq r, \ p > 2)
\]
by means of the following general result. Let \( p \) be a fixed prime \( \geq 2 \). Then if
\[
(1.1) \quad \sum_{n=0}^m a_n \frac{e^n}{n!} = \sum_{k=0}^m A_k (e^p - 1)^p,
\]
where the \( a_n, A_k \) are integral \( \pmod{p^r} \), it follows that
\[
(1.2) \quad \sum_{r=0}^n (-1)^r \binom{n}{r} a_{n+np-1} = 0 \pmod{p^r} \quad (n \geq r).
\]

Indeed since (1.1) is equivalent to
\[
a_n = \sum_{k=0}^n A_k C_n^{(k)}, \quad \text{where} \quad C_n^{(k)} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n,
\]
we have
\[
\sum_{r=0}^n (-1)^r \binom{r}{a_{n+np-1}} = \sum_{k=0}^{n+np-1} A_k \sum_{j=0}^k (-1)^j \binom{k}{j} C_n^{(k-j)} = \sum_{k=0}^{n+np-1} A_k \sum_{j=0}^k (-1)^j \binom{k}{j} j^n (1-j^{p-1})
\]
and (1.2) follows by Fermat’s theorem.