ACTA ARITHMETICA
VI (1961)

On the problem of Gauss

by

O. M. Fomenko (Krasnodar, USSR)

Notation. For $B > 0$ the symbol $A \ll B$ will denote that $|A|/B \leqslant$ const. For real α the symbol $\{\alpha\}$ will denote the fractional part of α; next: $[\alpha] = \alpha - \{\alpha\}$, $\langle\alpha\rangle = \min(\{\alpha\}, 1-\{\alpha\})$, $\exp x = e^x$. The letter ε will denote an arbitrarily small positive number.

It is known that Gauss investigated the asymptotic behaviour of sums $\sum_{t=1}^{N} h(-t)$, where $h(-t)$ denotes the number of classes of purely radical quadratic forms of the negative determinant $-t$. The best estimation of the remainder term in this problem has been obtained by I. M. Vinogradov. In paper [1] he showed it to be of the order

$$N^{0.7-\delta+\varepsilon} \text{ where } \quad \delta = \tfrac{1}{405},$$

and in paper [2] he lowered it to

$$N^{11/16+\varepsilon}.$$

In an analogous way one can show that the remainder term in the asymptotic formula which expresses the number of integer points of the domain $x^2+y^2+z^2 \leqslant a^2$ is of the order $a^{11/8+\varepsilon}$.

Paper [2] gives some indications for further lowering of the estimation of the order of remainder terms in these two important problems. Following these indications we shall show that one can obtain the following orders:

$$N^{701/1020+\varepsilon} \quad \text{and} \quad a^{701/510+\varepsilon}.$$

We shall follow the methods of papers [1] and [2] replacing the considerations of those papers by shorter arguments. The following lemma, proved in [1], will be useful for our purpose:

Lemma. *Let A, q and r be real numbers satisfying the relations $A > 0$, $r-q > 0$; let k be an integer $\geqslant 2$ and let $\varkappa = 2^k$. Next, let $f(x)$ be a real* *function defined on the interval $q \leqslant x \leqslant r$, differentiable k times and satisfying the relation*

$$A^{-1} \ll f^{(k)}(x) \ll A^{-1}.$$

Then:

$$\sum_{q<x\leqslant r} \exp\left(2\pi i f(x)\right) \ll (r-q)\left(A^{-1/(\varkappa-2)} + \left(\frac{(r-q)^k}{A}\right)^{-2/\varkappa} + (r-q)^{-2/\varkappa}\right).$$

We shall denote by $N^{\alpha+\varepsilon}$ the order of the remainder term in the problem of Gauss. From the considerations of paper [1] it follows that it sufficies to estimate the expression

$$B = \sum_{m=1}^{\infty} C_m W_m,$$

where

$$W_m = \sum_{x>\sqrt{n}+1}^{\leqslant\sqrt{4n/3}} W_{m,x}, \qquad W_{m,x} = \sum_{y>-\sqrt{x^2-n}}^{\leqslant\sqrt{x^2-n}} \exp\left(2\pi i m \frac{n+y^2}{x}\right);$$

the number C_m dependes only on m, $C_m \ll Z_m$, $\Delta = n^{\alpha-1}$ and

$$Z_m = \begin{cases} 1/m & \text{if} \quad m \leqslant \Delta^{-1}, \\ 1/\Delta^2 m^3 & \text{if} \quad m > \Delta^{-1}. \end{cases}$$

It is shown in paper [2] that for $m \leqslant \sqrt{n}$

$$W_m \ll m\sqrt{n} + \sqrt{n}(\log n)^2;$$

for $m > \sqrt{n}$ we have a trivial evaluation $W_m \ll n$. Let us now estimate B. We have

$$\sum_{m>0}^{\leqslant n^{\alpha-0.5}} \frac{m\sqrt{n}}{m} \ll n^{\alpha};$$

$$\sum_{m>n^{2.5-3\alpha}}^{\leqslant n^{0.5}} \frac{m\sqrt{n}}{\Delta^2 m^3} \ll n^{\alpha};$$

$$\sum_{m>n^{0.5}} \frac{n}{\Delta^2 m^3} \ll n^{2(1-\alpha)};$$

$n^{2(1-\alpha)} < n^{\alpha}$ if $\alpha > \frac{2}{3}$, which will be assumed throughout. Thus

$$\sum_{m=1}^{n^{0.5}} Z_m \sqrt{n}(\log n)^2 \ll \sqrt{n}(\log n)^3 + \sqrt{n}(\log n)^2.$$

If follows that

$$B = B_0 + O(n^{\alpha})$$

where

$$B_0 = \sum_{m>n^{\alpha-0.5}}^{\leqslant n^{2.5-3\alpha}} C_m \sum_{u\geqslant -m}^{\leqslant m} \sum_{v\geqslant (u^2+3m^2)/4m}^{\leqslant m} \frac{2im\sqrt{n}}{4mv-u^2} \exp\left(2\pi i \sqrt{n(4mv-u^2)}\right).$$

The sum B_0 may be split into $\ll \log n$ sums U_M of the form

$$U_M = \sum_{m>M_0}^{\leqslant M} C_m \sum_{u} \sum_{v} \frac{2\pi i m\sqrt{n}}{4mv-u^2} \exp\left(2\pi i \sqrt{n(4mv-u^2)}\right)$$

where M_0 and M are integers with

$$n^{\alpha-0.5} \leqslant M_0 < M \leqslant n^{2.5-3\alpha}, \qquad M \leqslant \sqrt{\tfrac{3}{2}} M_0.$$

It is more convenient to replace the inequalities which give the domain of summation with respect to u and v by

$$m^2-u^2 \geqslant 0, \qquad m-v \geqslant 0, \qquad 4mv-u^2-3m^2 \geqslant 0.$$

We observe that

$$U_M = \frac{i\sqrt{n}}{6M^5} \sum_{m>\sqrt{2/3}M}^{\leqslant M} \sum_{u\geqslant -M}^{\leqslant M} \sum_{v>\sqrt{3/8}M}^{\leqslant M} \sum_{s_1=0}^{M^2} \sum_{s_2=0}^{M} \sum_{s_3=0}^{M^2} \sum_{k_1=1}^{2M^2} \sum_{k_2=1}^{2M} \sum_{k_3=1}^{3M^2} R,$$

$$R = C_m \frac{m}{4mv-u^2} \exp\Bigg(2\pi i \sqrt{n(4mv-u^2)} + 2\pi i\left(\frac{m^2-u^2-s_1}{2M^2}k_1 + \frac{m-v-s_2}{2M}k_2 + \frac{4mv-u^2-3m^2-s_3}{3M^2}k_3\right)\Bigg).$$

Let U_{M,k_1,k_2,k_3} be the part of the sum U_M which corresponds to a given k_1, k_2, k_3. It is shown in [2] that

$$U_{M,k_1,k_2,k_3} \ll M^{-4} Z_M n^{1/2+\varepsilon'} T'_{k_1,k_2,k_3} \Omega,$$

$$\Omega = \sum_{z>2M^2}^{\leqslant 4M^2} \left| \sum_{u>-M}^{\leqslant M} \frac{\exp\left(2\pi i\left(-k_1u^2/2M^2 - k_3u^2/3M^2 + \sqrt{n(z-u^2)}\right)\right)}{z-u^2} \right|,$$

$$T'_{k_1,k_2,k_3} = \min\left(M^2, \frac{1}{\langle k_1/2M^2\rangle}\right) \min\left(M, \frac{1}{\langle k_2/2M\rangle}\right) \min\left(M^2, \frac{1}{\langle k_3/3M^2\rangle}\right)$$

and

$$\sum_{k_1}\sum_{k_2}\sum_{k_3} T'_{k_1,k_2,k_3} \ll M^5(\log n)^3.$$

For our next considerations we shall need the evaluation of the sum

$$K_{u_1,u} = \sum_{z>2M^2}^{\leqslant 4M^2} \frac{\exp\left(2\pi i\sqrt{n}\left(\sqrt{z-u_1^2}-\sqrt{z-u^2}\right)\right)}{(z-u_1^2)(z-u^2)}$$

for integer u_1 and u where $-M \leqslant u_1 \leqslant M$, $-M \leqslant u \leqslant M$. By putting $t = u^2-u_1^2$ we shall restrict ourselves to the case $t \geqslant 0$ (the case $t<0$ can be reduced to the case $t>0$ in an obvious way). For $t = 0$ we easily obtain

$$K_{u_1,u} \ll M^{-2}.$$

For $t>0$, as it is shown in [2], we have

$$(1) \qquad K_{u_1,u} \ll \frac{n^{1/4}t^{1/2}}{M^{9/2}} + \frac{1}{n^{1/4}t^{1/2}M^{3/2}}.$$

Another estimation of $K_{u_1,u}$ will also be convenient. It is shown in [1] that

$$K_{u_1,u} = K'_{u_1,u} + O(M^{-1,5}n^{-1/4}t^{-1/2}),$$

where

$$K'_{u_1,u} = \sum_{w>-f'(4M^2)}^{\leqslant -f'(2M^2)} \frac{1+i}{\sqrt{2}}\cdot\frac{\exp\left(2\pi i\left(f(z_w)+wz_w\right)\right)}{(z_w-u_1^2)(z_w-u^2)\sqrt{f''(z_w)}},$$

and

$$f(z) = \sqrt{n}\left(\sqrt{z-u_1^2}-\sqrt{z-u^2}\right)$$

with z_w defined by the relation

$$f'(z_w) = -w.$$

Next, we shall evaluate the sum

$$E = \sum_{w>q}^{\leqslant r} \exp\left(2\pi i F(w)\right)$$

where $q = -f'(4M^2)$; $r = -f'(2M^2)$, $F(w) = f(z_w)+wz_w$. We find

$$\sqrt{n}\,t/M^3 \ll r-q \ll \sqrt{n}\,t/M^3.$$

Putting

$$\frac{1}{\sqrt{z_w-u_1^2}} = \xi_1, \qquad \frac{1}{\sqrt{z_w-u^2}} = \xi$$

we easily find that

$$\frac{1}{n^{(k-1)/22}\,\xi(\xi-\xi_1)^{k-1}} \ll F^{(k)}(w) \ll \frac{1}{n^{(k-1)/2}\,\xi^2(\xi-\xi_1)^{k-1}}.$$

Since $M^{-1} \ll \xi \ll M^{-1}$, $t/M^3 \ll \xi-\xi_1 \ll t/M^3$, we have

$$\frac{M^{3k-1}}{n^{(k-1)/2}\,t^{k-1}} \ll F^{(k)}(w) \ll \frac{M^{3k-1}}{n^{(k-1)/2}\,t^{k-1}}.$$

Under suitable restrictions concerning t we may assume that $r-q\to\infty$ as $n\to\infty$. Let us apply our lemma to the sum E. After the reduction we get:

$$E \ll \frac{n^{(\varkappa-k-1)/2(\varkappa-2)}\,t^{(\varkappa-k-1)/(\varkappa-2)}}{M^{(3\varkappa-3k-5)/(\varkappa-2)}} + \frac{n^{(\varkappa-2)/2\varkappa}\,t^{(\varkappa-2)/\varkappa}}{M^{(3\varkappa-6)/\varkappa}}.$$

If follows easily that

$$(2) \qquad K_{u_1,u} \ll \frac{n^{(\varkappa-4k)/4(\varkappa-2)}\,t^{(\varkappa-2k)/2(\varkappa-2)}}{M^{(9\varkappa-6k-16)/2(\varkappa-2)}} + \frac{n^{(\varkappa-4)/4\varkappa}\,t^{(\varkappa-4)/2\varkappa}}{M^{(9\varkappa-12)/2\varkappa}}.$$

Let us first consider the part Ω_0 of Ω defined as:

$$\Omega_0 = \sum_{z>2M^2}^{\leqslant 4M^2}\left|\sum_{u\geqslant -h}^{\leqslant h}\frac{\exp\left(2\pi i\left(-k_1u^2/2M^2-k_3u^2/3M^2+\sqrt{n(z-u^2)}\right)\right)}{z-u^2}\right|$$

(the definition of h is given below; now we shall mention that $h<M$). We have

$$\Omega_0 \ll n^{\varepsilon'/2}\left(h^{1/2}+n^{1/8}h^{3/2}/M^{5/4}\right).$$

Let $\Omega' = \Omega-\Omega_0$. Then

$$\Omega' = 2\sum_{z>2M^2}^{\leqslant 4M^2}\left|\sum_{u>h}^{\leqslant M}\frac{\exp\left(2\pi i\left(-k_1u^2/2M^2-k_3u^2/3M^2+\sqrt{n(z-u^2)}\right)\right)}{z-u^2}\right|.$$

Dividing interval $h<u\leqslant M$ into intervals $h<u\leqslant 2h$, $2h<u\leqslant 3h, \ldots, [M/h]h<u\leqslant M$, we represent Ω' in the form

$$\Omega' = 2\sum_{l=1}^{[M/h]}\Omega_{lh},$$

where

$$\Omega_{lh} = \sum_{z>2M^2}^{\leqslant 4M^2} \left| \sum_{u>lh}^{lh+h'} \frac{\exp\left(2\pi i\left(-k_1u^2/2M^2+k_3u^2/3M^2+\sqrt{n(z-u^2)}\right)\right)}{z-u^2} \right|,$$

and $h' \leqslant h$.

We find

$$\Omega_{lh} \ll M\sqrt{Q_{lh}},$$

where

$$Q_{lh} = \sum_{u_1>lh}^{\leqslant lh+h'} \sum_{u>lh}^{\leqslant lh+h'} K_{u_1,u}.$$

For $t = 0$ the number of solutions of $u^2-u_1^2 = t$ will be $\ll h$. For a given $\xi > 0$ the number of solutions of $u-u_1 = \xi$ will be $\ll h$, and the value of $t = (u_1+\xi)^2-u_1^2$ which corresponds to this ξ will satisfy the relation

$$2lh\xi < t \leqslant 2(l+3)h\xi.$$

The part of Q_{lh} which corresponds to the given $\xi > 0$ will be $\ll hK_{u_1,u}$. Thus (we apply the evaluation (1) to the first sum, and evaluation (2) to the second sum):

$$Q_{lh} \ll hM^{-2} + \sum_{\xi=1}^{[hn^{-x}]} h\left(\frac{n^{1/4}(lh\xi)^{1/2}}{M^{9/2}} + \frac{1}{n^{1/4}M^{3/2}(lh\xi)^{1/2}}\right) +$$

$$+ \sum_{\xi=[hn^{-x}]}^{h} h\left(\frac{n^{(\varkappa-2k)/4(\varkappa-2)}(lh\xi)^{(\varkappa-2k)/2(\varkappa-2)}}{M^{(9\varkappa-6k-16)/2(\varkappa-2)}} + \frac{n^{(\varkappa-4)/4\varkappa}(lh\xi)^{(\varkappa-4)/2\varkappa}}{M^{(9\varkappa-12)/2\varkappa}}\right)$$

$$\ll hM^{-2} + \frac{n^{1/4}l^{1/2}h^3n^{-3x/2}}{M^{9/2}} + \frac{hn^{-x/2}}{n^{1/4}l^{1/2}M^{3/2}} +$$

$$+ \frac{n^{(\varkappa-2k)/4(\varkappa-2)}l^{(\varkappa-2k)/2(\varkappa-2)}h^{(3\varkappa-2k-4)/(\varkappa-2)}}{M^{(9\varkappa-6k-16)/2(\varkappa-2)}} + \frac{n^{(\varkappa-4)/4\varkappa}l^{(\varkappa-4)/2\varkappa}h^{(3\varkappa-4)/\varkappa}}{M^{(9\varkappa-12)/2\varkappa}};$$

the numerical value of x will become clear in the sequel. Now we shall easily find the estimation of Ω'. We have:

$$\Omega' \ll Mh^{-1/2} + n^{1/8}h^{1/4}n^{-3x/4} + n^{(\varkappa-2k)/8(\varkappa-2)}h^{(\varkappa-2k)/4(\varkappa-2)}M^{k/(\varkappa-2)} + \\ + n^{(\varkappa-4)/8\varkappa}h^{(\varkappa-4)/4\varkappa}M^{2/\varkappa}.$$

Let us make the first and the last term of the same order. To do that we put

$$h = [M^{4(\varkappa-2)/(3\varkappa-4)}n^{-(\varkappa-4)/2(3\varkappa-4)}].$$

Now we have

$$\Omega \ll \frac{n^{\varepsilon'/2+1/8}h^{3/2}}{M^{5/4}} + Mh^{-1/2} + n^{1/8}h^{1/4}n^{-3x/4} + n^{(\varkappa-2k)/8(\varkappa-2)}h^{(\varkappa-2k)/4(\varkappa-2)}M^{k/(\varkappa-2)},$$

whence

$$U_M \ll Mn^{1/2+\varepsilon''}Z_M\left(\frac{n^{1/8}h^{3/2}}{M^{5/4}} + Mh^{-1/2} + n^{1/8}h^{1/4}n^{-3x/4} + \\ + n^{(\varkappa-2k)/8(\varkappa-2)}h^{(\varkappa-2k)/4(\varkappa-2)}M^{k/(\varkappa-2)}\right).$$

Let $M \leqslant \Delta^{-1} = n^{1-\alpha}$. Then

$$U_M \ll n^{1/2+\varepsilon''}\left(\frac{n^{1/8}h^{3/2}}{M^{5/4}} + Mh^{-1/2} + n^{1/8}h^{1/4}n^{-3x/4} + \\ + n^{(\varkappa-2k)/8(\varkappa-2)}h^{(\varkappa-2k)/4(\varkappa-2)}M^{k/(\varkappa-2)}\right). \tag{3}$$

Now we shall find the values of α for which we have the inequality

$$n^{1/2}Mh^{-1/2} \leqslant n^{\alpha}.$$

After some computations we get

$$\alpha \geqslant \tfrac{11}{16} - 1/16(\varkappa-1).$$

The inequality

$$n^{(\varkappa-2k)/8(\varkappa-2)}h^{(\varkappa-2k)/4(\varkappa-2)}M^{k/(\varkappa-2)} \leqslant n^{\alpha}$$

holds if

$$\alpha \geqslant \frac{11}{16} - \frac{3k\varkappa-20\varkappa+24}{16(4\varkappa^2+(k-12)\varkappa+8)}.$$

From the two inequalities for α we see that the best results are obtained for $k = 8$. In this case

$$\alpha \geqslant \frac{11}{16} - \frac{1}{4080} = \frac{701}{1020}.$$

Let us put $\alpha = 701/1020$. Now it is easy to show that we can take, for instance, $x = 0.001$. Comparing the first and the third term in parentheses in (3) we see that the first term is negligible as compared with the third. Thus, for $M \leqslant \Delta^{-1}$:

$$U_M \ll n^{701/1020+\varepsilon''}.$$

We easily see that the same result holds also for $M > \Delta^{-1}$. Hence

$$B \ll n^{701/1020+\varepsilon_1}.$$

Thus, as is shown in [1], one can obtain

$$\sum_{t=1}^{N} h(-t) = \frac{4\pi}{21\zeta(3)} N^{3/2} - \frac{2}{\pi^2} N + O(N^{701/1020+\varepsilon}).$$

References

[1] I. M. Vinogradov (И. М. Виноградов), *Улучшение остаточного члена одной асимптотической формулы,* (in Russian), Izv. Akad. Nauk SSSR, Ser. Mat., 13 (1949), pp. 97-110.

[2] — *Улучшение асимптотических формул для числа целых точек в области трех измерений,* (in Russian), ibid. 19 (1955), pp. 3-9.

Reçu par la Rédaction le 23. 3. 1960

ACTA ARITHMETICA
VI (1961)

Local relation of Gauss sums

by

T. KUBOTA (Nagoya, Japan)

Among many other important properties of Gauss sums it is known that the Gauss sum $\tau(\chi)$ of a congruence character χ of an algebraic number field F is essentially the same thing as the constant factor $w(\chi)$ appearing in the functional equation of Hecke's L-function defined by the character χ. Thus interpreted, the Gauss sum $\tau(\chi)$ is very naturally decomposed into its local components $\tau_{\mathfrak{p}}(\chi)$, where $\mathfrak{p}$ means a finite or infinite place of F (see Hasse [4]). We call $\tau_{\mathfrak{p}}(\chi)$ a *local Gauss sum*. The aim of the present note is to investigate some arithmetic attributes of the local Gauss sum.

Let us first consider the factor set

$$j_{\mathfrak{p}}(\chi, \psi) = \frac{\tau_{\mathfrak{p}}(\chi)\tau_{\mathfrak{p}}(\psi)}{\tau_{\mathfrak{p}}(\chi\psi)}$$

between local Gauss sums. It is well known that in many cases such a factor set becomes a so-called Jacobi sum (Hasse [3], Weil [7]). But, in the general case of local Gauss sums, in particular in the case where the conductors of χ, ψ are divisible by a higher power of $\mathfrak{p}$, there is no so simple expression of $j_{\mathfrak{p}}(\chi, \psi)$ as ordinary Jacobi sums. We shall prove, however, the formulas (5), (12) of § 1, which show that $j_{\mathfrak{p}}(\chi, \psi)$ is in every case transformed into a *generalized Jacobi sum*.

In § 2, we deal with the explicit determination of the value of $j_{\mathfrak{p}}(\chi, \psi)$, restricting χ, ψ to quadratic characters. In general, the problem of this kind necessarily concerns a "Grössencharakter" (Weil [7]). But, if χ, ψ are quadratic, then the square of the generalized Jacobi sum $j_{\mathfrak{p}}(\chi, \psi)$ is a natural number which is easily determined and the sign of $j_{\mathfrak{p}}(\chi, \psi)$ itself is, as the formula (16) of § 2 shows, given by the quadratic norm residue symbol.

The formula (16) is equivalent to a splitting formula (17) of the quadratic norm residue symbol. For prime ideals prime to 2, the formula (16) (or equivalently (17)) is easily proved by a simple computation, and for prime ideals dividing 2, (17) is an almost immediate consequence of