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Congruence properties of certain polynomial sequences

by

L. CARLITZ (Durham, North Carolina)

1. Introduction. Consider the recurrence

$$u_{n+1} = f(n)u_n + g(n)u_{n-1}, \tag{1.1}$$

where $f(n)$, $g(n)$ are polynomials in n (and possibly some additional indeterminates) with integral coefficients. Moreover we assume that

$$u_0 = 1, \quad u_1 = f(0), \quad g(0) = 0. \tag{1.2}$$

Thus the sequence $\{u_n; n = 0, 1, 2, \ldots\}$ is uniquely determined by (1.1) and (1.2).

The writer has proved ([1], Theorem 1) that if $m \geqslant 1$, $r \geqslant 1$, then u_n satisfies the congruence

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm} u_{(r-s)m} \equiv 0 \pmod{m^{r_1}} \tag{1.3}$$

for all $n \geqslant 0$, where

$$r_1 = [(r+1)/2], \tag{1.4}$$

the greatest integer $\leqslant (r+1)/2$. The principal object of the present paper is to show that, with the same hypotheses, u_n satisfies the simpler congruences

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm} u_m^{r-s} \equiv 0 \pmod{m^{r_1}}, \tag{1.5}$$

where again r_1 is defined by (1.4). In addition we show that (1.5) implies

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm} u_{k+(r-s)m} \equiv 0 \pmod{m^{r_1}} \tag{1.6}$$

for all $n \geqslant 0$, $k \geqslant 0$; for $k = 0$, (1.6) evidently reduces to (1.3). For a more general result see Theorem 2 below.

We remark that in particular the Hermite and Laguerre polynomials satisfy (1.1) and (1.2) when $f(n)$ and $g(n)$ are properly specialized. Thus (1.5) applies and we obtain

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}(2x)^{(r-s)m}H_{n+sm}(x)\equiv 0 \pmod{m^{r_1}}, \tag{1.7}$$

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}(-x)^{(r-s)m}A_{n+sm}^{(\lambda)}(x)\equiv 0 \pmod{m^{r_1}}, \tag{1.8}$$

where

$$A_n^{(\lambda)}(x)=n!\,L_n^{(\lambda)}(x) \tag{1.9}$$

and $L_n^{(\lambda)}(x)$ is the Laguerre polynomial in the usual notation ([4], Chapter 5); the parameter λ is a rational number that is integral $(\operatorname{mod} m)$. The writer [2] has given a simpler proof of (1.7) and (1.8) by making use of certain explicit formulas available for the Hermite and Laguerre polynomials.

2. It will be convenient to replace (1.1), (1.2) by

$$u_{n+1}(x)=\big(x+f(n)\big)u_n(x)+g(n)u_{n-1}(x), \tag{2.1}$$

$$u_0(x)=1,\quad u_1(x)=x+f(0),\quad g(0)=0; \tag{2.2}$$

as above it is assumed that $f(n)$, $g(n)$ are polynomials in n with integral coefficients. Clearly $u_n(x)$ is a polynomial in x of degree n with integral coefficients; the highest coefficient $=1$. Also it is evident that many sequences of orthogonal polynomials are included in the present discussion.

Since (2.1) implies

$$xu_n(x)=u_{n+1}(x)-f(n)u_n(x)-g(n)u_{n-1}(x),$$

it is clear that

$$x^m u_n(x)=\sum_{s=-m}^{m}A_{ms}(n)u_{n+s}(x)\quad (m,n=0,1,2,\ldots),$$

where the $A_{ms}(n)$ are polynomials in n with integral coefficients. Consequently

$$u_m(x)u_n(x)=\sum_{s=-m}^{m}B_{ms}(n)u_{n+s}(x), \tag{2.3}$$

where the $B_{ms}(n)$ are polynomials in n with integral coefficients. In (2.3) we think of m as fixed while $n=0,1,2,\ldots$; also we take

$$B_{ms}(n)=0\quad (s<-n).$$

We now rewrite (2.3) as

$$u_m(x)u_n(x)-u_{m+n}(x)=\sum_{s=-m}^{m-1}B_{ms}(n)u_{n+s}(x) \tag{2.4}$$

and apply the case $r=1$ of (1.3), that is,

$$u_{m+n}(x)\equiv u_m(x)u_n(x) \pmod{m}.$$

Then (2.4) becomes

$$\sum_{s=-m}^{m-1}B_{ms}(n)u_{n+s}(x)\equiv 0 \pmod{m}. \tag{2.5}$$

We shall require the following

LEMMA. *Let $u_0(x), u_1(x), \ldots, u_n(x)$ denote a set of polynomials in x with integral coefficients and highest coefficients $=1$; also let*

$$\deg u_s(x)=s\quad (0\leqslant s\leqslant n).$$

Assume that $A_0, A_1, \ldots, A_n$ are integers such that

$$\sum_{s=0}^{n}A_s u_s(x)\equiv 0 \pmod{m}; \tag{2.6}$$

then

$$A_s\equiv 0 \pmod{m}\quad (0\leqslant s\leqslant n). \tag{2.7}$$

We remark that if $u(x)$ is a polynomial with integral coefficients, the statement $u(x)\equiv 0 \pmod{m}$ means that each coefficient of $u(x)$ is divisible by m. Then if we put

$$u_s(x)=\sum_{j=0}^{s}a_{sj}x^j\quad (a_{ss}=1,\ 0\leqslant s\leqslant n),$$

(2.6) becomes

$$\sum_{s=0}^{n}A_s\sum_{j=0}^{s}a_{sj}x^j\equiv 0 \pmod{m},$$

so that

$$\sum_{j=0}^{n} A x^j \sum_{s=j}^{n} A_s a_{sj} \equiv 0 \pmod{m}.$$

Consequently, by the above remark,

(2.8) $$\sum_{s=j}^{n} A_s a_{sj} \equiv 0 \pmod{m} \qquad (0 \leqslant s \leqslant n).$$

Since the matrix (a_{sj}) is triangular and $a_{ss} = 1$ for $0 \leqslant s \leqslant 1$, it is clear that (2.8) implies (2.7). This completes the proof of the lemma.

Applying the lemma to (2.5) we immediately obtain

(2.9) $$B_{ms}(n) \equiv 0 \pmod{m} \qquad (-m \leqslant s \leqslant m-1).$$

In the next place we define the operator Δ by means of

(2.10) $$\Delta\varphi_n = u_m(x)\varphi_n - \varphi_{n+m}$$

and generally

(2.11) $$\Delta^r\varphi_n = u_m(x)\Delta^{r-1}\varphi_n - \Delta^{r-1}\varphi_{n+m},$$

where φ_n is an arbitrary function of n. Clearly (2.10) and (2.11) imply

(2.12) $$\Delta^r\varphi_n = \sum_{s=0}^{r} (-1)^s \binom{r}{s} u_m^{r-s}(x)\varphi_{n+sm}.$$

Applying Δ^{r-1} to both members of (2.4) we get

(2.13) $$\Delta^r u_n(x) = \sum_{s=-m}^{m-1} \Delta^{r-1}\{B_{ms}(n)u_{n+s}(x)\}.$$

In addition to Δ^r we also require the operator δ^r defined by

(2.14) $$\delta^r\varphi_n = \sum_{s=0}^{r} (-1)^s \binom{r}{s}\varphi_{n+sm}.$$

Since (2.14) is equivalent to

(2.15) $$\varphi_{n+km} = \sum_{j=0}^{k} (-1)^j \binom{k}{j}\delta^j\varphi_n,$$

we get

$$\Delta^{r-1}\{B_{ms}(n)u_{n+s}(x)\} = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} u_m^{r-1-k}(x) B_{ms}(n+km) u_{n+km}(x)$$
$$= \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} u_m^{r-1-k}(x) u_{n+km}(x) \sum_{j=0}^{k} (-1)^j \binom{k}{j} \delta^j B_{ms}(n)$$
$$= \sum_{j=0}^{r-1} \binom{r-1}{j} \delta^j B_{ms}(n) \sum_{k=j}^{r-1} (-1)^{k-j} \binom{r-1-j}{k-j} u_m^{r-1-k}(x) u_{n+km}(x)$$
$$= \sum_{j=0}^{r-1} \binom{r-1}{j} \delta^j B_{ms}(n) \sum_{k=0}^{r-1-j} (-1)^k \binom{r-1-j}{k} u_m^{r-1-j-k}(x) u_{n+jm+km}(x)$$
$$= \sum_{j=0}^{r-1} \binom{r-1}{j} \delta^j B_{ms}(n) \cdot \Delta^{r-1-j} u_{n+jm}(x).$$

Thus (2.13) becomes

(2.16) $$\Delta^r u_n(x) = \sum_{s=-m}^{m-1} \sum_{j=0}^{r-1} \delta^j B_{ms}(n) \Delta^{r-1-j} u_{n+jm}(x).$$

We shall now prove (1.5) by an induction with respect to r. For $r = 1$, the result is the case $r = 1$ of (1.3). We accordingly assume that (1.5) holds up to and including the value $r-1$. Also since $B_{ms}(n)$ is a polynomial in n with integral coefficients, it follows from (2.14) that

(2.17) $$\delta^j B_{ms}(n) \equiv 0 \pmod{m^j}.$$

Consider a typical term

$$A_{js} = \delta^j B_{ms}(n) \Delta^{r-1-j} u_{n+jm}(x)$$

in the right member of (2.16). For $j = 0$, we use (2.9) to get

(2.18) $$A_{0s} \equiv 0 \pmod{m^{1+[r/2]}},$$

by the inductive hypothesis. For $j \geqslant 1$, we employ (2.17) to get

(2.19) $$A_{js} \equiv 0 \pmod{m^{j+[(r-j)/2]}}.$$

Since

$$1 + [\tfrac{1}{2}r] \geqslant [\tfrac{1}{2}(r+1)], \quad j + [\tfrac{1}{2}(r-j)] \geqslant [\tfrac{1}{2}(r+1)] \qquad (1 \leqslant j \leqslant r),$$

it evidently follows from (2.16), (2.18) and (2.19) that

(2.20) $$\Delta^i u_n(x) \equiv 0 \pmod{m^{r_1}}.$$

This completes the proof of (1.5).

We may state

THEOREM 1. *Let $f(n)$, $g(n)$ denote polynomials in n with coefficients that are integral* $(\mathrm{mod}\, m)$, *where m is a fixed integer $\geqslant 1$. Define the sequence of polynomials $\{u_n(x)\}$ by means of* (2.1) *and* (2.2). *Then $u_n(x)$ satisfies the congruence* (2.20) *for all $n \geqslant 0$, $r \geqslant 1$, where*

$$\Delta^r u_n(x) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} u_m^{r-s}(x)\, u_{n+sm}(x) \quad \textit{and} \quad r_1 = [(r+1)/2].$$

3. The proof of (1.6) depends upon the following identity. Put

$$U_{nk}^{(r)}(x) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} u_{n+sm}(x)\, u_{k+(r-s)m}(x). \tag{3.1}$$

Then we have

$$U_{nk}^{(r)}(x) = \sum_{s=0}^{r} (-1)^{r-j} \binom{r}{s} \Delta^s u_n(x) \cdot \Delta^{r-s} u_k(x). \tag{3.2}$$

Indeed it follows easily from the definition of $\Delta^s u_n(x)$ that

$$u_{n+rm}(x) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} u_m^{r-s}(x)\, \Delta^s u_n(x). \tag{3.3}$$

Then (3.1) becomes

$$\begin{aligned} U_{nk}^{(r)}(x) &= \sum_{s=0}^{r} (-1)^s \binom{r}{s} \sum_{j=0}^{s} (-1)^j \binom{s}{j} u_m^{s-j}(x)\, \Delta^j u_n(x) \cdot u_{k+(r-s)m}(x) \\ &= \sum_{j=0}^{s} \binom{r}{j} \Delta^j u_n(x) \sum_{s=j}^{r} (-1)^{s-j} \binom{r-j}{s-j} u_m^{s-j}(x)\, u_{k+(r-s)m}(x) \\ &= \sum_{j=0}^{s} \binom{r}{j} \Delta^j u_n(x) \sum_{s=0}^{r-j} (-1)^s \binom{r-j}{s} u_m^{s}(x)\, u_{k+(r-j-s)m}(x) \\ &= \sum_{j=0}^{s} (-1)^{r-j} \binom{r}{j} \Delta^j u_n(x) \cdot \Delta^{r-j} u_k(x). \end{aligned}$$

This evidently proves (3.2).

Now by Theorem 1, we have

$$\Delta^s u_n(x) \equiv 0 \ (\mathrm{mod}\, m^{[(s+1)/2]}), \tag{3.4}$$

$$\Delta^{r-s} u_k(x) \equiv 0 \ (\mathrm{mod}\, m^{[(r-s+1)/2]}), \tag{3.5}$$

for all $n \geqslant 0$, $k \geqslant 0$, $0 \leqslant s \leqslant r$. Since

$$[\tfrac{1}{2}(s+1)] + [\tfrac{1}{2}(r-s+1)] \geqslant [\tfrac{1}{2}(r+1)],$$

it follows from (3.2). (3.4) and (3.5) that

$$U_{rk}^{(r)}(x) \equiv 0 \ (\mathrm{mod}\, m^{r_1}) \tag{3.6}$$

for all $n \geqslant 0$, $k \geqslant 0$. This completes the proof of (1.6).

A more general result can be obtained by first generalizing the identity (3.2). Let $n_1, \ldots, n_k$ be arbitrary non-negative integers and $\lambda_1, \ldots, \lambda_k$ arbitrary parameters. Put

$$U_k^{(r)} = U_{n_1 \ldots n_k}^{(r)}(x) = \sum_{s_1+\ldots+s_k=r} \frac{r!}{s_1! \ldots s_k!} \lambda_1^{s_1} \ldots \lambda_k^{s_k} \prod_{j=1}^{k} u_{n_j+s_j m}(x). \tag{3.7}$$

This can be written more compactly in the symbolic form

$$U_k^{(r)} = u_1^{n_1} \ldots u_k^{n_k} (\lambda_1 u_1^m + \ldots + \lambda_k u_k^m)^n, \tag{3.8}$$

where it is understood that, after expanding the right member by the multinomial theorem, each $u_j^{n_j+ms_j}$ is replaced by $u_{n_j+s_j m}(x)$.

Using (3.2), we get

$$\begin{aligned} U_k^{(r)} &= \sum_{s_1+\ldots+s_k=r} \frac{r!}{s_1! \ldots s_k!} \lambda_1^{s_1} \ldots \lambda_k^{s_k} \times \\ &\quad \times \prod_{j=1}^{k} \sum_{t_j=0}^{s_j} (-1)^{t_j} \binom{s_j}{t_j} u_m^{s_j-t_j} \Delta^{t_j} u_{n_j} \\ &= \sum_{t_1+\ldots+t_k \leqslant r} (-1)^t \frac{r!}{t_1! \ldots t_k! (r-t)!} \lambda_1^{t_1} \ldots \lambda_k^{t_k} \Delta^{t_1} u_{n_1} \ldots \Delta^{t_k} u_{n_k} \times \\ &\quad \times \sum_{s_1, \ldots, s_k} \frac{(r-t)!}{(s_1-t_1)! \ldots (s_k-t_k)!} (\lambda_1 u_m)^{s_1-t_1} \ldots (\lambda_k u_m)^{s_k-t_k}, \end{aligned}$$

where $t = t_1 + \ldots + t_k$. We therefore get the identity

$$\begin{aligned} U_k^{(r)} &= \sum_{t_1+\ldots t_k \leqslant r} (-1)^t \frac{r!}{t_1! \ldots t_k! (r-t)!} \lambda_1^{t_1} \ldots \lambda_k^{t_k} \times \\ &\quad \times (\lambda_1 + \ldots + \lambda_k)^{r-t} u_m^{r-t} \Delta^{t_1} u_{n_1} \ldots \Delta^{t_k} u_{n_k}. \end{aligned} \tag{3.9}$$

In the identity (3.9) the λ_j are arbitrary quantities. We shall now assume that each λ_j is integral $(\mathrm{mod}\, m)$ and moreover

$$\lambda_1 + \ldots + \lambda_k \equiv 0 \ (\mathrm{mod}\, m). \tag{3.10}$$

Applying Theorem 1 we get

$$\Delta^{t_1}u_{n_1}\dots\Delta^{t_k}u_{n_k}\equiv 0\ (\mathrm{mod}\, m^e),$$

where

$$e=[\tfrac{1}{2}(t_1+1)]+\dots+[\tfrac{1}{2}(t_k+1)].$$

Therefore, using (3.10), it follows that the indicated summand in the right member of (3.8)

$$\equiv 0\ (\mathrm{mod}\, m^{r-(t_1+\dots+t_k)+e}).$$

Since

$$t_j-[\tfrac{1}{2}(t_j+1)]\leqslant \tfrac{1}{2}t_j \quad (j=1,\dots,k),$$

it is clear that

$$r-(t_1+\dots+t_k)+e\geqslant r-\tfrac{1}{2}(t_1+\dots+t_k)\geqslant \tfrac{1}{2}r.$$

It follows that

$$U_k^{(r)}\equiv 0\ (\mathrm{mod}\, m^{r_1}). \tag{3.11}$$

We may now state

THEOREM 2. *Let the sequence* $\{u_n(x)\}$ *be defined as in Theorem* 1. *Define*

$$U_k^{(r)}=U_{n_1\dots n_k}^{(r)}(x)$$

by means of (3.7), *where* $n_1,\dots,n_k$ *are arbitrary integers* $\geqslant 0$ *and* $\lambda_1,\dots,\lambda_k$ *are integral* $(\mathrm{mod}\, m)$ *and in addition satisfy* (3.10). *Then* $U_k^{(r)}$ *satisfies* (3.11) *with* $r_1=[(r+1)/2]$.

In particular for $r=2$, $\lambda_1=1$, $\lambda_2=-1$, Theorem 2 reduces to (3.6).

We remark that the congruence (3.11) was suggested by the following congruence for the Bernoulli numbers proved by Vandiver [5]:

$$h_1^{n_1}\dots h_k^{n_k}(\lambda_1 h_1^{p-1}+\dots+\lambda_k h_k^{p-1})^r\equiv 0\ \big(\mathrm{mod}\,(p^r,p^{n_1-1},\dots,p^{n_k-1})\big) \tag{3.12}$$

$$(n_i\not\equiv 0\ (\mathrm{mod}\, p-1),\ i=1,\dots,k),$$

where the left member is expended by the multinomial theorem and B_n/n substituted for h_i^n in the result; B_m is the Bernoulli number in the even suffix notation, the λ_j are rational integers such that

$$\lambda_1+\dots+\lambda_k\equiv 0\ (\mathrm{mod}\, p)$$

and p is an odd prime. For example, when $r=2$, (3.12) implies in particular

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{B_{m+(r-s)(p-1)}}{m+(r-s)(p-1)}\,\frac{B_{n+s(p-1)}}{n+s(p-1)}\equiv 0\ (\mathrm{mod}\, p^r)$$

provided $p-1\nmid m$, $p-1\nmid n$, $m>r$, $n>r$. The congruence (3.12) was later generalized by the present writer [3].

While (3.11) superficially resembles (3.12), it should be noted that the congruences differ widely in certain respects.

Returning to Theorem 2, we remark that the λ_j may contain indeterminates, or again may be algebraic numbers; all that is required is that each λ_j is integral $(\mathrm{mod}\, m)$ and that (3.10) is satisfied.

We also remark if some sequences $\{u_n(x)\}$ satisfies

$$\Delta^r u_n(x)\equiv 0\ (\mathrm{mod}\, m^r), \tag{3.13}$$

then exactly as in the proof of Theorem 2, we get

$$U_k^{(r)}\equiv 0\ (\mathrm{mod}\, m^r); \tag{3.14}$$

observe that in both (3.13) and (3.14) the modulus is m^r rather than m^{r_1}.

4. As remarked in the Introduction, Theorem 1 and 2 apply to the Hermite polynomial $H_n(x)$ and the modified Laguerre polynomial $\Lambda_n^{(\lambda)}(x)$ as defined by (1.9). Since

$$H_m(x)\equiv(2x)^m,\qquad \Lambda_m^{(\lambda)}(x)\equiv(-x)^m\ (\mathrm{mod}\, m),$$

we get (1.7) and (1.8).

Another interesting example is furnished by the polynomial $f_n(x)$ defined as follows. Put

$$\frac{(1+t)^u}{(1-t)^{u+1}}=\sum_{n=0}^{n}A_n(u)t^n,\qquad f_n(x)=n!\,A_n(u)\qquad (x=2u+1).$$

Then $f_n(x)$ satisfies the recurrence

$$f_{n+1}(x)=xf_n(x)+n^2 f_{n-1}(x);$$

also $f_0(x)=1$, $f_1(x)=x$. Thus (2.1) and (2.2) are satisfied. It is proved in [3] that

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}f_{n+sm}(x)f_{(r-s)m}(x)\equiv 0\ (\mathrm{mod}\, m^r). \tag{4.1}$$

It is also proved that

$$f_m(x)\equiv(x-1)(x-3)\dots(x-2m+1)\ (\mathrm{mod}\, m).$$

Consequently Theorem 1 yields

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}\{(x-1)(x-3)\dots(x-2m+1)\}^{r-s}f_{n+sm}(x)\equiv 0\ (\mathrm{mod}\, m^{r_1}). \tag{4.2}$$

Note that in (4.1) the modulus is m^r while in (4.2) it is only m^{r_1}. As remarked at the end of § 3, the hypothesis (3.13) implies in particular

$$U_2^{(r)} \equiv 0 \pmod{m^r},$$

but the converse is apparently not true.

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DUKE UNIVERSITY

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On the average number of direct factors of a finite abelian group

by

E. Cohen (Knoxville, Tenn.)

1. Introduction. For positive integers n, let $\tau(n)$ denote the number of divisors of n, and let $t(n)$ denote the number of decompositions of n into two relatively prime factors. In this paper we prove analogues for the finite abelian groups of the classical results of Dirichlet and Mertens on the average order of $\tau(n)$ and $t(n)$. We recall Dirichlet's formula [4], with $x \geqslant 2$,

$$D(x) \equiv \sum_{n\leqslant x}\tau(n) = x(\log x+2\gamma-1)+O(\sqrt{x}), \tag{1.1}$$

γ denoting Euler's constant, and Mertens's estimate [8],

$$D^*(x) \equiv \sum_{n\leqslant x} t(n) = ax(\log x+2\gamma-1)+2bx+O(\sqrt{x}\log x), \tag{1.2}$$

where $a = \eta(2)$, $b = \eta'(2)$, $\eta(s) = 1/\zeta(s)$, $\zeta(s)$ denoting the Riemann zeta-function, $s > 1$. For proofs and discussions of (1.1) and (1.2) we mention [1], §§ 13.2, 13.5, 13.9, [3], p. 282-283, 289, [7], p. 665-666.

The functions $\tau(n)$ and $t(n)$ can be generalized from the (multiplicative) semigroup J^* of the integers n to the semigroup X of the finite abelian groups with respect to the direct product. A general discussion of functions defined in X appears in [2]. For groups G, H contained in X, denote by (G, H) the group of maximal order in X which is simultaneously a direct factor of G and H. Denoting by E_0 the identity of X, we say that G and H are *relatively prime* if $(G, H) = E_0$. A direct factor D of G will be called *unitary* if $D\times E = G$, $(D, E) = E_0$.

For groups G in X, let $\tau(G)$ denote the number of direct factors of G in X, or equivalently, the total number of decompositions, $G = D\times E$, in X. Analogously, let $t(G)$ denote the number of *unitary* factors of G in X, that is, the total number of direct decompositions of G into two relatively prime factors of X. In view of the isomorphism [2] of J^* with the