and so the number of solutions is $1+\chi(-3)$. In the second sum there is exactly one u for each $v \neq p-1$. Hence the sum is $p-2$. Therefore we have

$$\sigma_3 = \chi(-1)p\Omega+p^2\big(1+\chi(-3)\big)-p(p-2) = p^2\chi(-3)+p\{2+\chi(-1)\Omega\}.$$

Separating the cases $p = 6n\pm 1$ and substituting from Lemma 1 we have (2). (3) now follows from (1). This completes the proof of Theorem 1.

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On new "explicit formulas" in prime number theory II

by

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1. The first part of this paper has been concerned with new explicit formulas for

$$\psi_0(x) = \frac{\psi(x-0)+\psi(x+0)}{2}, \quad \text{where} \quad \psi(x) = \sum_{n\leqslant x}\Lambda(n) \equiv \sum_{p^m\leqslant x}\log p,$$

depending upon the zeros of the partial sums $U_N(s) = \sum_{n\leqslant N} 1/n^s$ of the zeta-series. The following formula has been established ([2], Theorem):

$$\psi_0(x) = \frac{\log N!}{N} - \sum_{\varrho}\frac{x^\varrho}{\varrho}, \tag{1.1}$$

$\varrho = \beta+i\gamma$ running through the zeros of $U_N(s)$, $2\leqslant x\leqslant N$, and N being sufficiently large. In the particular case of $N = [e^x]$ we have obtained

$$\psi_0(x) = x - \sum_{|\gamma|\leqslant x^{16},\, \beta\geqslant -1}\frac{x^\varrho}{\varrho} + O(\log x). \tag{1.2}$$

It seems to be worth while to generalise (1.1), (1.2) and find similar formulas depending upon the zeros of other Dirichlet-polynomials approximating to $\zeta(s)$. The most interesting case is that of the Riesz means

$$R_N(s) = \sum_{n\leqslant N}\left(1-\frac{\log n}{\log N}\right)n^{-s}, \qquad s = \sigma+it,$$

considering that they converge to $\zeta(s)$ in the closed half-plane $\sigma\geqslant 1$, $s\neq 1$ (see [3] and [4]). We are now going to study that case. We shall, in fact, find some analogies with (1.1), (1.2) and at the same time touch on the distribution of zeros of $R_N(s)$. It seems plausible that $R_N(s)$ do not vanish in the whole half-plane $\sigma\geqslant 1$. Yet, for the time being, we are only able to determine a certain portion of this half-plane which is free of the zeros of $R_N(s)$. We may note in passing that the regions announced in Lemma 2 and Lemma 4 might be considerably improved if we made use of better estimations of $\zeta(s)$. This, however, is of no importance in the sequel.

We make no claim of having discovered the formulas (2.1) and (2.2) of Lemma 1. They have probably been known for some time, yet we have not been able to trace them in literature.

The present paper has been as well as the previous one much influenced by Professor P. Turán. I wish to express my sincere indebtedness to him again.

2. LEMMA 1. *We have*

$$R_N(s) = \zeta(s) + \frac{\zeta'(s)}{\log N} + \frac{N^{1-s}}{(1-s)^2 \log N} + O\left(\frac{\log N}{N^{1/8}}\right) \tag{2.1}$$

and

$$R'_N(s) = \zeta'(s) + \frac{\zeta''(s)}{\log N} + 2\frac{N^{1-s}}{(1-s)^3 \log N} - \frac{N^{1-s}}{(1-s)^2} + O\left(\frac{\log^2 N}{N^{1/8}}\right) \tag{2.2}$$

in the domain $\sigma \geqslant 1$, $|t| \leqslant N$, $s = \sigma + it \neq 1$.

Proof. We note that (2.1) is evident for $\sigma > 1 + \frac{1}{5}$ since in that case

$$R_N(s) = \zeta(s) + \frac{\zeta'(s)}{\log N} - \sum_{n>N}\left(1 - \frac{\log n}{\log N}\right) n^{-s} = \zeta(s) + \frac{\zeta'(s)}{\log N} +$$

$$+ O\left(\frac{1}{\log N}\int_N^\infty \frac{\log y}{y^\sigma}\,dy\right) = \zeta(s) + \frac{1}{\log N}\zeta'(s) + O(N^{-1/5})$$

$$= \zeta(s) + \frac{1}{\log N}\zeta'(s) + \frac{N^{1-s}}{(1-s)^2 \log N} + O\left(\frac{\log N}{N^{1/8}}\right).$$

Thus we may confine ourselves to $\sigma \leqslant 1 + \frac{1}{5}$. We start from the generalized Perron formula (see [1], p. 50)

$$\log N \cdot R_N(s) = \frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty} \frac{\zeta(s+w)}{w^2} N^w\,dw \tag{2.3}$$

$$= \frac{1}{2\pi i}\int_{1/4-2Ni}^{1/4+2Ni} \frac{\zeta(s+w)}{w^2} N^w\,dw + O\left(\frac{1}{N^{3/4}}\right).$$

Then a simple application of the theorem of residues, applied to the rectangle Q with vertices $\pm\frac{1}{4} \pm 2Ni$, shows that

$$\frac{1}{2\pi i}\int_{(Q)} \frac{\zeta(s+w)}{w^2} N^w\,dw = \zeta(s)\log N + \zeta'(s) + \frac{N^{1-s}}{(1-s)^2}.$$

Further, owing to the familiar inequality [1]

$$|\zeta(u+iv)| \leqslant c_1 v^{(1-u)/2} \log|v| \qquad (u > 0,\ |v| \geqslant 2), \tag{2.4}$$

we can estimate

$$\left|\frac{1}{2\pi i}\int_{-1/4\pm 2Ni}^{1/4\pm 2Ni} \frac{\zeta(s+w)}{w^2} N^w\,dw\right| \leqslant c_2 \frac{N^{1/8+1/4}\log N}{N^2} = O\left(\frac{1}{N}\right)$$

$$\left(\tfrac{3}{4} \leqslant \Re(s+w),\ N \leqslant |\Im(s+w)| \leqslant 3N\right),$$

and

$$\left|\frac{1}{2\pi i}\int_{-1/4-2Ni}^{-1/4+2Ni} \frac{\zeta(s+w)}{w^2} N^w\,dw\right| \leqslant c_3 N^{-1/4} N^{1/8} \int_{-\infty}^{+\infty} \frac{dv}{v^2 + \frac{1}{16}} = O\left(\frac{\log N}{N^{1/8}}\right)$$

$$\left(\tfrac{3}{4} \leqslant \Re(s+w) \leqslant 1 - \tfrac{1}{20},\ |\Im(s+w)| \leqslant 3N\right).$$

Hence and by (2.3) we obtain

$$\frac{1}{2\pi i}\int_{(Q)} \frac{\zeta(s+w)}{w^2} N^w\,dw = \log N \cdot R_N(s) + O\left(\frac{\log N}{N^{1/8}}\right)$$

and formula (2.1) follows.

If we differentiate (2.3) and start from the equality

$$\log N \cdot R'_N(s) = \frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty} \frac{\zeta'(s+w)}{w^2} N^w\,dw,$$

using further the estimate

$$|\zeta'(u+iv)| \leqslant c_4 v^{(1-u)/2} \log^2|v| \qquad (u > 0,\ |v| \geqslant 2), \tag{2.5}$$

in place of (2.4), we shall similarly get formula (2.2).

LEMMA 2. *For N sufficiently large $R_N(s)$ has no zeros in the region* $\sigma \geqslant 1$, $|t| \leqslant N$.

Proof. We shall begin by proving that $R_N(s) \neq 0$ for $\sigma \geqslant 1$, $1 \leqslant |t| \leqslant N$. Formula (2.1) gives

$$R_N(s) = \zeta(s)\left(1 + \frac{\zeta'}{\zeta}(s) \cdot \frac{1}{\log N}\right) + O\left(\frac{1}{\log N}\right).$$

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[1] All the properties of the zeta-function which are used in this paper are well known and may be found in the monograph [6]. $c_1, c_2, \ldots$ denote positive numerical constants throughout.

Applying the well-known estimates

$$\frac{1}{\zeta(s)} = o\bigl(\log(|t|+1)\bigr), \quad \sigma \geqslant 1, \quad |t| \geqslant 1, \tag{2.6}$$

$$\frac{\zeta'}{\zeta}(s) = o\bigl(\log(|t|+1)\bigr), \quad \sigma \geqslant 1, \quad |t| \geqslant 1, \tag{2.7}$$

we get

$$|R_N(s)| \geqslant \frac{c_5}{\log N} \quad \text{for} \quad \sigma \geqslant 1, \quad 1 \leqslant |t| \leqslant N.$$

Next we consider the region $|1-s| \leqslant D/\log N$, $\sigma \geqslant 1$, where $D = 2+\varepsilon$ ($\varepsilon > 0$) is a certain number to be defined later. Writing $\zeta(s) = 1/(s-1) + O(1)$, $\zeta'(s) = -1/(s-1)^2 + O(1)$ and developing $N^{1-s}/(1-s)^2\log N$ in Laurent series with respect to $(1-s)$ we obtain from (2.1)

$$R_N(s) = \frac{\log N}{2} + \frac{(1-s)}{6}\log^2 N\left(1+\frac{1-s}{4}\log N\right) + \\ + \frac{(1-s)^3\log^4 N}{5!} + \frac{(1-s)^4\log^5 N}{6!} + \ldots + O(1),$$

$$|R_N(s)| \geqslant \Re R_N(s) \geqslant \frac{\log N}{2} + \Re\left\{\frac{1-s}{6}\log^2 N\left(1+\frac{(1-s)\log N}{4}\right)\right\} - \\ -\left(\frac{D^3}{5!}+\frac{D^4}{6!}+\ldots\right)\cdot\log N + O(1)$$

$$\geqslant \frac{\log N}{2} + \left\{\Re\left(\frac{1-s}{6}\log^2 N\right)\Re\left(\frac{1-s}{4}\log N+1\right) - \right. \\ \left. -\Im\left(\frac{1-s}{6}\log^2 N\right)\Im\left(\frac{1-s}{4}\log N+1\right)\right\} - \left(\frac{D^3}{5!}+\frac{D^4}{6!}+\ldots\right)\log N + O(1).$$

Also

$$\Re\left(\frac{1-s}{6}\log^2 N\right)\Re\left(1+\frac{1-s}{4}\log N\right) \\ = \frac{(1-\sigma)\log^2 N}{6}\left(1+\frac{1-\sigma}{4}\log N\right) \geqslant -\frac{\log N}{6},$$

$$\Im\left(\frac{1-s}{6}\log^2 N\right)\Im\left(1+\frac{1-s}{4}\log N\right) = \frac{t^2\log^3 N}{24},$$

whence

$$\Re\left(\frac{1-s}{6}\log^2 N\right)\Re\left(1+\frac{1-s}{4}\log N\right) - \Im\left(\frac{1-s}{6}\log^2 N\right)\Im\left(\frac{1-s}{4}\log N+1\right) \\ \geqslant \frac{-\log N}{6} - \frac{D^2\log N}{24} = -\frac{\log N}{3} + o(\log N)$$

(the last symbol "o" referring to $\varepsilon \to 0$). Hence

$$|R_N(s)| \geqslant \frac{\log N}{6} - \log N\left(\frac{2^3}{5!}+\frac{2^4}{6!}+\ldots\right) + o(\log N)$$

$$= \log N\left(\frac{1}{6} - \frac{1}{4}(e^2-7)+o(1)\right) \geqslant \log N\left(\frac{1}{6}-\frac{1}{4}\,0\cdot 4+o(1)\right) \geqslant \frac{\log N}{20}$$

with a suitably fixed $\varepsilon > 0$.

Lastly, we shall treat the case $\sigma \geqslant 1$, $|t| \leqslant 1$, $D/\log N \leqslant |1-s|$. Suppose, in addition, that $|1-s| \leqslant 1/\sqrt[3]{\log N}$. Then we have by (2.1)

$$|R_N(s)-\zeta(s)| \leqslant \frac{2}{|1-s|^2\log N} + O\left(\frac{1}{\log N}\right),$$

$$|R_N(s)| = |R_N(s) - \zeta(s) + \zeta(s)| \geqslant |\zeta(s)| - |R_N(s)-\zeta(s)|$$

$$\geqslant \frac{1}{|1-s|} - \frac{2}{|1-s|^2\log N} + O(1) = \frac{1}{|1-s|}\left(1-\frac{2}{|1-s|\log N)}\right) + O(1)$$

$$> \sqrt[3]{\log N}\left(1-\frac{2}{D}\right) + O(1) > c_6\sqrt[3]{\log N},$$

as $D > 2$. For $|1-s| > 1/\sqrt[3]{\log N}$ we clearly get

$$|R_N(s) - \zeta(s)| \leqslant \frac{2}{|1-s|^2\log N} + O\left(\frac{1}{\log N}\right) \leqslant \frac{c_7}{\sqrt[3]{\log N}},$$

and as $|\zeta(s)| > c_8$ we finally obtain $|R_N(s)| > c_9$. The proof is finished.

However, Lemma 2 does not help much in our further considerations. We must prove that $R_N(s)$ do not vanish in a certain half-plane. In fact, we have the following

LEMMA 3. *Let $N > c_{10}$. If $\sigma \geqslant 1+2\frac{\log\log N}{\log N}$, then*

$$|R_N(s)| > \frac{c_{11}}{\sigma-1}\left(\frac{\log\log N}{\log N}\right)^2 \tag{2.8}$$

(and thus $R_N(s) \neq 0$ in this half-plane).

Proof. Write

$$r_N(s) = \sum_{n>N} \left(1-\frac{\log n}{\log N}\right) n^{-s},$$

so that

$$R_N(s) = \zeta(s) + \frac{1}{\log N}\zeta'(s) - r_N(s).$$

We clearly have

$$|r_N(s)| \leqslant \sum_{n>N} \frac{\log n}{\log N}\cdot\frac{1}{n^\sigma} \leqslant \frac{1}{\log N}\int_N^\infty \frac{\log y}{y^\sigma}\,dy$$

$$= \frac{N^{1-\sigma}}{\log N(\sigma-1)}\left(\log N + \frac{1}{\sigma-1}\right) < \frac{c_{12}}{(\sigma-1)\log^2 N}.$$

Further, as

$$\left|1+\frac{1}{\log N}\cdot\frac{\zeta'}{\zeta}(s)\right| \geqslant 1 - \frac{1}{\log N}\cdot\frac{\zeta'}{\zeta}(\sigma) \geqslant 1 - \frac{c_{13}}{(\sigma-1)\log N}$$

$$\geqslant 1 - \frac{c_{14}}{\log\log N} > \frac{1}{2}$$

and

$$\left|\frac{1}{\zeta(s)}\right| \leqslant \frac{c_{15}}{\min(\sigma-1,1)},$$

we get

$$\left|\frac{1}{\zeta(s)+\frac{1}{\log N}\zeta'(s)}\right| = \frac{1}{|\zeta(s)|}\cdot\frac{1}{\left|1+\frac{1}{\log N}\cdot\frac{\zeta'}{\zeta}(s)\right|} \leqslant \frac{c_{16}}{\min(\sigma-1,1)},$$

so that for $1+2\log\log N/\log N \leqslant \sigma \leqslant 2$

$$|R_N(s)| > c_{17}(\sigma-1) - c_{18}\frac{1}{(\sigma-1)\log^2 N}$$

$$= \frac{1}{(\sigma-1)\log^2 N}\left(c_{17}(\sigma-1)^2\log^2 N - c_{18}\right) > \frac{c_{19}}{\sigma-1}\left(\frac{\log\log N}{\log N}\right)^2.$$

For $\sigma > 2$, (2.8) is evident.

LEMMA 4. *For sufficiently large N we have*

$$\left|\frac{R_N'}{R_N}(s)\right| \leqslant c_{20}(\log\log N)^3 \tag{2.9}$$

in the domain $\sigma \geqslant 1$, $1 \leqslant |t| \leqslant \log^4 N$.

Proof: We proceed as in the proof of Lemma 2. We have

$$|R_N'(s)| \leqslant c_{21}(\log\log N)^2,$$

considering that

$$|\zeta'(\sigma+it)| \leqslant c_{22}\log^2(|t|+1), \qquad |\zeta''(\sigma+it)| \leqslant c_{23}\log^3(|t|+1).$$

On the other hand (2.6) gives

$$\left|\frac{1}{\zeta(\sigma+it)}\right| \leqslant c_{24}\log\log N,$$

which implies that

$$|R_N(s)| \geqslant \frac{c_{25}}{\log\log N},$$

so that (2.9) is established.

LEMMA 5. *We have for sufficiently large N*

$$\lim_{X\to\infty}\frac{1}{2X}\int_{-X}^{X}\left|\frac{R_N'}{R_N}(1+\tfrac{1}{4}+it)\right|^2 dt \leqslant c_{26}.$$

Proof. We use simplified formulas (2.1), (2.2) valid in the domain $\sigma \geqslant 1+\frac{1}{4}$, $-\infty < t < +\infty$ and get

$$|R_N'(1+\tfrac{1}{4}+it)| \leqslant |\zeta'(1+\tfrac{1}{4}+it)| + \frac{c_{27}}{\log N},$$

$$|R_N(1+\tfrac{1}{4}+it)| \geqslant |\zeta(1+\tfrac{1}{4}+it)| - \frac{c_{28}}{\log N} \geqslant c_{29}|\zeta(1+\tfrac{1}{4}+it)|.$$

Hence

$$\frac{1}{2X}\int_{-X}^{X}\left|\frac{R_N'}{R_N}(1+\tfrac{1}{4}+it)\right|^2 dt = O\left(\frac{1}{2X}\int_{-X}^{X}\left|\frac{\zeta'}{\zeta}(1+\tfrac{1}{4}+it)\right|^2 dt + 1\right) = O(1)$$

(in fact, as we know, $\lim_{X\to\infty}\frac{1}{2X}\int_{-X}^{X}\left|\frac{\zeta'}{\zeta}(1+\frac{1}{4}+it)\right|^2 dt = \sum_n \frac{\Lambda^2(n)}{n^{2+1/2}}$).

LEMMA 6. *The function $\frac{R_N'}{R_N}(s)$ may be developed in a Dirichlet series $\sum_{n=1}^{\infty}\frac{-d_n}{n^s}$ in the half-plane $\sigma > 1+2\frac{\log\log N}{\log N}$ $(N > c_{10})$.*

Further, if $n = p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$, $n \leqslant N$, *then*

$$d_n = \Lambda(n) - \frac{k!}{k}\cdot\frac{\log n}{\log^k N}\cdot\log p_1\log p_2\ldots\log p_k \tag{2.10}$$

and for $n \leqslant x \leqslant c_{30}\log N$

$$d_n = \Lambda(n) + O\left(\frac{\log^2 x}{x}\right). \tag{2.11}$$

Proof. As to the convergence of $\sum_{n=1}^{\infty} -d_n/n^s$, see [2], proof of Lemma 4. In order to establish (2.10) we notice that $R_N(s)$ is the N-th partial sum of the Dirichlet series for $\zeta(s)+\zeta'(s)/\log N$, so that $\sum_{n\leqslant N} -d_n/n^s$ is the N-th partial sum of the Dirichlet series for

$$\frac{\zeta'(s)+\dfrac{1}{\log N}\cdot\zeta''(s)}{\zeta(s)+\dfrac{1}{\log N}\cdot\zeta'(s)},$$

which is equal to

$$\frac{\zeta'}{\zeta}(s)+\left(\frac{\zeta'(s)+\dfrac{1}{\log N}\zeta''(s)}{\zeta(s)+\dfrac{1}{\log N}\zeta'(s)}-\frac{\zeta'}{\zeta}(s)\right)$$

$$=\frac{\zeta'}{\zeta}(s)+\frac{1}{\log N}\cdot\frac{\zeta(s)\zeta''(s)-(\zeta'(s))^2}{\zeta(s)}\cdot\frac{1}{\zeta(s)+\dfrac{\zeta'(s)}{\log N}}$$

$$=\frac{\zeta'}{\zeta}(s)+\frac{1}{\log N}\left(\frac{\zeta'}{\zeta}(s)\right)'\cdot\frac{1}{1+\dfrac{1}{\log N}\cdot\dfrac{\zeta'}{\zeta}(s)}$$

$$=\frac{\zeta'}{\zeta}(s)+\left\{\frac{1}{\log N}\cdot\frac{\zeta'}{\zeta}(s)-\frac{1}{2}\left(\frac{1}{\log N}\cdot\frac{\zeta'}{\zeta}(s)\right)^2+\frac{1}{3}\left(\frac{1}{\log N}\cdot\frac{\zeta'}{\zeta}(s)\right)^3-\ldots\right\}'.$$

This gives (2.10). The asymptotic formula (2.11) is obvious for $k=1$ and for $k\geqslant 2$ it follows by the estimation

$$\frac{k!}{k}\cdot\frac{\log n}{\log^k N}\log p_1\log p_2\ldots\log p_k = O\left(\log x\left(\frac{k\log x}{\log N}\right)^k\right)$$

$$= O\left(\log x\left(\frac{c_{31}\log^2 x}{x}\right)^2\right) = O\left(\frac{\log^5 x}{x^2}\right).$$

LEMMA 7. a. *There exists a sequence of numbers* $T_0, T_1, T_2, \ldots$ *such that*

1. $n \leqslant T_n \leqslant n+1$,
2. $\left|\dfrac{R_N'}{R_N}(s)\right| \leqslant c_{32}\log^2 N$ *for* $-\frac{1}{2}\leqslant\sigma\leqslant 2$, $t = T_n$.

b. *For every* $m = 1, 2, \ldots$ *there exists a sequence* $T_0^{(m)}, T_1^{(m)}, \ldots$ *such that*

1. $n \leqslant T_n^{(m)} \leqslant n+1$,
2. $\left|\dfrac{R_N'}{R_N}(s)\right| \leqslant c_{33}m^2\log^2 N$ *for* $-m-\frac{1}{2}\leqslant\sigma\leqslant -m+1$, $t = T_n^{(m)}$.

c. *For every* $m = 1, 2, \ldots$ *there exists a sequence* $S_0^{(m)}, S_1^{(m)}, \ldots$ *such that*

1. $-m+\frac{1}{2}\leqslant S_n^{(m)}\leqslant -m+1$,
2. $\left|\dfrac{R_N'}{R_N}(s)\right| \leqslant c_{34}m^2\log^2 N$ *for* $\sigma = S_n^{(m)}$, $n\leqslant t\leqslant n+1$.

d. *For every* $m = 1, 2, \ldots$ *there exists a sequence* $\tilde{S}_0^{(m)}, \tilde{S}_1^{(m)}, \ldots$ *such that*

1. $-m\leqslant \tilde{S}_n^{(m)}\leqslant -m+1$,
2. $\left|\dfrac{R_N'}{R_N}(s)\right| \leqslant c_{35}m^2\log^2 N$ *for* $\sigma = \tilde{S}_n^{(m)}$, $n\leqslant t\leqslant n+\frac{3}{2}$.

e. *For every* $m = 1, 2, \ldots$ *there exists a sequence* $\tilde{T}_0^{(m)}, \tilde{T}_1^{(m)}, \ldots$ *such that*

1. $n\leqslant \tilde{T}_n^{(m)}\leqslant n+\frac{1}{2}$,
2. $\left|\dfrac{R_N'}{R_N}(s)\right| \leqslant c_{36}m^2\log^2 N$ *for* $-m\leqslant\sigma\leqslant -m+1$, $t = \tilde{T}_n^{(m)}$.

We omit the proof, which is analogous to that of Lemma 3 in [2].

3. THEOREM I. *Let* $N > c_{10}$ *be an integer. Write*

$$\Phi(x) = \sum_{n\leqslant x} d_n,\qquad \Phi_0(x) = \frac{\Phi(x-0)+\Phi(x+0)}{2}.$$

Let x *range in the interval* $[2, c_{30}\log N]$. *Then*

$$\Phi_0(x) = \frac{\sum_{n\leqslant N}\log n\log(N/n)}{\sum_{n\leqslant N}\log(N/n)} - \sum_{\varrho}\frac{x^\varrho}{\varrho}, \tag{3.1}$$

where $\varrho = \beta + i\gamma$ *run through the zeros of* $R_N(s) = \sum_{n\leqslant N}(1-\log n/\log N)n^{-s}$ *and* $\sum_{\varrho} x^{\varrho}/\varrho = H_N(x)$ *denotes the limit of* $H_N(x,T) = \sum_{|\gamma|\leqslant T} x^{\varrho}/\varrho$ *as* $T\to\infty$.

Writing further

$$P_N(x,T) = H_N(x) - H_N(x,T),$$

we have

$$|P_N(x,T)| \leqslant \begin{cases} c_{37}\dfrac{x^2}{T}\left(\dfrac{\log^{14}N}{(\log\log N)^6}+\dfrac{\log^6 N}{\|x\|}\right) & \text{if} \quad \|x\|\neq 0,\\ c_{37}\dfrac{x^2}{T}\cdot\dfrac{\log^{14}N}{(\log\log N)^6} & \text{if} \quad \|x\| = 0,\\ c_{37}\left(\dfrac{x^2}{T}\cdot\dfrac{\log^{14}N}{(\log\log N)^6}+\log x\right) & \text{always,}\end{cases}$$

where $\|x\|$ *is the distance of* x *from the nearest integer.*

Proof. We follow the same procedure as in [2], considering the integral

$$\frac{1}{2\pi i}\int_{C_q^T}\frac{x^s}{s}\left(-\frac{R_N'}{R_N}(s)\right)ds,$$

where C_q^T is as before the contour consisting of the segment

$$\left[2+6\frac{\log\log N}{\log N}-iT', 2+6\frac{\log\log N}{\log N}+iT'\right]$$

and of three polygonal lines given by Lemma 7. Cauchy's theorem then gives

$$(3.2)\quad \frac{1}{2\pi i}\int_{C_q^T}\frac{x^s}{s}\left(-\frac{R_N'}{R_N}(s)\right)ds = \frac{\sum_{n\leqslant N}\log(N/n)\log n}{\sum_{n\leqslant N}\log(N/n)} - \sum_{\varrho \text{ inside } C_q^T}\frac{x^{\varrho}}{\varrho}$$
$$= \frac{\sum_{n\leqslant N}\log(N/n)\log n}{\sum_{n\leqslant N}\log(N/n)} - H_N(x,T) + O\left(\frac{x}{T}\log^3 N\right);$$

on the other hand, we obtain by a direct evaluation

$$(3.3)\quad \frac{1}{2\pi i}\int_{C_q^T}\frac{x^s}{s}\left(-\frac{R_N'}{R_N}(s)\right)ds = \Phi_0(x) + O\left(\frac{q^2\log^2 N}{x^{q-1}}\log T\right) + O\left(\frac{x^2\log^2 N}{T}\right) + O\left(\sum_{n=1}^{\infty}|u_n|\right),$$

where for $n\neq x$

$$(3.4)\quad |u_n| \leqslant |d_n|\frac{x^2\log^6 N}{\pi T n^{2+6\log\log N/\log N}}\cdot\frac{n+x}{|n-x|}$$

and for $n = \nu = \nu(x)$, defined by $\nu-\frac{1}{2} < x \leqslant \nu+\frac{1}{2}$, we have

$$(3.5)\quad |u_\nu| \leqslant \begin{cases} c_{38}\dfrac{x^2}{T\|x\|}\log^6 N & \text{if} \quad \|x\|\neq 0,\\ c_{38}\dfrac{\log x}{T} & \text{if} \quad \|x\| = 0,\\ c_{38}\log x & \text{always.}\end{cases}$$

(3.4) gives (cf. [2], (3.10))

$$\sum_{n\neq\nu}|u_n| \leqslant \frac{x^2\log^6 N}{\pi T}\left(5\sum_{n=1}^{\infty}\frac{|d_n|}{n^{2+6\log\log N/\log N}} + 2\sum_{r=1}^{[x]}\frac{\max_{1\leqslant n\leqslant 3x/2}|d_n|\cdot\frac{5}{2}x}{(\frac{1}{2}x)^2\cdot\frac{1}{2}r}\right)$$

whence, noting the inequalities

$$\left|\frac{R_N'}{R_N}(s)\right| \leqslant c_{39}\frac{\log^4 N}{(\log\log N)^3} \quad \text{for} \quad \sigma\geqslant 1+2\frac{\log\log N}{\log N} \quad \text{(see (2.8))}$$

and

$$\max_{1\leqslant n\leqslant 3x/2}|d_n| \leqslant c_{40}\log x \quad \text{(see (2.11))},$$

we easily get (cf. [2], 3)

$$(3.6)\quad \sum_{n\neq\nu}|u_n| \leqslant c_{41}\frac{x^2\log^{14}N}{T(\log\log N)^6}.$$

Putting (3.2), (3.3) and (3.6) together and letting q tend to infinity, we obtain

$$(3.7)\quad \Phi_0(x) = \frac{\sum_{n\leqslant N}\log n\log(N/n)}{\sum_{n\leqslant N}\log(N/n)} - H_N(x,T) + O(|u_\nu|) + O\left(\frac{x^2}{T}\cdot\frac{\log^{14}N}{(\log\log N)^6}\right).$$

Then (3.5) and (3.7) give the result announced in Theorem 1. Formula (3.1) clearly follows on letting in turn T tend to infinity.

THEOREM II. *Let* $x\geqslant 2$. *Then*

$$\psi_0(x) = x - \sum_{|\gamma|\leqslant x^3, \beta\geqslant -1}\frac{x^{\varrho}}{\varrho} + O(\log^2 x),$$

where $\varrho = \beta+i\gamma$ *denote the zeros of* $R_N(s) = \sum_{n\leqslant N}(1-\log n/\log N)n^{-s}$ *and* $N = [e^x]$.

Proof. We shall slightly modifyt he previous proof. Namely, we start from the integral $\frac{1}{2\pi i}\int \frac{x^s}{s}\left(-\frac{R_N'}{R_N}(s)\right)ds$, taken over the contour $\bar{C}_q^T$ differing from that of the latter proof only in its right-hand side segment, which is now $[2+\frac{1}{2}-iT', 2+\frac{1}{2}+iT']$. Then, if $T\leqslant N$, we have in view of Lemma 2

$$\frac{1}{2\pi i}\int\limits_{\bar{C}_q^T}\frac{x^s}{s}\left(-\frac{R_N'}{R_N}(s)\right)ds = \frac{\sum\limits_{n\leqslant N}\log n\log(N/n)}{\sum\limits_{n\leqslant N}\log(N/n)} - H_N(x,T) + O\left(\frac{x\log N}{T}\right),$$

and also

$$\frac{1}{2\pi i}\int\limits_{\bar{C}_q^T}\frac{x^s}{s}\left(-\frac{R_N'}{R_N}(s)\right)ds$$

$$= \Phi_0(x) + O\left(\frac{q^2\log^2 N}{x^{q-1}}\log T\right) + O\left(\left|\int\limits_{iT'}^{2+1/2+iT'}\frac{R_N'}{R_N}(\sigma+it)\frac{x^s}{s}ds\right|\right) + O\left(\sum_{n=1}^{\infty}|u_n|\right)$$

where

$$\sum_{n=1}^{\infty}|u_n| \leqslant \frac{x^{2+1/2}}{\pi T}\left(5\sum_{n=1}^{\infty}\frac{|d_n|}{n^{2+1/2}} + 2\sum_{r=1}^{[x]}\frac{\max\limits_{1\leqslant n\leqslant 3x/2}|d_n|\cdot\frac{5}{2}x}{(\frac{1}{2}x)^2\cdot\frac{1}{2}r}\right) + c_{42}\log x.$$

Now (see [5], p. 307 and Lemma 5)

$$\sum_{n=1}^{\infty}\frac{|d_n|}{n^{2+1/2}} \leqslant \sum_{n=1}^{\infty}\frac{|d_n|^2}{n^{2+1/2}} + O(1)$$

$$= \lim_{X\to\infty}\frac{1}{2X}\int\limits_{-X}^{X}\left|\frac{R_N'}{R_N}(1+\tfrac{1}{4}+it)\right|^2 dt + O(1) = O(1).$$

Further, by Lemmas 4 and 7 we obtain

$$\left|\int\limits_{iT'}^{2+1/2+iT'}\frac{R_N'}{R_N}(\sigma+it)\frac{x^s}{s}ds\right| \leqslant \left|\int\limits_{iT'}^{1+iT'}\right| + \left|\int\limits_{1+iT'}^{2+1/2-iT'}\right|$$

$$\leqslant c_{43}\left(\frac{x}{T}\log^2 N + \frac{x^{2+1/2}}{T}(\log\log N)^3\right), \quad \text{if only} \quad T\leqslant \log^4 N.$$

Hence

$$\frac{\sum\limits_{n\leqslant N}\log n\log(N/n)}{\sum\limits_{n\leqslant N}\log(N/n)} - H_N(x,T)$$

$$= \Phi_0(x) + O\left(\frac{x^{2+1/2}(\log\log N)^3}{T} + \frac{x\log^2 N}{T} + \frac{x^{2+1/2}}{T}\log^2 x + \log x\right).$$

But as

$$\frac{\sum\limits_{n\leqslant N}\log n\log(N/n)}{\sum\limits_{n\leqslant N}\log(N/n)} = \log N + O(1)$$

and by (2.11)

$$\Phi_0(x) = \psi_0(x) + O(\log^2 x),$$

we obtain

$$\psi_0(x) = x - \sum_{|\gamma|\leqslant T}\frac{x^\varrho}{\varrho} + O(\log^2 x) + O\left(\frac{x^3}{T} + \frac{x^{2+1/2}\log^3 x}{T}\right).$$

It remains to put $T = x^3$ and repeat the closing argument from the proof of the Corollary in [2].

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