On some problems of the arithmetical theory of continued fractions

by

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§ 1. For a given quadratic surd \( \xi \) let us denote by

\[
[b_0, b_1, \ldots, b_{n-1}, \overline{b_0, b_1, \ldots, b_{n-1}}]
\]

its expansion into an arithmetical continued fraction, by \( \text{lp} \xi \)—the length of the shortest period of this expansion, by \( \text{lap} \xi \)—the number of terms before the period. For some polynomials \( f(n) \) assuming only integral values (so-called integer-valued polynomials) there are known formulæ for the expansion of \( \sqrt{f(n)} \) into continued fractions such that the partial quotients are also integer-valued polynomials and \( \text{lp} \sqrt{f(n)} \) is independent of \( n \) (cf. [3], [6]). Recently H. Schmidt has proved ([3], Satz 10) that

If \( h \) is an integer \( \neq 0, \pm 1, \pm 2, \pm 4 \), then for each \( n \geq n_0 \) the set of all integers \( n \geq n_0 \) cannot be decomposed into a finite number of classes, so that the relation

\[
\sqrt{h^2 + \xi} = \left[ p_0(n), \ldots, p_k(n), \ldots \right], \quad n \geq n_0, \quad n \in K,
\]

holds for each class \( K \) (\( p_n \) are polynomials assuming integral values for \( n \in K \), \( k \) depends only upon \( K \)).

This theorem suggests the following problem \( P \).

\( P. \) Decide for a given integer-valued polynomial \( f(n) \) whether

\[
\lim_{n \to \infty} \text{lp} \sqrt{f(n)} < \infty.
\]

An investigation of this problem is the main aim of the present paper.

In \( \S 2 \) we investigate the relation between \( \text{lp} \xi \) and \( \text{lp}(p\xi + r)/(q\xi + s) \), where \( p, q, r, s \) are integers.

In \( \S 3 \) we give a negative solution of the problem \( P \) for polynomials of odd degree and for a large class of polynomials of even degree.
In § 4 after more accurate study of the behaviour of the function \(lp\sqrt{s^{n}+h}\) and on the base of results of § 2 we give a complete solution of the problem \(P\) for polynomials of the second degree.

We shall use the following notation; \(\xi, \zeta, \xi'\) will denote either rational numbers or quadratic surds; in the latter case \(\eta, \eta', \eta''\) will be corresponding conjugate numbers. Putting

\[
\xi = [b_{0}, b_{1}, b_{2}, \ldots]
\]

we shall assume simultaneously

\[
\begin{align*}
A_{-1} &= 1, & A_{0} &= b_{0}, & A_{n} &= b_{n}A_{n-1} + A_{n-2}, \\
B_{-1} &= 0, & B_{0} &= 1, & B_{n} &= b_{n}B_{n-1} + B_{n-2},
\end{align*}
\]

(1)

whence \([b_{0}, b_{1}, \ldots, b_{n}] = A_{n}/B_{n}\) and

\[
\zeta = [b_{0}, b_{1}, b_{2}, \ldots] \quad \text{(cf. [2], p. 24 and 34).}
\]

For rational \(\xi\) we put \(lp\xi = 0\) and

\[
lp\xi = \begin{cases} 1 & \text{if } \xi \text{ is an integer,} \\ 0 & \text{if } \xi = [b_{0}, b_{1}, \ldots, b_{n-1}], \ b_{n-1} > 1
\end{cases}
\]

(the so-called normal expansion).

\[\text{§ 2. Lemma 1. Let } b > 1, \ h \text{ and } a \text{ be positive integers. If}
\]

\[(2')\]

\[
\xi = [b_{0}, b_{1}, \ldots, b_{k}]
\]

or

\[(2'')\]

\[
\xi = [b_{0}, b_{1}, \ldots, b_{n-1}, \overline{b_{n}, b_{n+1}, \ldots, b_{n+k-1}}],
\]

where

\[
b_{i} < b \quad (1 \leq i \leq h-1),
\]

then

\[
i \leq B_{i} \leq B_{i}' \quad (0 \leq i \leq h-1).
\]

Moreover, if for some integers \(p\) and \(r\)

\[
(5)
\]

\[
\xi' = (p\xi + r)/a,
\]

then

\[
(6)
\]

\[
lp\xi' < 2ab^{h}.
\]

On the arithmetical theory of continued fractions

Proof. Formula (4) follows by easy induction from (1) and (2). Hence for rational \(\xi\) we immediately get the remaining part of the lemma.

In fact, putting

\[
\xi' = [b_{0}', b_{1}', \ldots, b_{n-1}'] = A_{n-1}'/B_{n-1}',
\]

we have in view of (5)

\[
\frac{A_{n-1}'}{B_{n-1}'} = \frac{pA_{n-1} + rB_{n-1}}{sB_{n-1}};
\]

then \(lp\xi' = k' \leq B_{n-1}' \leq sB_{n-1} = ab^{h-1} < 2ab^{h}.

In the case \(p = 0\) we have likewise

\[
\frac{A_{n-1}}{B_{n-1}} = \frac{r}{s}
\]

whence \(lp\xi' = k' \leq B_{n-1}' \leq s < 2ab^{h}.

One can therefore assume that \(\xi\) is irrational and \(p \neq 0\). It follows from (2'') that

\[
\zeta = [b_{0}, b_{1}, \ldots, b_{k-1}, \xi_{k}],
\]

\(\xi_{k}\), which has a pure period in its expansion, is by a well-known theorem, a reduced surd, i.e.

\[
(7) \quad \xi_{k} > 1, \quad 0 > \eta_{k} > -1.
\]

On the basis of well-known formulae (cf. [2], § 13, (7)) we have:

\[
\xi = \frac{A_{k-1} + (-1)^{k-1}B_{k-1}}{B_{k-1}} = \frac{A_{k-1} + (-1)^{k-1}B_{k-1}}{B_{k-1}},
\]

\[
\eta = \frac{A_{k-1} + (-1)^{k-1}B_{k-1}}{B_{k-1}} = \frac{A_{k-1} + (-1)^{k-1}B_{k-1}}{B_{k-1}},
\]

whence

\[
|\xi - \eta| = \frac{|1 - \eta_{k}/\xi_{k}|}{|B_{k-1} + B_{k-2}/\xi_{k}|} \cdot |B_{k-1}/\eta_{k} + B_{k-2}/\eta_{k}|.
\]

Since, in view of (7),

\[
0 < -\eta_{k}/\xi_{k}, \quad 0 < B_{k-1} + B_{k-2}/\xi_{k} < B_{k-1} + B_{k-2} \leq B_{k},
\]

\[-B_{k-1} < B_{k-1}/\eta_{k} + B_{k-2} < B_{k-1} < B_{k-1},
\]

we get by (4)

\[
|\xi - \eta| > 1/B_{k-1}/B_{k} > 1/b^{h-1}
\]

and by (5)

\[
|\xi' - \eta'| = \frac{|p|}{s} |\xi - \eta| > \frac{1}{ab^{h-1}}.
\]
If \( \xi' > \eta' \), we assume \( k' = 2mb^b - 2 \). Therefore, in view of (4),

\[
B_{k'-1}^b B_{k'-2}^b > k' (k' - 1) = (2mb^b - 2)(2mb^b - 3) > \frac{1}{|\xi' - \eta'|}.
\]

We shall prove that \( \xi'' \) is a reduced surd. It follows from the formula

\[
\eta'' = \frac{A_{k'-1}^b}{B_{k'-1}^b} + \frac{(-1)^{k'-1}}{A_{k'-1}^b B_{k'-2}^b + B_{k'-3}^b}
\]

that

\[
B_{k'-1}^b (B_{k'-2}^b \eta'' + B_{k'-3}^b) = \frac{(-1)^{k'}}{A_{k'-1}^b B_{k'-2}^b - \eta''}.\]

Since \( k' = 2mb^b - 2 \) is even, we have

\[A_{k'-1}^b (B_{k'-2}^b - \eta'') > \xi' - \eta'' > 0.\]

The last two formulae together give

\[\frac{1}{\xi' - \eta''} > (B_{k'-1}^b \eta'' + B_{k'-3}^b) B_{k'-2}^b > 0.\]

We then get, on the one hand,

\[0 < B_{k'-1}^b \eta'' + B_{k'-3}^b, \quad \text{whence} \quad \eta'' > -B_{k'-1}^b / B_{k'-2}^b > -1;\]

on the other hand, in view of (8),

\[B_{k'-1}^b B_{k'-2}^b > B_{k'-1}^b (B_{k'-2}^b \eta'' + B_{k'-3}^b), \quad \text{whence} \quad \eta'' < 0.\]

Therefore \( \eta'' > 1 \) and since \( \xi'' > 1 \), the surd \( \xi'' \) is reduced

(\( k' = 2mb^b - 2 \)).

In the case \( \eta' > \xi' \) we prove similarly that the surd \( \xi'' \) is reduced

for \( k' = 2mb^b - 1 \). Since a reduced surd gives in its expansion a pure period, we have in both cases

\[\text{lap} \xi' = k < 2mb^b, \quad \text{q.e.d.}\]

Remark. Inequalities (4) and (6) can be greatly improved; however, it is without any importance for the applications intended.

In the following we shall profit by a theorem used in the investigation of Hurwitz's continued fractions and due to A. Hurwitz and A. Châtelet. We quote this theorem according to Perron's monograph ([2], Satz 4.1) with slight changes in his notation to avoid confusion with ours.

\[
\xi' = \frac{p_k x' + r_k}{s_k} (p_k, r_k, s_k - \text{integers}, p_k > 0, s_k > 0, p_k s_k = d > 1).
\]

For any index \( v \geq 1 \) the number

\[
\frac{p_v A_{v-1}^b + r_v B_{v-1}^b}{s_v B_{v-1}^b}
\]

can be developed in an arithmetical continued fraction \([d_v, d_{v-1}, \ldots, d_0]\); and besides the number of its terms can be chosen so that \( \mu = v \pmod{2} \); let \( C_v, D_v \) be the numerators and denominators of its convergents, so that in particular

\[
\frac{p_v A_{v-1}^b + r_v B_{v-1}^b}{s_v B_{v-1}^b} = \frac{C_{v-1}}{D_{v-1}}.
\]

Then there exist three uniquely determined integers \( p_1, r_1, s_1 \) such that the formula

\[
\left[ \frac{p_1}{s_1} \frac{r_1}{s_1} \right] = \frac{C_{v-1}}{D_{v-1}} \frac{C_{v-2}}{D_{v-2}} \left[ \frac{p_1}{s_1} \frac{r_1}{s_1} \right]
\]

holds and besides

\[p_1 > 0, \quad s_1 > 0, \quad p_1 s_1 = d, \quad -s_1 \leq r_1 \leq p_1;\]

\[\xi' = [d_v, d_{v-1}, \ldots, d_0, \xi'], \quad \text{where} \quad \xi' = \frac{p_1 x' + r_1}{s_1}.
\]

The theorem quoted obviously preserves its validity for \( d = 1 \) as well as for rational \( \xi' \) in the latter case under the condition \( v \leq \text{lap} \xi'. \)

On the basis of Lemma 1 and theorem H we shall show

THEOREM 1. For arbitrary positive integers \( m \) and \( d \) there exists a number \( M = M(m, d) \) such that if \( \text{lap} \xi' \leq M \) and

\[
\xi' = \frac{p_k x' + r_k}{s_k} (p_k, r_k, s_k - \text{integers}, p_k > 0, p_k s_k = d)
\]

then \( \text{lap} \xi' \leq M_0 \).

Proof. We shall prove it by induction with respect to \( m \). For \( m = 1 \) the theorem follows immediately from Lemma 1, whence after the substitution \( b = 2, k = 1 \) (assumption (3) being satisfied in emptiness),

\[x = p, r = r_0, s = s_0 \]

we get

\[\xi' < 4s_0.\]
Assume now that the theorem is valid for \( m = h - 1 \) \( (h > 1) \); we shall show that it is valid for \( m = h \).

By hypothesis there exists a number \( M(h-1, d) \) such that if \( \text{lap} \xi \leq h - 1 \) and \( \xi = (p \xi + r)/s \) \((p, r, s — \text{integers}, p > 0, s > 0, ps = d)\), then

\[
\text{lap} \xi' \leq M(h - 1, d).
\]

Let \( M = 2M(h-1, d) + 2^{h+1}d^{h+1} \). The proof will be complete if we show that for any \( \xi \) such that \( \text{lap} \xi \leq h \) the number \( \xi' \) defined by (9) satisfies the inequality

\[
\text{lap} \xi' \leq M.
\]

Since \( M(h-1, d) < M \), we can assume that \( \text{lap} \xi = h \) and that \( \xi \) is given by one of the formulae (2).

If for each \( i \leq h < b \leq 2d \), then putting in Lemma 1 \( b = 2d, p = p_i, r = r_i, s = s_i \), we get

\[
\text{lap} \xi' \leq 2s_i(2d)^i \leq 2^{h+1}d^{h+1} \leq M.
\]

It remains to consider the case where for some \( v < h \): \( b_v \geq 2d \).

We then have

\[
(10) \quad \xi = [b_0, b_1, \ldots, b_{h-1}, b_h], \quad \xi > b_v \geq 2d.
\]

In virtue of theorem H there exist integers \( p_i, r_i, s_i \) such that

\[
(11) \quad p_i > 0, \quad s_i > 0, \quad p_i r_i = d_i - s_i \leq r_i \leq p_i,
\]

\[
(12) \quad p_i [b_0, b_1, \ldots, b_{h-1}] + r_i = [d_0, d_1, \ldots, d_{h-1}],
\]

\[
(13) \quad \xi' = \frac{p_i \xi + r_i}{s_i} = [d_0, d_1, \ldots, d_{h-1}, \xi'_1], \quad \xi'_v = \frac{p_v \xi + r_v}{s_v}.
\]

From (10) and (11) we get

\[
\xi'_v > \xi_v / (d_v - 1) \geq \xi / (d - 1) \geq 1,
\]

which together with formula (13) proves that numbers \( d_0, d_1, \ldots, d_{h-1} \) are the initial partial quotients of the number \( \xi' \). Hence

\[
\text{lap} \xi' \leq \mu + \text{lap} \xi'_v.
\]

Meanwhile, by (12)

\[
\mu \leq 1 + \text{lap} p_i [b_0, b_1, \ldots, b_{h-1}] + r_i
\]

and since \( \text{lap} [b_0, b_1, \ldots, b_{h-1}] \leq v < h \), we have in virtue of the inductive assumption

\[
(15) \quad \mu \leq 1 + M(h-1, d).
\]

On the other hand, since \( \text{lap} \xi = \text{lap} \xi' - v < h \), we have

\[
(16) \quad \text{lap} \xi'_v \leq M(h-1, d)
\]

and finally by (14), (15), (16) we get

\[
\text{lap} \xi' \leq 1 + 2M(h-1, d) \leq M, \quad \text{q. e. d.}
\]

**Corollary.** For any positive integer \( m \) and arbitrary integers \( d \) and \( q \) there exists a number \( M = M(m, d, q) \) such that

\[
\text{lap} \xi \leq m, \quad \xi' = \frac{p \xi + r}{q \xi + s} \quad (p, r, s — \text{integers}, q \xi + s \neq 0)
\]

and \( ps - qr = d \), then \( \text{lap} \xi' \leq M \).

**Proof.** The case \( d = 0 \) is trivial; thus let \( d \neq 0 \). It is easy to verify the equality (cf. [3], p. 56):

\[
- [b_0, b_1, b_2, b_3, \ldots] = \left\{ \begin{array}{ll}
[b_0 + 1, b_1 - 1, b_2, b_3, \ldots] & \text{for } b_1 > 1, \\
[b_0 + 1, b_1 + 1, b_2, b_3, \ldots] & \text{for } b_1 = 1,
\end{array} \right.
\]

whence

\[
(17) \quad \text{lap}(-\xi) = 3 + \text{lap} \xi.
\]

If \( q = 0 \), then \( s \neq 0 \) and we have

\[
\xi' = \frac{\text{sgn} p \cdot |p| \xi + r \cdot \text{sgn} p}{\text{sgn} s \cdot |s|};
\]

the corollary follows therefore directly from Theorem 1 and formula (17).

If \( q \neq 0 \), then

\[
\xi' = \text{sgn} q \cdot \frac{r^{-1} + p}{|q|}, \quad \xi = \frac{\text{sgn} q \cdot |q| \xi + s + q}{|q| d}
\]

and we obtain the corollary applying Theorem 1 successively to the numbers \( \xi \) and \( \xi' \) using formula (17) and the obvious inequality

\[
\text{lap} \xi' \leq 1 + \text{lap} \xi.
\]

**Lemma.** Let \( b, h, p \) and \( s \) be positive integers. If \( \xi \) is given by (2') and \( \xi' \) by (8) and if

\[
(18) \quad b_h < b \quad (b \leq i < h + h - 1),
\]

then \( \text{lap} \xi' \leq 8(ps)^2 b^h \).
Proof. It follows from (\(\mathcal{E}'\)) that
\[
\xi_k = \{b_{k1}, b_{k2}, \ldots, b_{kL-1}, \xi_{kL}\};
\]
the number \(\xi_k\) satisfies therefore the equation
\[
B_{k-1,1}x^2 + (B_{k-2,1} - A_{k-1,1})x - A_{k-2,1} = 0,
\]
where numbers \(A_{jk}\) and \(B_{jk}\) are respectively the numerator and the denominator of the \(k\)th convergent of \([b_{k1}, b_{k2}, \ldots]\).

Denoting by \(\Delta\) the discriminant of the equation (19) we have
\[
\Delta = (B_{k-1,1} - A_{k-1,1})^2 + 4B_{k-2,1}A_{k-2,1} = (d_{k-1,1} + B_{k-1,1})^2 + 4(-1)^{k-1},
\]
and since from (18) easily follows
\[
A_{k-1,1} < b^k, \quad B_{k-2,1} < b^{k-1},
\]
we get
\[
\Delta \leq 4b^{2k}.
\]

It follows from the formulae
\[
\mathcal{E}' = \frac{p}{q} + r, \quad \xi_k = \frac{A_{k-1,1} \xi_{k-1} + A_{k-2,1}}{B_{k-1,1} \xi_{k-1} + B_{k-2,1}}
\]
and from equation (19) for \(\xi_k\) that the number \(\xi_k\) satisfies the equation
\[
Ax^2 + Bx + C = 0,
\]
where integers \(A, B, C\) are defined by the formula
\[
\begin{pmatrix}
2A & B \\
B & 2C
\end{pmatrix}
= \begin{pmatrix}
x & 0 \\
-r & p
\end{pmatrix}
\begin{pmatrix}
B_{k-2,1} - A_{k-2,1} \\
B_{k-1,1} - A_{k-1,1}
\end{pmatrix}
\begin{pmatrix}
2B_{k-1,1}^2 - A_{k-1,1} \\
B_{k-2,1} - A_{k-2,1}
\end{pmatrix}
\times
\begin{pmatrix}
B_{k-1,1} - A_{k-1,1} \\
B_{k-2,1} - A_{k-2,1}
\end{pmatrix}
\begin{pmatrix}
B_{k-2,1} - A_{k-2,1} \\
B_{k-1,1} - A_{k-1,1}
\end{pmatrix}
\begin{pmatrix}
x - r \\
0
\end{pmatrix}.
\]

On the other hand, as can easily be seen from Lagrange's proof of his well-known theorem about periodical expansions of quadratic surds (cf. [2], pp. 66-68), if \(\mathcal{E}'\) is a root of equation (21), then
\[
lp \mathcal{E}' \leq 2\mathcal{E}';
\]
where \(\Delta'\) is the discriminant of that very equation. But, as follows from (22),
\[
\Delta' = \begin{vmatrix}
2A & B \\
B & 2C
\end{vmatrix}
= (ps)^2(A_{k-1,1}B_{k-2,1} - B_{k-1,1}A_{k-2,1}) = (ps)^2 d.
\]

The last two formulæ together with (30) finally give
\[
lp \mathcal{E}' \leq 8(p^2s)^{1/2}, \quad q.e.d.
\]

**Theorem 2.** For arbitrary integers \(n > 0\) and \(d\), there exists a number \(N = N(n, d)\) such that if
\[
(23) \quad lp \xi < n, \quad \xi = \frac{p\xi + r}{q\xi + s} \quad (p, q, r, s \text{— integers, } q\xi + s \neq 0)
\]
and \(ps - qr = d\), then
\[
lp \xi < N.
\]

**Proof.** The case of \(\xi\)—rational or \(d = 0\) is trivial; let \(\xi\) be a quadratic surd, \(d \neq 0\). On the basis of Theorem 1 there exists a number \(M(n, d)\) such that, if \(p, r, s\)—integers, \(p > 0, s > 0, \, ps = |d|\) and \(lp\xi < n\), then
\[
\lambda p\xi = (p^2s)^{1/2} < M(n, d).
\]

Let \(N = M(n, d)/(|d| + 1)^3 + 2^{2n+2}d^{2n+2}\). We shall show that if conditions (23) hold, then
\[
lp \xi < N.
\]

Put
\[
\beta = q(p, q), \quad \delta = -p(p, q).
\]
Since \((p, \beta) = 1\), there exist integers \(a, \gamma\) such that
\[
(24) \quad a\beta - \beta\gamma = sgn d.
\]

Putting
\[
(25) \quad \xi' = \frac{p\xi + r}{q\xi + s},
\]
we get
\[
\xi' = \frac{p_0\xi + r_0}{s_0},
\]
where the integers \(p_0, r_0, s_0\) are defined by the formula
\[
\begin{pmatrix}
p_0 & r_0 \\
0 & s_0
\end{pmatrix}
= \begin{pmatrix}
\alpha & \gamma \\
\beta & \delta
\end{pmatrix}
\begin{pmatrix}
p & r \\
q & s
\end{pmatrix}.
\]

Thus
\[
(26) \quad p_0s_0 = (a\beta - \beta\gamma)(ps - qr) = d\text{sgn }d = |d|.
\]

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In view of formulae (24) and (25), the surds \( \xi' \) and \( \xi'' \) are equivalent, whence

\[
1p \xi' = 1p \xi'' = 1p \xi' + r_0.
\]

Changing, if necessary, the signs of \( p_0, r_1, s_0 \) we can therefore assume that

\[
\xi' = \frac{p_1 \xi + r_0}{s_0}, \quad p_0 > 0, \quad s_0 > 0, \quad p_0 s_0 = d > 0.
\]

Let \( \xi \) be given by formula (24'), where \( k \leq n \). If for each \( i \) such that \( k \leq i \leq h + k - 1 \) we have \( b_i < 2d \), then, putting in Lemma 2 \( b = 2d \), \( p = p_0, r = r_0 \), we get

\[
1p \xi < 8 (p_0 s_0)^2 (2d)^{2l} \leq 8d^2 (2d)^{2l} = 2^{2m+2m+2} \leq N.
\]

It remains to consider the case, where for some \( v \geq h \) holds \( b_v \geq 2d \). We then have

\[
\xi = [b_0, b_1, \ldots, b_{v-1}, b_v, \ldots, b_{h-1}, b_h, \ldots, b_{h+1}, \ldots, b_{h+v-1}, \ldots, b_{h+v}].
\]

Using theorem \( H \) we get

\[
\xi' = [a_0, a_1, \ldots, a_{s-1}, \xi'_1], \quad \xi'' = \xi'_1 = \frac{p_1 \xi + r_1}{s_1},
\]

where

\[
(26) \quad p_1 > 0, \quad s_1 > 0, \quad p_1 s_1 = d, \quad -s_1 \leq r_1 \leq p_1,
\]

and for all \( i \geq 1 \)

\[
(28) \quad \xi'_i = [a_{i-1}, a_i, \ldots, a_{s_{i-1}-1}, \xi'_{i+1}],
\]

\[
(29) \quad \xi''_i = [a_{i-1}, a_i, \ldots, a_{s_{i-1}-1}, \xi''_{i+1}], \quad \xi''_{i+1} = \frac{p_{i+1} \xi + r_{i+1}}{s_{i+1}},
\]

\[
(30) \quad p_{i+1} > 0, \quad s_{i+1} > 0, \quad p_{i+1} s_{i+1} = d, \quad -s_{i+1} \leq r_{i+1} \leq p_{i+1}.
\]

In view of the inequality \( \xi > b_v \geq 2d \), it follows from (27) and (30) that \( b_i > 1 \) \((i = 1, 2, \ldots)\); the number \( \xi' \) has therefore the following expansion into an arithmetical continued fraction:

\[
\xi' = [a_0, a_1, \ldots, a_{s-1}, \xi'_1, a_2, \ldots, a_{s_2-1}, a_{s_2}, \ldots, a_{s_3-1}, a_{s_3}, \ldots, a_{s_{i-1}}, a_{s_{i-1}}],
\]

It follows from (27) and (30) that the number of all possible different systems \( (p_i, r_i, s_i) \) does not exceed \( d(d+2) \). Thus, among the systems \( (p_i, r_i, s_i) \) \((i = 1, 2, \ldots, d+1)\) there must be at least two identical ones; there exist therefore positive integers \( g < j \leq d+1 \) such that

\[
p_g = p_j, \quad r_g = r_j, \quad s_g = s_j.
\]

On the basis of (26) and (29) it follows hence that

\[
\xi'_g = \xi'_j,
\]

thus

\[
1p \xi' \leq \mu_j - \mu_g = \sum_{k=g+1}^{j-1} (\mu_k - \mu_g).
\]

On the other hand, in virtue of formula (28), the definition of the number \( M(n, [d]) \) and the condition \( k \leq n \),

\[
\mu_{j+1} - \mu_j \leq 1 + \text{lap} p_j [b_j, \ldots, b_{j+1-1}, b_{j+1} + r_j] \leq M(n, [d]),
\]

In view of \( j-g < (d+1)^2 \), we thus get

\[
1p \xi'' \leq (j-g) M(n, [d]) \leq (d+1)^2 M(n, [d]) \leq N, \quad \text{q.e.d.}
\]

Remark. As can easily be seen, we use in the proof given above only a special case of Theorem 1. We proved it in full generality only for a more complete characterisation of the relation between continued fractions and rational homographic transformations.

\section{3. Lemma 3. If \( \xi^{(n)} \to \xi \) (\( \xi^{(n)} \) are quadratic surds, \( \xi \) an arbitrary irrational number) and \( \xi^{(n)} \neq \xi \), then \( \lim (\text{lap} \xi^{(n)} - 1p \xi^{(n)}) = \infty \).}

\textbf{Proof.} If formula (32) does not hold, the sequence \( \xi^{(n)} \) contains a subsequence for which

\[
1p \xi^{(n)} + \text{lap} \xi^{(n)} \leq L < \infty.
\]

Proving Lemma 3 by reduction to absurdity we can therefore assume at once that inequality (33) holds. Let

\[
\xi^{(n)} = [b_0^{(n)}, b_1^{(n)}, \ldots, b_{s_0-1}^{(n)}, b_s^{(n)}, b_{s_1}^{(n)}, \ldots, b_{s_2-1}^{(n)}, b_{s_2}^{(n)}, \ldots, b_{s_{i-1}}^{(n)}, b_{s_{i-1}}^{(n)}],
\]

\[
h_n = \text{lap} \xi^{(n)}, \quad h_n = 1p \xi^{(n)}, \quad \xi = [b_0, b_1, \ldots, b_s].
\]
Since $\zeta$ is irrational, we have

$$\lim b_i = b_i \quad (i = 0, 1, 2, \ldots);$$

thus for every $i$ there exists an $n_i$ such that

$$b_i = b_i \quad (n \geq n_i).$$

By (33) we have $h_i + k_n \leq L$. Putting $K = L!$ we have, for every $n$, $h_i \leq L$, $k_n \leq K$, whence

$$b_i = b_i \quad (i \geq L, \quad n \geq 1, \quad t > 0).$$

Let $M = \max(n_0, n_1, \ldots, n_{L-1})$. We shall show that, contrary to assumption (31), for $n > M$, $\varepsilon_i = \zeta$.

In fact, by (34) we have for $n > M$

$$b_i = b_i \quad (0 < i < L+K).$$

Assume now that $j \geq L+K$. We obviously have $j = tK+i$, where $t$ is an integer $\geq 0$, $L < i < L+K$ and according to (35)

$$b_i = b_i \quad (n \geq 1).$$

Put $m = \max(M, n)$. By (36) we have

$$b_i = b_i \quad (0 < i < L+K, \quad n > M).$$

Applying successively formulae (37), (38), (37) and (35) we get for $n > M$

$$b_i = b_i = b_i = b_i = b_i \quad (j \geq L+K),$$

whence by (36) it follows at last that for $n > M$

$$\varepsilon_i = \zeta, \quad q. e. d.$$

Remark. One can easily deduce from the lemma proved above Sats 11 and Sats 12 of [3]. There is no inverse implication, but the argumentation given above is a direct generalization of the method used by Schmidt in his proofs.

**Theorem 3.** Let $f(x) = a_0x^p + a_1x^{p-1} + \ldots + a_p$ be an integer-valued polynomial with $a_0 > 0$. If

1. $p \equiv 1 \pmod 2$ or
2. $p \equiv 0 \pmod 2$ and $a_0$ is not a rational square,

then

$$\lim_{n \to \infty} \sqrt[p]{f(n)} = \infty.$$

Proof. In view of Lemma 3 and the equality $\det \sqrt[p]{f(n)} = 1$, it is sufficient to show that the set $F$ of all the residues mod 1 of numbers $\sqrt[p]{f(n)},$

$n = 1, 2, \ldots$ has at least one irrational point of accumulation. We shall prove more: that the set $F$ is dense in $(0, 1).$

In case 1 put $p = 2m+1$, $m > 0$. As can easily be seen, we have in the environment of $\infty$

$$\frac{d^m \sqrt[p]{f(n)}}{d n^m} \sim \sqrt[p]{\frac{a_0 + \frac{1}{2} f(n)}{k!}} f(n)^{m-k+1}.$$

On the other hand, by a well-known theorem of the theory of finite differences (cf. [4], p. 230, th. 221), we have

$$Af(x) = Af(x + \theta k dx), \quad 0 < \theta < 1,$$

where $g(x)$ is an arbitrary real function with the $k$th derivative continuous in the interval $(x, x + k dx)$. Putting

$$g(x) = \sqrt[p]{f(n)}, \quad dx = 1,$$

we obtain by a comparison of the preceding two formulae

$$A^m \sqrt[p]{f(n)} \sim \sqrt[p]{\frac{a_0 + \frac{1}{2} f(n)}{k!} f(n)^{m-k+1}},$$

whence for sufficiently large $x$

$$A^m \sqrt[p]{f(n)} \sim \sqrt[p]{\frac{a_0 + \frac{1}{2} f(n)}{k!} f(n)^{m-k+1}},$$

$$A^{m+1} \sqrt[p]{f(n)} \sim \sqrt[p]{\frac{a_0 + \frac{1}{2} f(n)}{k!} f(n)^{m-k+1}},$$

thus

$$A^m \sqrt[p]{f(n)} \to \infty, \quad A^{m+1} \sqrt[p]{f(n)} \to 0.$$

The density of the set $F$ follows immediately in virtue of a theorem of Cassia (I, p. 152).

In case 2, we have, as can easily be seen

$$f(x) = u(x) + v(x),$$

where $u$ and $v$ are polynomials with coefficients from $K(\sqrt[p]{a_0})$ and

$$\deg u < \deg f, \quad \deg v = \deg f + 1.$$

Putting $p = 2m$, $u(n) = a_0n^m + a_1n^{m-1} + \ldots + a_m$, we find from formulae (39) that $a_0 = a_n$, whence according to the assumption about
It follows that $a_0$ is irrational. In virtue of a well-known theorem of Weyl, the set of all the residues mod 1 of numbers $u(n)$ $(n = 1, 2, \ldots)$ is dense in $(0, 1)$. Since, in view of (39)

$$\lim |f'(x) - u(x)| = 0,$$

the set $F$ has the same property, q. e. d.

**Remark.** In both cases, 1 and 2, it is easy to give examples of polynomials $f(x)$ such that

$$\lim \sup |f'(x)| < \infty.$$

(40)

It suffices to assume $f_1(x) = x$, $f_2(x) = 2x^2$. The proof of inequality (40) for the polynomial $f_1(x)$ is immediate; for the polynomial $f_2(x)$ we use the fact that for an infinite sequence of positive integers $a_k$ is $f_2(a_k) = y_k^2 + 1$ (where $y_k$ — integers), whence in view of the expansion

$$y_k^2 + 1 = (y_k^2, 2y_k)$$

it follows that

$$\lim \sup |f_2^'(a_k)| = 1.$$

§ 4. **Lemma 4.** Let $f(n)$ be an integer-valued polynomial and let

$$\left \lfloor \frac{1}{u_1(n)} \right \rfloor + \left \lfloor \frac{1}{u_2(n)} \right \rfloor + \ldots + \left \lfloor \frac{1}{u_\ell(n)} \right \rfloor + 1$$

(41)

where $u_j$ are polynomials of a positive degree with rational coefficients and

$$\lim_{n \to \infty} u_j(n) = \infty.$$

Put

$$T_{-1}(n) = 1, \quad T_{-1}(n) = u_0(n), \quad T_{-1}(n) = u_0(n)T_{-1}(n) + T_{-1}(n),$$

(43)

$$U_{-1}(n) = 0, \quad U_{-1}(n) = 1, \quad U_{-1}(n) = u_1(n)U_{-1}(n) + U_{-1}(n),$$

(44)

$$\sqrt{f(n)} = \xi = \{b_0, b_1, b_2, \ldots\} \quad b_\ell \quad \text{— integers}.$$

Then, for every $j$ and $n > u_j(f)$, there exists a $k = k(j, n)$ such that

$$\frac{A_k}{B_k} = T_{-1}(n),$$

(45)

$$\xi_{k+1}(n) = (-1)^{j-k} \frac{U_{j}(n)}{B_k} w(n) + (-1)^{j-k} \frac{U_{j}(n)U_{j-1}(n) - B_k B_{k-1}}{B_k^2}.$$

(46)

**Proof.** Since the polynomials $u_j$ have rational coefficients, there exists a positive integer $m$ such that

$$T_{j}(n) = P(n)m, \quad U_{j}(n) = Q(n)m,$$

where $P(n)$, $Q(n)$ are polynomials with integral coefficients.

From formulae (41) and (43) we get

$$\sqrt{f(n)} = \frac{T_{j}(n)}{U_{j}(n)} + \frac{(-1)^{j-k}}{U_{j}(n)U_{j}(n)w(n) + U_{j-1}(n)}.$$

whence in view of (47)

$$\left | \frac{\sqrt{f(n)} - P(n)}{Q(n)} \right | = \frac{1}{Q(n)} \left | \frac{w(n) + U_{j-1}(n)}{U_{j}(n)} \right |.$$

Since in view of (42) and (43)

$$w(n) + \frac{U_{j-1}(n)}{U_{j}(n)} \rightarrow \infty,$$

we have for sufficiently large $n$

$$\left | \frac{\sqrt{f(n)} - P(n)}{Q(n)} \right | < \frac{1}{2Q(n)}.$$

In virtue of a well-known theorem (cf. [2]), Satz 2.14), $P(n)/Q(n)$ is therefore equal to some convergent of expansion (44). Then, for some $k$, equality (45) holds and since

$$\sqrt{f(n)} = \frac{A_k}{B_k} + \frac{(-1)^{j-k}}{B_k \xi_{k+1}(n) + B_{k-1}^2},$$

we get also (46) in view of (48).

**Definition.** For a given prime $p$ and a given rational number $r \neq 0$ we shall denote by $\exp(p, r)$ the exponent with which $p$ comes into the canonical expansion of $r$.

**Lemma 5.** Suppose we are given a prime $p$ and integers $n$ and $k$, both $\neq 0$. Let then

$$P_{-1} = k, \quad P_0 = n, \quad P_r = 2nP_{r-1} + hP_{r-1},$$

(49)

$$Q_{-1} = 0, \quad Q_0 = 1, \quad Q_r = 2nQ_{r-1} + hQ_{r-1}.$$ 

If $\exp(p, h) > 2\exp(p, 2n)$, then for every integer $\nu \geq 0$

$$\exp(p, P_\nu) = \exp(p, n) + \nu \exp(p, 2n),$$

(50)

$$\exp(p, Q_\nu) = \nu \exp(p, 2n).$$
Proof by induction with respect to \( r \). For \( r = 0 \) the lemma follows directly from formulae (49).

For \( r = 1 \) we have \( P_r = 2n^2 + h \), \( Q_r = 2n \); thus \( \exp(p, Q_r) = \exp(p, 2n) \). Since, by hypothesis,

\[
\exp(p, h) > 2\exp(p, 2n) \geq \exp(p, 2n^2),
\]

it follows that

\[
\exp(p, P_r) = \exp(p, 2n^2) = \exp(p, n) + \exp(p, 2n)
\]

and formulae (50) hold also for \( r = 1 \).

Assume now that the lemma is right for the numbers \( r - 2 \) and \( r - 1 \) (\( r \geq 2 \)); we shall show its validity for \( r \).

It follows easily from the inductive assumption that

\[
e_{r-1} = \exp(p, 2n_{r-1}) = \exp(p, n) + \exp(p, 2n),
\]

\[
e_r = \exp(p, h_{r-1}) = \exp(p, n) + (r - 2)\exp(p, 2n) + \exp(p, h),
\]

\[
e_r = \exp(p, 2n_{r-1}) = r\exp(p, 2n),
\]

\[
e_r = \exp(p, h_{r-1}) = (r - 2)\exp(p, 2n) + \exp(p, h).
\]

In view of the inequality \( \exp(p, h) > 2\exp(p, 2n) \) we therefore have \( e_1 < e_2 < e_3 \), whence it follows by (49) that

\[
\exp(p, P_r) = e_r = \exp(p, n) + r\exp(p, 2n),
\]

\[
\exp(p, Q_r) = e_r = r\exp(p, 2n), \quad q. e. d.
\]

**Theorem 4.** Suppose we are given an integer \( h \neq 0 \). Denote by \( E \) the set of all integers \( n \) such that \( h \mid 4n^2 \). We have

\[
\lim_{n \to \infty} \frac{\ln \sqrt{n^2 + h}}{n} = \infty,
\]

\[
\lim_{n \to \infty} \frac{\ln \sqrt{n^2 + h}}{n} < \infty.
\]

**Proof.** We begin with a proof of equality (51). Choose an arbitrary \( g \); we shall show that for sufficiently large \( n \in E \)

\[
\ln \sqrt{n^2 + h} > g.
\]

It is easy to verify the identity

\[
\sqrt{n^2 + h} = n + \frac{1}{2n\sqrt{h}} + \frac{1}{n + \sqrt{n^2 + h}},
\]

from which we immediately obtain

\[
\sqrt{n^2 + h} = n + \frac{1}{2n\sqrt{h}} + \frac{1}{2n\sqrt{h}} + \ldots + \frac{1}{2n\sqrt{h}} + \frac{1}{n + \sqrt{n^2 + h}}.
\]

Put in Lemma 4

\[
u_r = \begin{cases} \frac{2n}{h} & (n \text{ odd} \leq 2g - 1), \\ \frac{2n}{h} & (n \text{ even} < 2g - 1). \end{cases}
\]

Comparing polynomials \( T_r, U_r \) determined for these \( n \), by formulae (42) and polynomials \( P_r, Q_r \) defined by (49), we find by an easy induction

\[
T_r = P_r, U_r = Q_r h^{-(p-1)/2}, \quad U_r = Q_r h^{-(p-1)/2},
\]

whence

\[
\frac{T_r}{U_r} = \frac{P_r}{Q_r}.
\]

Assume now that \( n \in E \), \( n \) so large that \( \sqrt{n^2 + h} \) is irrational, and

\[
\sqrt{n^2 + h} = \xi = [a_0, a_1, \ldots].
\]

In virtue of Lemma 4 for sufficiently large \( n \) for each \( i \leq g \) there exists a \( k_i \) such that

\[
A_{k_i} = \frac{T_{n-1}}{B_{k_i}},
\]

\[
B_{k_i} = (-1)^{k_i} \frac{U_{n-1} + U_{n+1} - U_{n-2} - U_{n+2}}{B_{k_i}}.
\]

Since \( n \in E \), there exists a prime \( p \) such that

\[
\exp(p, h) > 2\exp(p, 2n)
\]

and in virtue of Lemma 5

\[
\exp(p, P_{n-1}) = \exp(p, 2n) + (2i-1)\exp(p, 2n),
\]

\[
\exp(p, Q_{n-1}) = (2i-1)\exp(p, 2n).
\]
In view of (55) and (57), we therefore have
\[ \exp \left( p, \frac{A_n}{B_n} \right) = \exp \left( p, \frac{V_{n-1}}{Q_{n-1}} \right) = \exp(p, n) \]
and since the fraction \( A_n/B_n \) is irreducible, it follows that
\[ \exp(p, B_n) = 0. \]
On the basis of (54) and (60) we get hence
\[ \exp \left( p, \frac{U_{n-1}}{B_n} \right) = \exp(p, U_{n-1}) = (2^4 - 1)\exp(p, 2n) \]
\[ -i\exp(p, h) = -\exp(p, 2^4) - i(\exp(p, h) - 2\exp(p, 2n)). \]
Then, in view of inequality (59), the numbers \( \exp(p, U_{n-1}/B_n) \) are for
\( i = 1, 2, \ldots, g \) all different; since \( V^{n+1 + h} \) is irrational and (58) holds, the numbers \( z_{i,1} \), have the same property. Since \( k_1 + 1 > 1 = \exp V^{n+1 + h} \), at least \( g \) different complete quotients occur in the period of expansion
(56); we then have \( \exp(p, g) \), which completes the proof of (51).

In order to prove formula (52) we shall use Theorem 2. From that theorem follows the existence of a number \( N = N(h) \) such that if for positive integers \( D_1, D_2 \) and \( l \)
\[ V^{D_1} = \frac{1}{V^{D_1}}, \quad 0 < l \leq |h| \quad \text{and} \quad \exp(p, D_1) \leq 12, \]
then \( \exp V^{D_2} \leq N. \)

We shall show that for sufficiently large \( n \in E \)
\[ \exp V^{n+1 + h} \leq N. \]

In fact, since \( n \in E, h \in E \), there exist—as can easily be seen—integers \( a, \beta \neq 0 \) and positive integer \( x \) such that
\[ 2a = \alpha x, \quad h = \beta x. \]

We obviously have
\[ \exp V^{n+1 + h} = \frac{1}{\exp V^{(a\beta)^2+4a}}, \quad |\beta| \leq |h|. \]

On the other hand, as can be verified, the following expansions hold for \( x \geq 5: \)
\[ a > 0, \quad x—\text{even} \]
\[ V^{(a\beta)^2+4a} = [ax, \frac{1}{2} x, 2ax]; \]
\[ a > 0 \text{ even}, \quad x—\text{odd} \]
\[ V^{(a\beta)^2+4a} = [ax, \frac{1}{2} (ax-1), 1, 1, \frac{1}{2} (ax-2), 1, 1, \frac{1}{2} (x-1), 2ax]; \]
\[ a > 0 \text{ odd}, \quad x—\text{odd} \]
\[ V^{(a\beta)^2+4a} = [ax, \frac{1}{2} (x-1), 1, 1, \frac{1}{2} (ax-1), 1, 1, \frac{1}{2} (x-1), 2ax]; \]
\[ a < 0, \quad x—\text{even} \]
\[ V^{(a\beta)^2+4a} = [ax, \frac{1}{2} (x-1), 1, 1, \frac{1}{2} (ax+1), 2ax, \frac{1}{2} (ax-1), 1, 1, \frac{1}{2} (x-1), 2ax]; \]
\[ a < 0 \text{ even}, \quad x—\text{odd} \]
\[ V^{(a\beta)^2+4a} = [ax, \frac{1}{2} (x-1), 1, 1, \frac{1}{2} (ax+1), 2ax, \frac{1}{2} (x-1), 1, 1, \frac{1}{2} (ax-1), 2ax]; \]
\[ a < 0 \text{ odd}, \quad x—\text{odd} \]
\[ V^{(a\beta)^2+4a} = [ax, \frac{1}{2} (x-1), 1, 1, \frac{1}{2} (ax+1), 2ax, \frac{1}{2} (x-1), 1, 1, \frac{1}{2} (ax-1), 2ax]. \]

Thus, we always have \( \exp(p, V^{(a\beta)^2+4a}) \leq 12 \) and formula (61) follows immediately from (62) and the definition of \( N. \)

**Theorem 5.** Let \( f(n) = a_n + bn + c, \)
\( a, b, c—\text{integers, } a > 0, \)
\( A = |a|4(2a^2, b)^3. \)

**Proof.** We obviously have
\[ V^{(a\beta)^2+4a} = \frac{1}{2a} V^{(2a^2+b)^2} - A \]
and in virtue of Theorem 2 inequality (63) is equivalent to the following
\[ \lim \exp V^{(2a^2+b)^2} - A < \infty. \]

But in virtue of Theorem 4 the last inequality holds if and only if for some \( n_0 \)
\[ A = |a|4(2a^2+b)^3 \text{ for } n > n_0. \]

We have
\[ 4(2a^2+b)^3 = 4(2a^2, b)^3 \left( \frac{2a^2}{2a^2, b} + \frac{b}{2a^2, b} \right)^2. \]
Since the arithmetical progression
\[ \frac{2a^r - b}{(2a^r - b)} \quad (n = 0, 1, \ldots) \]
whose first term and difference are relatively prime, contains infinitely many numbers coprime with \( d \), divisibility (64) is a necessary and sufficient condition of (65) and therefore also of (63), q. e. d.

Theorems 3 and 5 give together a complete solution of the problem P for polynomials of the second degree (the case \( d = 0 \) is trivial).

In order to obtain by a similar method a complete solution for polynomials of higher degree, it would be necessary to have for \( \psi(n) \) an expansion analogous to (53), i.e. an expansion of form (41) and then to know whether it is periodical.

Now, for polynomials \( f(n) \) of the form
\[ \alpha n^m + \alpha_1 n^{m-1} + \ldots + \alpha_m \]
an expansion of form (41) is uniquely determined. In fact, let
\[ \psi(n) = u_0(n) + \frac{1}{u_1(n)} + \frac{1}{u_2(n)} + \ldots + \frac{1}{u_{r-1}(n)} \]
and put for \( i \leq j \)
\[ \psi_i(n) = u_0(n) + \frac{1}{u_1(n)} + \ldots + \frac{1}{u_i(n)} \]

Since \( u_0(n) \) is the unique polynomial \( g \) such that \( \lim_{n \to \infty} |\psi(n) - g(n)| = 0 \), we have \( u_0(n) = u(n) \), where \( u(n) \) is defined by formulae (39).

Suppose now that we have determined polynomials \( u, \ldots, u_{r-1}(n) \);

we easily find
\[ u_i(n) = \psi_i(n) + p_i(n) \]
where \( p_i, q_i \) are polynomials with rational coefficients, and then \( u_i(n) \) is uniquely determined by the conditions
\[ u(n) + p_i(n) = q_i(n) u_i(n) + r_i(n), \quad \text{degree } r_i < \text{degree } q_i. \]

The construction of the sequence \( u_i(n) \) is therefore easy, but the decision whether the sequence thus determined \( u_i \) is periodical presents a considerable difficulty even for polynomials of degree 4.

Thus, the investigation of the problem P has led us to the following problem P1.