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Determination of Fermat varieties with trivial Hasse–Witt map (An application of the Farey series)

by

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1. Introduction. In the present paper, by making use of Farey series (cf. Section 4), we shall determine the Fermat varieties with trivial Hasse–Witt map. The precise statement is given below.

Let $F_{n,N,p} \subset \boldsymbol{P}^{n+1}$ denote the n-dimensional Fermat variety of degree N over an algebraically closed field k of characteristic $p > 0$:

$$F_{n,N,p} = \{(x_0, x_1, \ldots, x_{n+1}) \in \boldsymbol{P}^{n+1} \mid x_0^N + x_1^N + \ldots + x_{n+1}^N = 0\}.$$

We suppose that $n \geqslant 1$, $N \geqslant 2$ and $p \nmid N$; hence $F_{n,N,p}$ is a smooth variety. Let

$$\mathscr{F}\colon \mathcal{O}_{F_{n,N,p}} \to \mathcal{O}_{F_{n,N,p}}$$

denote the Frobenius map, i.e., the pth power map of the structure sheaf $\mathcal{O}_{F_{n,N,p}}$. We put $\mathcal{O} = \mathcal{O}_{F_{n,N,p}}$ and consider the cohomology group $H^n(F_{n,N,p}, \mathcal{O})$. Then $\mathscr{F}$ induces the p-linear endomorphism of the finite-dimensional vector space $H^n(F_{n,N,p}, \mathcal{O})$ over k, which we call the *Hasse–Witt map* of $F_{n,N,p}$.

The aim of this paper is to prove the following:

THEOREM. *If $N \leqslant n+1$ then $H^n(F_{n,N,p}, \mathcal{O}) = \{0\}$. Assume $N > n+1$. Then the Hasse–Witt map of $F_{n,N,p}$ is zero if and only if there exist positive integers k, l and m satisfying*

(1) $$1 \leqslant k \leqslant n, \quad 1 \leqslant l \leqslant k-1, \quad 1 \leqslant m \leqslant n-k+1,$$

(2) $$(k, l) = 1,$$

(3) $$\{p\}_N \text{ is of the form } (lN-m)/k$$

where $\{p\}_N$ denotes the remainder of dividing p by N.

After recalling an algorithm for computing the Hasse–Witt matrix of a Fermat hypersurface in Section 2, we shall prove the Theorem in Sections 3, 4.

In case $n \leqslant 2$, N. Suwa and K. Toki obtained the above result (cf. [3]). Moreover, N. Suwa conjectured that a criterion for the Hasse–Witt map of the

Fermat variety $F_{n,N,p}$ to be zero can be written as a simple congruence relation among n, N and p. We formulated his conjecture in the above form and proved it.

We wish to thank Professor N. Suwa for suggesting the problem. His advice was indispensable for the second author to perform this task.

2. The Hasse–Witt matrix of Fermat varieties. Following N. Koblitz [2], we recall fundamental facts about the Hasse–Witt matrix of Fermat varieties. Let $F = F_{n,N,p} \subset \boldsymbol{P}^{n+1}$ be the n-dimensional Fermat hypersurface of degree N over an algebraically closed field k of characteristic $p > 0$ defined by the equation $h = 0$, where

$$h = x_0^N + x_1^N + \dots + x_{n+1}^N \in k[x_0, x_1, \dots, x_{n+1}].$$

Then we have a commutative diagram of structure sheaves

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_{\boldsymbol{P}^{n+1}}(-N) & \to & \mathcal{O}_{\boldsymbol{P}^{n+1}} & \to & \mathcal{O}_F & \to & 0 \\ & & \downarrow h^{p-1}\mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \\ 0 & \to & \mathcal{O}_{\boldsymbol{P}^{n+1}}(-N) & \to & \mathcal{O}_{\boldsymbol{P}^{n+1}} & \to & \mathcal{O}_F & \to & 0, \end{array}$$

where $\mathcal{F}$ is the pth power map.

The resulting long exact sequence gives an isomorphism of cohomology groups

$$H^n(F, \mathcal{O}_F) \xrightarrow{\sim} H^{n+1}(\boldsymbol{P}^{n+1}, \mathcal{O}_{\boldsymbol{P}^{n+1}}(-N))$$

so that the Frobenius map on $H^n(F, \mathcal{O}_F)$ corresponds to the map on $H^{n+1}(\boldsymbol{P}^{n+1}, \mathcal{O}_{\boldsymbol{P}^{n+1}}(-N))$ induced by the composition

$$\mathcal{O}_{\boldsymbol{P}^{n+1}}(-N) \xrightarrow{\mathcal{F}} \mathcal{O}_{\boldsymbol{P}^{n+1}}(-pN) \xrightarrow{h^{p-1}} \mathcal{O}_{\boldsymbol{P}^{n+1}}(-N).$$

Set

$$I = \{1, 2, \dots, N-1\},$$

$$W = \{w = (w_0, w_1, \dots, w_{n+1}) \in I^{n+2} \mid \sum_{i=0}^{n+1} w_i \equiv 0 \pmod N\},$$

$$W_i = \{w = (w_0, w_1, \dots, w_{n+1}) \in W \mid |w| = (i+1)N\},$$

where $|\ |: W \to \boldsymbol{Z}$ is defined by $|w| = \sum_{i=0}^{n+1} w_i$.

Let $\{\ \}: \boldsymbol{Z} \to I_0 = \{0, 1, \dots, N-1\}$ be defined by $\{z\} \equiv z \pmod N$. Then the group $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$ acts on W by

$$z \cdot w = (\{zw_0\}, \{zw_1\}, \dots, \{zw_{n+1}\})$$

for any z in $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$. As is easily seen, $\{1/X^w \mid w \in W_0\}$ forms a basis for $H^{n+1}(\boldsymbol{P}^{n+1}, \mathcal{O}_{\boldsymbol{P}^{n+1}}(-N))$, where $X^w = \prod_{i=0}^{n+1} x_i^{w_i}$.

The *Hasse–Witt matrix* $\mathrm{HW}(F_{n,N,p})$ of $F_{n,N,p}$ with respect to this basis is given by $(h_{v,w})_{v,w \in W_0}$, where

$$h_{v,w} = \begin{cases} \text{a non-zero element of } \boldsymbol{F}_p & \text{if } w = p \cdot v, \\ 0 & \text{otherwise.} \end{cases}$$

We know that

$$\dim H^{n+1}(\boldsymbol{P}^{n+1}, \mathcal{O}_{\boldsymbol{P}^{n+1}}(-N)) = \binom{N-1}{n+1}.$$

Thus we have the following Proposition which reduces our problem to an arithmetical one.

PROPOSITION. *If* $N \leqslant n+1$ *then* $H^n(F_{n,N,p}, \mathcal{O}) = \{0\}$. *Assume* $N \geqslant n+2$. *Then*

$$\mathrm{HW}(F_{n,N,p}) = 0$$

if and only if $p \cdot w \notin W_0$ *for every* $w \in W_0$.

3. Proof of the "if" part in the Theorem. First we shall prove the "if" part which is easier than the "only if" part. In this section, $\{a\}_N$ or $\{a\}$ is the remainder of dividing a by N.

Suppose the characteristic p of the ground field satisfies the conditions in the Theorem, i.e., $\{p\}_N$ can be written in the form

$$\{p\}_N = (lN - m)/k,$$

where $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant k-1$, $1 \leqslant m \leqslant n-k+1$ and $(k, l) = 1$. Let $w = (w_0, w_1, \dots, w_{n+1})$ be any element in W_0, i.e., $\sum_{i=0}^{n+1} w_i = N$ and $0 < w_i < N$. Changing the suffix, we may assume

$$w_0 \equiv w_1 \equiv \dots \equiv w_{\alpha_0} \equiv 0 \pmod k,$$

$$w_{\alpha_0+1} \equiv w_{\alpha_0+2} \equiv \dots \equiv w_{\alpha_0+\alpha_1} \equiv 1 \pmod k, \dots,$$

$$w_{\alpha_0+\alpha_1+\dots+\alpha_{i-1}+1} \equiv w_{\alpha_0+\alpha_1+\dots+\alpha_{i-1}+2} \equiv \dots \equiv w_{\alpha_0+\alpha_1+\dots+\alpha_i} \equiv i \pmod k, \dots,$$

$$w_{\alpha_0+\alpha_1+\dots+\alpha_{k-2}+1} \equiv w_{\alpha_0+\alpha_1+\dots+\alpha_{k-2}+2} \equiv \dots \equiv w_{\alpha_0+\alpha_1+\dots+\alpha_{k-1}} \equiv k-1 \pmod k.$$

Then we have

$$1 + \alpha_0 + \sum_{i=1}^{k-1} \alpha_i = n+2 \tag{3.1}$$

and

$$lNw_j - mw_j \equiv 0 \pmod k \quad \text{for } 0 \leqslant j \leqslant n+1,$$

$$ilN - mw_j \equiv 0 \pmod k \quad \text{for } \beta_i + 1 \leqslant j \leqslant \beta_{i+1},$$

where $\beta_i = \alpha_0 + \alpha_1 + \dots + \alpha_{i-1}$ $(1 \leqslant i \leqslant k)$.

Define rational numbers $\delta_0, \delta_1, \ldots, \delta_{n+1}$ by

$$\{pw_j\} = \delta_j N - mw_j/k \quad \text{for } 0 \leqslant j \leqslant \alpha_0,$$

$$\{pw_j\} = (\delta_j - 1)N + (il/k - [il/k])N - mw_j/k \quad \text{for } \beta_i + 1 \leqslant j \leqslant \beta_{i+1} \quad (1 \leqslant i \leqslant k-1).$$

Here $[\alpha]$ denotes the greatest integer not exceeding the real number α.

Remembering $\{p\}_N = (lN - m)/k$, we see that these δ_i's are positive integers.

Since

$$\sum_{j=0}^{\alpha_0} w_j + \sum_{i=1}^{k-1} \sum_{j=\beta_i+1}^{\beta_{i+1}} w_j = |w| = N$$

and

$$\sum_{j=0}^{\alpha_0} \delta_j + \sum_{i=1}^{k-1} \sum_{j=\beta_i+1}^{\beta_{i+1}} (\delta_j - 1) = \sum_{i=0}^{n+1} (\delta_i - 1) + (1 + \alpha_0),$$

we have

$$\begin{aligned} |p \cdot w| &= \sum_{j=0}^{n+1} \{pw_j\} \\ &= \sum_{j=0}^{\alpha_0} (\delta_j N - mw_j/k) + \sum_{i=1}^{k-1} \Big\{ \sum_{j=\beta_i+1}^{\beta_{i+1}} ((\delta_j - 1)N + (il/k - [il/k])N - mw_j/k) \Big\} \\ &= AN, \end{aligned}$$

where

$$A = \sum_{i=0}^{n+1} (\delta_i - 1) + (1 + \alpha_0) + \sum_{i=1}^{k-1} \alpha_i (il/k - [il/k]) - m/k. \tag{3.2}$$

To prove the "if" part, by the Proposition in Section 2, it suffices to show

$$|p \cdot w| > N, \quad \text{i.e.,} \quad A > 1.$$

If $(k, l) = 1$ and $il \equiv 0 \pmod k$, then $i \equiv 0 \pmod k$. Since $1 \leqslant i \leqslant k-1$ and $(k, l) = 1$, it follows that $il \not\equiv 0 \pmod k$; hence

$$il/k - [il/k] \geqslant 1/k. \tag{3.3}$$

Thus we have

$$\begin{aligned} A &\geqslant 1 + \alpha_0 + \Big(\sum_{i=1}^{k-1} \alpha_i\Big)/k - m/k && \text{(by (3.2), (3.3))} \\ &= (1+\alpha_0) - (1+\alpha_0)/k + \Big((1+\alpha_0) + \sum_{i=1}^{k-1} \alpha_i\Big)/k - m/k \\ &= (1 - 1/k)(1+\alpha_0) + (n+2)/k - m/k && \text{(by (3.1))} \\ &\geqslant (n+2-m)/k \geqslant (n+2-(n-k+1))/k = (k+1)/k > 1 && \text{(by } m \leqslant n-k+1). \end{aligned}$$

This establishes the proof of the "if" part in the Theorem.

4. Proof of the "only if" part in the Theorem.

4.1. Preliminaries. Before proving the "only if" part, we recall some fundamental properties of Farey series (cf. [1]). When a positive integer n is given, the *Farey series* of order n is the ascending series of irreducible fractions in the interval $[0, 1]$, whose denominators are positive integers $\leqslant n$. We denote by $\mathscr{F}_n$ the Farey series of order n.

EXAMPLE. $\mathscr{F}_6 = \{0/1, 1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 1/1\}$.

Let $B/A < D/C$ be two successive terms in $\mathscr{F}_n$. Then we call $(B+D)/(A+C)$ the *mediant* of the two fractions.

LEMMA 1. *Let $B/A < B'/A'$ be two successive terms in $\mathscr{F}_n$. Then*

$$A + A' \geqslant n+1 \quad \text{and} \quad AB' - A'B = 1.$$

For the proof, see [1].

As the following Lemmas 2, 3, 4 are easily proved, we shall omit their proofs. We call the irreducible fractions in the interval $[0, 1]$ the *Farey fractions*.

LEMMA 2. *Let $B/A < B'/A' < B''/A''$ be three successive terms in $\mathscr{F}_n$. Then*

$$B'/A' = (B+B'')/(A+A'').$$

LEMMA 3. *Let $B/A < D/C$ be two Farey fractions satisfying $AD - BC = 1$. Then the fraction $(B+D)/(A+C)$ satisfies*

$$(A+C)D - (B+D)C = (B+D)A - (A+C)B = 1;$$

hence $B/A < (B+D)/(A+C) < D/C$, and $(B+D)/(A+C)$ is also a Farey fraction.

LEMMA 4. *Let A, B, C, D be positive integers. Then the following are equivalent:*

(1) *$AD - BC = 1$, and $B/A < D/C$ are Farey fractions.*

(2) *$A(B+D) - B(A+C) = 1$, and $B/A < (B+D)/(A+C)$ are Farey fractions.*

(3) *$(A+C)D - (B+D)C = 1$, and $(B+D)/(A+C) < D/C$ are Farey fractions.*

In the proof of the "only if" part, we shall often use the following facts.

LEMMA 5. *Let $B/A < D/C$ be two Farey fractions with $AD - BC = 1$.*

(1) *If F/E is a Farey fraction satisfying $B/A < F/E < D/C$, then $E \geqslant A + C$ and the residue $\{FA\}_E$ of FA modulo E is equal to $FA - BE$.*

(2) *If α is a fraction of the form $(xB+yD)/(xA+yC)$ with relatively prime positive integers x and y, then α is a Farey fraction.*

(3) *Let α be a Farey fraction. Then α satisfies $B/A < \alpha < D/C$ if and only if α is of the form as in (2). In this case x and y are uniquely determined by α.*

Proof. Using $AD-BC=1$, we have

$$E-(A+C) = (AD-BC)E - AFC + AFC - (A+C)$$

$$= A(ED-FC-1) + C(AF-BE-1).$$

By $ED-FC \geqslant 1$, $AF-BE \geqslant 1$, we have the inequality in (1). By the same argument for the difference $E-(AF-BE)$ as above, we get

$$0 < AF-BE < E,$$

which shows the last property in (1). Now we shall verify (2). We have

$$B/A < (xB+yD)/(xA+yC) < D/C \leqslant 1/1$$

by $AD-BC=1$, $x>0$, $y>0$. Moreover, since $(x, y)=1$, we have $(xB+yD, xA+yC)=1$. Therefore α is a Farey fraction. The "if" part of (3) has been proven above. Conversely, let α be a Farey fraction of the form F/E. If we solve the equation in x, y

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} E \\ F \end{pmatrix}$$

using $AD-BC=1$, then we get

$$x = ED-FC > 0, \qquad y = AF-BE > 0.$$

By $(E, F)=1$, we have $(x, y)=1$. The uniqueness obviously holds. ∎

4.2. The sequence $\mathscr{S}_x(B'/A', B''/A'')$. Let $B'/A' < B''/A''$ be Farey fractions such that $A'B''-A''B'=1$. For a positive integer x, we denote by $\mathscr{S}_x(B'/A', B''/A'') = \mathscr{S}_x$ the sequence $\{(xB'+yB'')/(xA'+yA'')\}_{y=0,1,2,\ldots}$ of fractions between B'/A' and B''/A''. Some terms in $\mathscr{S}_x$ $(x \geqslant 2)$ may be reducible fractions. We introduce a new notation $\alpha_{x,y}$ specially for an irreducible fraction $(xB'+yB'')/(xA'+yA'')$. In this case x and y are relatively prime non-negative integers.

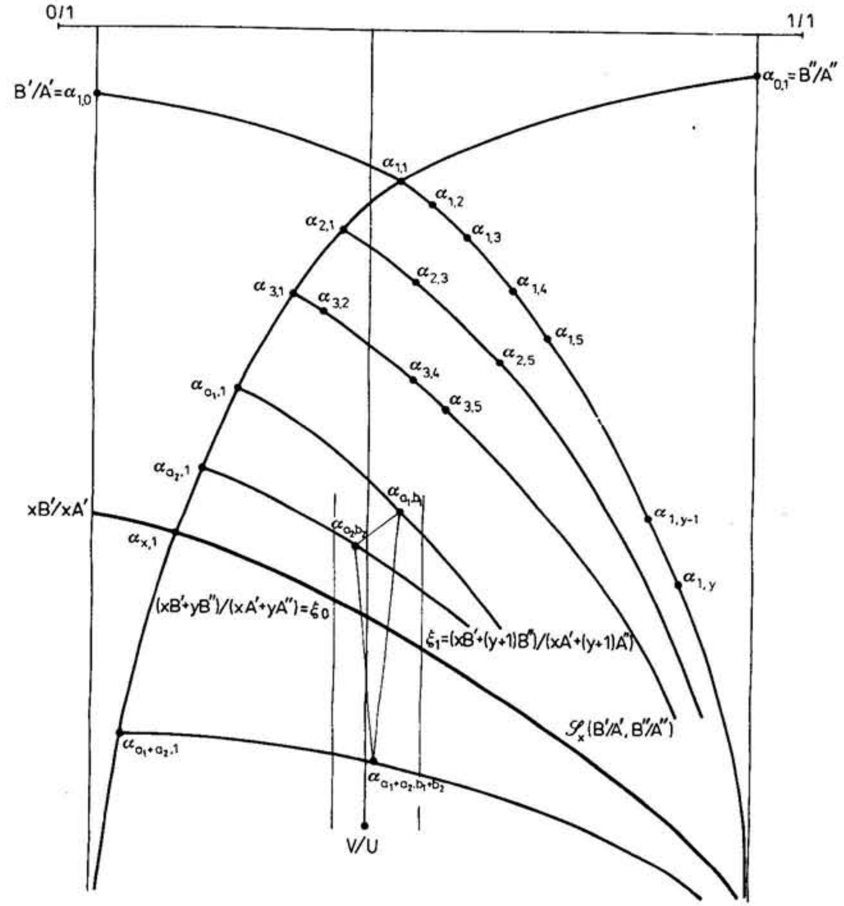

Fig. $(B'/A', B''/A'')$

Note that the denominators of $\alpha_{x,y}$'s become larger as $\alpha_{x,y}$'s go down, in the above figure.

As is easily seen, we have

$$\text{(4.2.1)} \qquad \begin{aligned} \alpha_{x,y} = \alpha_{x',y'} \quad &\Leftrightarrow \quad x = x' \text{ and } y = y', \\ \alpha_{x,y} < \alpha_{x',y'} \quad &\Leftrightarrow \quad xy' > x'y. \end{aligned}$$

Moreover, any $\alpha_{x,y}$ with $x, y > 0$ is equal to the mediant $\alpha_{x'+x'',y'+y''}$ of some successive pair $\alpha_{x',y'} < \alpha_{x'',y''}$ in $\mathscr{F}_{xA'+yA''-1}$, i.e., $x = x'+x''$, $y = y'+y''$, $x'y''-x''y'=1$,

$$\text{(4.2.2)} \qquad \alpha_{x',y'} < \alpha_{x,y} < \alpha_{x'',y''}.$$

Conversely, we have the following:

LEMMA 6. *Assume two Farey fractions* $\alpha_{x',y'} < \alpha_{x'',y''}$ *satisfy*

(4.2.3) $$x'y''-x''y' > 1.$$

Then there exists $\alpha_{x,y}$ *in* $\mathscr{F}_{\max\{x'A'+y'A'',x''A'+y''A''\}-1}$ *such that*

$$\alpha_{x',y'} < \alpha_{x,y} < \alpha_{x'',y''} \quad \text{and} \quad x \leqslant \max\{x', x''\}.$$

Proof. If $x' \leqslant x''$ then, by (4.2.1),

$$x''(y''-y') = x''y''-x''y' \geqslant x'y''-x''y' > 0 \quad \text{and} \quad x'' > 0.$$

Hence

$$(x''A'+y''A'')-(x'A'+y'A'') = (x''-x')A'+(y''-y')A'' > 0.$$

Thus we can divide the situation of Lemma 6 into the following cases:

(i) $x' \leqslant x''$ and $x'A'+y'A'' < x''A'+y''A''$,
(ii) $x' > x''$ and $x'A'+y'A'' > x''A'+y''A''$,
(iii) $x' > x''$ and $x'A'+y'A'' = x''A'+y''A''$,
(iv) $x' > x''$ and $x'A'+y'A'' < x''A'+y''A''$.

Here, by (4.2.3),

(4.2.4)
$$x'', y'' > 0 \text{ in (i) and (iv)},$$
$$x', y' > 0 \text{ in (ii)},$$
$$x', y' > 0 \text{ and } x'', y'' > 0 \text{ in (iii)}.$$

Now we shall prove the lemma in the case (i). By (4.2.4), we can use the fact (4.2.2). Thus we have a successive pair $\alpha_{a,b} < \alpha_{c,d}$ in $\mathscr{F}_{x''A'+y''A''-1}$ satisfying $\alpha_{x'',y''} = \alpha_{a+c,b+d}$. Then $\alpha_{x',y'} < \alpha_{a,b} < \alpha_{x'',y''}$ and $a \leqslant x''$. We take $\alpha_{x,y} = \alpha_{a,b}$. Similar proofs work in the other cases. ■

Let V/U be a Farey fraction satisfying $B'/A' < V/U < B''/A''$. Then, by Lemma 5 (3), we can write

(4.2.5) $$U = XA'+YA'', \quad V = XB'+YB'', \quad \text{i.e., } V/U = \alpha_{X,Y},$$
$$\text{where } X = UB''-VA'',\ Y = VA'-UB'.$$

We have two terms ξ_0 and ξ_1 in $\mathscr{S}_x(B'/A', B''/A'')$ such that $\xi_0 \leqslant V/U < \xi_1$, where

$$\xi_0 = (xB'+yB'')/(xA'+yA''), \quad \xi_1 = (xB'+(y+1)B'')/(xA'+(y+1)A'')$$

for some non-negative integer y.

LEMMA 7. *Let notations be as above.*

(1) *If* $V/U = \alpha_{X,Y}$ *lies on or above* $\mathscr{S}_x(B'/A', B''/A'')$ *in the Figure then* $X \leqslant x$.

(2) *If* $V/U = \alpha_{X,Y}$ *lies below* $\mathscr{S}_x(B'/A', B''/A'')$ *in the Figure (i.e.,* $X > x$*), then there exist* α_{a_1,b_1} *and* α_{a_2,b_2} *in the Figure with* $a_1b_2-a_2b_1 = 1$, $a_1, a_2 \leqslant x$, $a_1+a_2 > x$ *such that*

$$\bar{\xi}_0 \leqslant \alpha_{a_1,b_1} < V/U < \alpha_{a_2,b_2} \leqslant \bar{\xi}_1$$

where $\bar{\xi}_i$ *denotes the irreducible expression of* ξ_i $(i = 0, 1)$.

Moreover,

$$U > xA'+(y+1)A''.$$

Proof. By the definition of the sequence $\mathscr{S}_x(B'/A', B''/A'')$, we have the assertion (1). Let $X > x$. Take the nearest α_{a_1,b_1} and α_{a_2,b_2} to $V/U = \alpha_{X,Y}$ satisfying

(4.2.6) $$\bar{\xi}_0 \leqslant \alpha_{a_1,b_1} < V/U < \alpha_{a_2,b_2} \leqslant \bar{\xi}_1$$

with $a_1, a_2 \leqslant x$. Then $a_1+a_2 > x$, i.e., the mediant of α_{a_1,b_1} and α_{a_2,b_2} lies below $\mathscr{S}_x(B'/A', B''/A'')$ in the Figure. Moreover, by (4.2.1) and Lemma 6,

$$a_1b_2-a_2b_1 = 1.$$

By (4.2.6),

$$\alpha_{a_1,b_1} = (a_1B'+b_1B'')/(a_1A'+b_1A'') < V/U < \alpha_{a_2,b_2}$$
$$= (a_2B'+b_2B'')/(a_2A'+b_2A''),$$

and by Lemma 5 (1),

$$U \geqslant (a_1A'+b_1A'')+(a_2A'+b_2A'')$$
$$= (a_1+a_2)A'+(b_1+b_2)A'' > xA'+(b_1+b_2)A''.$$

On the other hand, in (4.2.6),

$$xb_1-ya_1 \geqslant 0 \quad \text{since } \bar{\xi}_0 \leqslant \alpha_{a_1,b_1}$$

and

$$xb_2-ya_2 > 0 \quad \text{since } \bar{\xi}_0 < \alpha_{a_2,b_2}.$$

Hence

$$b_1+b_2 > y(a_1+a_2)/x > y \quad \text{and} \quad U > xA'+(y+1)A''. \ ■$$

4.3. Main Lemmas I, (II.1), (II.2). In this section we shall prove three lemmas which will be directly used to prove the "only if" part in the Theorem. Let n be a positive integer. Let V/U be a Farey fraction. Now assume $U \geqslant n+2$. We can take the successive pair $B/A < D/C$ in the Farey series $\mathscr{F}_n$ satisfying

(4.3.0) $$B/A < V/U < D/C.$$

If $B/A = 0/1$, then the successor D/C of $0/1$ in $\mathscr{F}_n$ must be $1/n$. Then, by (4.2.5), we have $V/U = \alpha_{X,Y}$ where

(4.3.1) $$X = U - nV \quad \text{and} \quad Y = V.$$

Lemma I. *Let notations be as above, i.e., $0/1 < V/U < 1/n$. Then there is an integer $y \geqslant 0$ satisfying*

(4.3.2) $$y/(yn+1) < V/U \leqslant (y+1)/((y+1)n+1).$$

Moreover, we have the following:

(1) *If equality holds in* (4.3.2) *then*

$$y \geqslant 1 \quad \textit{and} \quad V = (U-1)/n.$$

(2) *If equality does not hold in* (4.3.2) *then*

$$U > n+(yn+1), \quad \{V(yn+1)\}_U = V(yn+1) - yU \quad \textit{and}$$
$$(y+1)U - ((y+1)n+1)V > 0.$$

Proof. We easily see the existence of a non-negative integer y satisfying (4.3.2). Both outersides of (4.3.2) are successive terms in $\mathscr{S}_1(0/1, 1/n)$. If equality does hold, then y must be positive since $U \geqslant n+2$. Moreover, V/U lies on $\mathscr{S}_1(0/1, 1/n)$. Therefore $X = 1$ and $V = (U-1)/n$ by (4.3.1). Suppose that equality does not hold. Then the third inequality in (2) obviously holds. By Lemma 5 (1), we get the other relations in (2). ∎

Next, we assume $B/A \neq 0/1$, i.e., $0/1 < B/A < V/U < D/C \leqslant 1/1$. Then $A \neq C$ since $AD - BC = 1$. And it is easily seen that, for any positive integer i,

$$iA < C \quad \text{if and only if} \quad iB < D.$$

Set $a = [C/A]$. If $a \geqslant 1$ then we define positive integers E_i, F_i ($i = 0, 1, \ldots, a$) by $E_i = C-(a-i)A$, $F_i = D-(a-i)B$. Since $A(D-iB)-B(C-iA) = 1$, we have $(E_i, F_i) = 1$ and

$$E_iF_{i-1} - E_{i-1}F_i = E_i(F_i - B) - (E_i - A)F_i = AF_i - BE_i > 0 \quad (i = 0, 1, \ldots, a).$$

Moreover, since $(C-iA)-(D-iB) = ((C-iA)(A-B)-1)/A \geqslant 0$, we have $E_i \geqslant F_i$. Then they form a subseries of $\mathscr{F}_n$:

$$D/C = F_a/E_a < F_{a-1}/E_{a-1} < \ldots < F_0/E_0 \leqslant 1/1,$$

and $E_0 < A$, $F_0 \leqslant B$.

If $a \geqslant 1$ (resp. $a = 0$), we put $E = E_0$, $F = F_0$ (resp. $E = C$, $F = D$). Moreover, we put

(4.3.3) $$A_{-1} = A - E \quad (> 0), \qquad B_{-1} = B - F \quad (\geqslant 0).$$

Then we easily see that

$$AF_i - BE_i = A_{-1}F - B_{-1}E = AF - BE = 1 \quad \text{for } 1 \leqslant i \leqslant a.$$

And the fraction B_{-1}/A_{-1} is a term in $\mathscr{F}_n$ satisfying $0/1 \leqslant B_{-1}/A_{-1} \leqslant B/A$.

Let $M = n-E+1$ and $K = n-C+1$. Then $1 \leqslant K \leqslant M \leqslant n$ and $MB+AF = KB+AD = (n+1)B+1$, $MA+AE = KA+AC = (n+1)A$. We shall divide the situation of V/U into two cases:

Case (II.1). $B/A < V/U \leqslant ((n+1)B+1)/((n+1)A)$;

Case (II.2). $((n+1)B+1)/((n+1)A) < V/U < D/C$.

We notice here that the fraction

$$((n+1)B+1)/((n+1)A) = (MB+AF)/(MA+AE) = (KB+AD)/(KA+AC)$$

may be reducible.

First, we consider Case (II.1). In this case we shall take $(B/A, F/E)$ as $(B'/A', B''/A'')$ in §4.2.

By the argument before Lemma 7, we have ξ_0 and ξ_1 in $\mathscr{S}_M(B/A, F/E)$ satisfying

(4.3.4) $$\xi_0 = (MB+jF)/(MA+jE) \leqslant V/U < \xi_1$$
$$= (MB+(j+1)F)/(MA+(j+1)E)$$

for some non-negative integer j. Then, by (4.2.5),

(4.3.5) $$V/U = \alpha_{X,Y} \quad \text{where } X = UF - VE,\ Y = VA - UB.$$

Lemma (II.1). *Let notations be as above.*

(1) *Assume that V/U lies on or above $\mathscr{S}_M(B/A, F/E)$ in* Fig $(B/A, F/E)$. *Then*

$$X \leqslant M.$$

(2) *Assume that V/U lies below $\mathscr{S}_M(B/A, F/E)$ in* Fig $(B/A, F/E)$ *(i.e., $X > M$). Then*

$$U > MA + (j+1)E.$$

(3) *We have the following:*

(II.1.0) $A-j-1 \geqslant 0$, $n+2-A+j > 0$.

(II.1.1) $(A-j-1)A_{-1} + (n+2-A+j)A = MA+(j+1)E$,
$(A-j-1)B_{-1} + (n+2-A+j)B = MB+(j+1)F-1$.

(II.1.2) $\{VA_{-1}\}_U = VA_{-1} - B_{-1}U$, $\{VA\}_U = VA - BU$.

(II.1.3) $(A-j-1)\{VA_{-1}\}_U + (n+2-A+j)\{VA\}_U < U$.

Proof. Replacing $(B'/A', B''/A'')$ by $(B/A, F/E)$, in Lemma 7, we get statements (1), (2). By the inequalities in Case (II.1) and (4.3.4), we have $0 \leqslant j \leqslant A-1$. Moreover, since $A \leqslant n$, we have (II.1.0). By (4.3.3), we get equalities (II.1.1). Since $B_{-1}/A_{-1} < V/U < F/E$ and $B/A < V/U < F/E$, we get (II.1.2) by Lemma 5 (1).

Now we shall show (II.1.3). Its left hand side is equal to

(4.3.6) $$(A-j-1)(VA_{-1}-B_{-1}U)+(n+2-A+j)(VA-BU)$$
$$= V((A-j-1)A_{-1}+(n+2-A+j)A)-U((A-j-1)B_{-1}+(n+2-A+j)B)$$
$$= U-[(MB+(j+1)F)U-(MA+(j+1)E)V]$$

by (II.1.1) and (II.1.2).

On the other hand, by (4.3.4), we have

(4.3.7) $$(MB+(j+1)F)U > (MA+(j+1)E)V.$$

By (4.3.6) and (4.3.7), we get (II.1.3). ∎

Second, we consider Case (II.2). By the argument before Lemma 7, we have ξ_0 and ξ_1 in $\mathscr{S}_K(B/A, D/C)$ satisfying

(4.3.8) $$\xi_0 = (KB+l_0D)/(KA+l_0C) \leqslant V/U < (KB+(l_0+1)D)/(KA+(l_0+1)C) = \xi_1$$

for some integer $l_0 \geqslant A$, uniquely determined by V/U. Now we define a positive integer i_0 by

(4.3.9) $$i_0 = \begin{cases} (l_0+1-A)/(n+1) & \text{if } l_0+1 \equiv A \pmod{n+1}, \\ [(l_0+1-A)/(n+1)]+1 & \text{if } l_0+1 \not\equiv A \pmod{n+1}. \end{cases}$$

We put $A_i = A+iC$ and $B_i = B+iD$ for each non-negative integer i. Then $A_iD-B_iC = 1$ for each i. Therefore each B_i/A_i is a Farey fraction and $B/A \leqslant B_i/A_i < D/C$ for each i. In particular, for $i = i_0$,

$$B/A \leqslant B_{i_0-1}/A_{i_0-1} < B_{i_0}/A_{i_0} < D/C.$$

Moreover,

$$A_{i_0}-(l_0+1)+i_0K \geqslant 0, \quad A_{i_0-1}-(l_0+1)+(i_0-1)K < 0,$$

because

$$A_{i_0}-(l_0+1)+i_0K = (n+1)(i_0-(l_0+1-A)/(n+1)) \geqslant 0$$

and

$$A_{i_0-1}-(l_0+1)+(i_0-1)K = (n+1)(i_0-1-(l_0+1-A)/(n+1))$$
$$= \begin{cases} -(n+1) < 0 & \text{if } l_0+1 \equiv A \pmod{n+1}, \\ (n+1)([(l_0+1-A)/(n+1)]-(l_0+1-A)/(n+1)) < 0 & \\ \qquad \text{if } l_0+1 \not\equiv A \pmod{n+1}. \end{cases}$$

Therefore

(4.3.10a) $$A_{i_0}-(l_0-i_0K)-1 \geqslant 0$$

and

(4.3.10b) $$(n+2-A_{i_0})+(l_0-i_0K) = -(A_{i_0-1}-(l_0+1)+(i_0-1)K) > 0.$$

Let $j_0 = l_0-i_0K$. Then, by (4.3.10b),

$$j_0 \geqslant A_{i_0}-(n+2)+1 = A+i_0C-(n+1) \geqslant A+C-(n+1).$$

Since $B/A < D/C$ are a successive pair in $\mathscr{F}_n$, we have $A+C \geqslant n+1$ by Lemma 1. Therefore $j_0 \geqslant 0$. We note that

(4.3.11) $$KB+l_0D = KB_{i_0}+j_0D, \quad KA+l_0C = KA_{i_0}+j_0C.$$

Therefore both ξ_0 and ξ_1 in (4.3.8) are in $\mathscr{S}_K(B_{i_0}/A_{i_0}, D/C)$. Three Farey fractions $B/A < B_{i_0}/A_{i_0} < D/C$ are terms in $\mathscr{F}_{\max(n, A_{i_0})}$ with $A_{i_0}D-B_{i_0}C = 1$.

Thus, for the three integers l_0, i_0 and j_0 defined above, using the two terms ξ_0 and ξ_1 in $\mathscr{S}_K(B_{i_0}/A_{i_0}, D/C)$, we have

(4.3.12) $$\xi_0 = (KB_{i_0}+j_0D)/(KA_{i_0}+j_0C) \leqslant V/U < (KB_{i_0}+(j_0+1)D)/(KA_{i_0}+(j_0+1)C) = \xi_1$$

by (4.3.8) and (4.3.11). In the following we take $(B_{i_0}/A_{i_0}, D/C)$ as $(B'/A', B''/A'')$ in §4.2. Then, by (4.2.5),

(4.3.13) $$V/U = \alpha_{X,Y} \quad \text{where } X = UD-VC,\ Y = VA_{i_0}-UB_{i_0}.$$

In Case (II.2), we look at (4.3.12) in place of (4.3.4) in Case (II.1). Then we can show the similar lemma to Lemma (II.1) by using the data $\{K, j_0, A_{i_0}, A_{i_0-1}, B_{i_0}, B_{i_0-1}, C, D\}$ in place of the data $\{M, j, A, A_{-1}, B, B_{-1}, E, F\}$ in Case (II.1).

LEMMA (II.2). *Let notations be as above.*

(1) *Assume that V/U lies on or above $\mathscr{S}_K(B_{i_0}/A_{i_0}, D/C)$ in* Fig $(B_{i_0}/A_{i_0}, D/C)$. *Then*

$$X \leqslant K.$$

(2) *Assume that V/U lies below $\mathscr{S}_K(B_{i_0}/A_{i_0}, D/C)$ in* Fig $(B_{i_0}/A_{i_0}, D/C)$ *(i.e., $X > K$). Then*

$$U > KA_{i_0}+(j_0+1)C.$$

(3) *We have the following:*

(II.2.0) $$A_{i_0}-j_0-1 \geqslant 0, \quad n+2-A_{i_0}+j_0 > 0.$$

(II.2.1) $$(A_{i_0}-j_0-1)A_{i_0-1}+(n+2-A_{i_0}+j_0)A_{i_0} = KA_{i_0}+(j_0+1)C,$$
$$(A_{i_0}-j_0-1)B_{i_0-1}+(n+2-A_{i_0}+j_0)B_{i_0} = KB_{i_0}+(j_0+1)D-1.$$

(II.2.2) $\{VA_{i_0-1}\}_U = VA_{i_0-1} - B_{i_0-1}U,$
$\{VA_{i_0}\}_U = VA_{i_0} - B_{i_0}U.$

(II.2.3) $(A_{i_0} - j_0 - 1)\{VA_{i_0-1}\}_U + (n+2-A_{i_0}+j_0)\{VA_{i_0}\}_U < U.$

Proof. Replacing $(B'/A', B''/A'')$ by $(B_{i_0}/A_{i_0}, D/C)$, in Lemma 7, we get statements (1), (2). By (4.3.10a) and (4.3.10b), we have (II.2.0). Since $A_{i_0-1} = A_{i_0} - C$ and $B_{i_0-1} = B_{i_0} - D$, we get (II.2.1). Since $B_{i_0-1}/A_{i_0-1} < V/U < D/C$ and $B_{i_0}/A_{i_0} < V/U < D/C$, we get (II.2.2) by Lemma 5 (1). As for (II.2.3), we write

$K, j_0, A_{i_0}, A_{i_0-1}, B_{i_0}, B_{i_0-1}, C, D$ for $M, j, A, A_{-1}, B, B_{-1}, E, F,$

respectively, in (4.3.6). Then, by (II.2.1) and (II.2.2), the left hand side of (II.2.3) is equal to

(4.3.14) $$U - [(KB_{i_0} + (j_0+1)D)U - (KA_{i_0} + (j_0+1)C)V].$$

On the other hand, by (4.3.12), we have

(4.3.15) $$(KB_{i_0} + (j_0+1)D)U > (KA_{i_0} + (j_0+1)C)V.$$

By (4.3.14) and (4.3.15), we get (II.2.3). ■

4.4. Proof. Now we shall prove the "only if" part, by using Lemmas I, (II.1), (II.2). We assume $N \geqslant n+2$ and p is a prime number with $p \nmid N$. In fact we shall show that either there exists $w = (w_0, w_1, \ldots, w_{n+1}) \in W_0$ (i.e., $\sum_{i=0}^{n+1} w_i = N$) with $p \cdot w \in W_0$ (i.e., $\sum_{i=0}^{n+1} \{pw_i\}_N = N$), or $\{p\} = \{p\}_N$ is of the form

$$\{p\} = (lN - m)/k$$

where $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant k-1$, $1 \leqslant m \leqslant n-k+1$ and $(k, l) = 1$ (cf. Proposition in Section 2).

We notice here that if $w = (w_0, w_1, \ldots, w_{n+1}) \in W_0$ and $\sum_{i=0}^{n} \{pw_i\}_N < N$ then $p \cdot w$ must belong to W_0.

Considering $\{p\}/N$ as a Farey fraction, we put $V = \{p\}$, $U = N$ and we apply Lemmas I, (II.1), (II.2).

(i) Case I: $0/1 < \{p\}/N < 1/n$ (i.e., $B/A = 0/1$ in (4.3.0)).

We apply Lemma I to $V/U = \{p\}/N$. We look at inequalities (4.3.2). If equality holds then $\{p\} = (N-1)/n$ (i.e., $k = n$, $l = 1$, $m = 1$). Otherwise we put

$$w = (1, \ldots, 1, yn+1, N-n-(yn+1)) \quad \text{with 1 repeated } n \text{ times.}$$

Then, by Lemma I (2), we have $w \in W_0$. Moreover, by Lemma I (2),

$$p \cdot w = (\{p\}, \ldots, \{p\}, \{\{p\}(yn+1)\}_N, N - n\{p\} - \{\{p\}(yn+1)\}_N) \in W_0,$$

because

$$n\{p\} + \{\{p\}(yn+1)\}_N = n\{p\} + \{p\}(yn+1) - yN = \{p\}((y+1)n+1) - yN$$
$$= N - ((y+1)N - ((y+1)n+1)\{p\}) < N.$$

Now assume $B/A \neq 0/1$ in (4.3.0).

(ii) Case (II.1): $B/A < \{p\}/N \leqslant ((n+1)B+1)/((n+1)A)$.

We look at inequalities (4.3.4). By (4.3.5), $X = NF - \{p\}E$. By Lemma (II.1), $X \leqslant M$ or $X > M$ where $M = n - E + 1$. If $X \leqslant M$ then, by Lemma (II.1) (1),

$$\{p\} = (FN - X)/E \quad \text{with } X \leqslant n - E + 1 \quad (\text{i.e., } k = E, l = F, m = X).$$

If $X > M$ then we put

(4.4.1) $$w = (A_{-1}, \ldots, A_{-1}, A, \ldots, A, N - (A-j-1)A_{-1} - (n+2-A+j)A)$$

with A_{-1} (resp. A) repeated $A-j-1$ (resp. $n+2-A+j$) times. Then, by Lemma (II.1) (2) (3), we have $w \in W_0$ and

(4.4.2) $$p \cdot w = (\{pA_{-1}\}_N, \ldots, \{pA_{-1}\}_N, \{pA\}_N, \ldots, \{pA\}_N,$$
$$N - (A-j-1)\{pA_{-1}\}_N - (n+2-A+j)\{pA\}_N) \in W_0.$$

Here we notice that $\{\{p\}A_{-1}\}_N = \{pA_{-1}\}_N$, $\{\{p\}A\}_N = \{pA\}_N$.

(iii) Case (II.2): $((n+1)B+1)/((n+1)A) < \{p\}/N < D/C$.

We look at inequalities (4.3.12). By (4.3.13), $X = ND - \{p\}C$. By Lemma (II.2), $X \leqslant K$ or $X > K$ where $K = n - C + 1$. If $X \leqslant K$ then, by Lemma (II.2) (1),

$$\{p\} = (DN - X)/C \quad \text{with } X \leqslant n - C + 1 \quad (\text{i.e., } k = C, l = D, m = X).$$

If $X > K$ then we put

(4.4.3) $$w = (A_{i_0-1}, \ldots, A_{i_0-1}, A_{i_0}, \ldots, A_{i_0},$$
$$N - (A_{i_0} - j_0 - 1)A_{i_0-1} - (n+2-A_{i_0}+j_0)A_{i_0})$$

with A_{i_0-1} (resp. A_{i_0}) repeated $A_{i_0} - j_0 - 1$ (resp. $n+2-A_{i_0}+j_0$) times. Then, by Lemma (II.2) (2) (3), we have $w \in W_0$ and

(4.4.4) $$p \cdot w = (\{pA_{i_0-1}\}_N, \ldots, \{pA_{i_0-1}\}_N, \{pA_{i_0}\}_N, \ldots, \{pA_{i_0}\}_N,$$
$$N - (A_{i_0} - j_0 - 1)\{pA_{i_0-1}\}_N - (n+2-A_{i_0}+j_0)\{pA_{i_0}\}_N) \in W_0.$$

Here we notice that $\{\{p\}A_{i_0-1}\}_N = \{pA_{i_0-1}\}_N$, $\{\{p\}A_{i_0}\} = \{pA_{i_0}\}_N$.

Thus we have completed the proof of the "only if" part in the Theorem.

4.5. Examples. In the case $N = n+2$ $(1 \leqslant n \leqslant 6)$, we see that if $\{p\}_N \neq 1$ then $\mathrm{HW}(F_{n,N,p}) = 0$, because

$n = 1$; $\{p\}_3 = 2 = N-1$.

$n = 2$; $\{p\}_4 = 3 = N-1$.

$n = 3$; $\{p\}_5 = 2 = (N-1)/2$,
or $\{p\}_5 = 3 = N-2$,
or $\{p\}_5 = 4 = N-1$.

$n = 4;\qquad \{p\}_6 = 5 = N-1.$

$n = 5;\qquad \{p\}_7 = 2 = (N-1)/3,$
or $\{p\}_7 = 3 = (N-1)/2,$
or $\{p\}_7 = 4 = N-3,$
or $\{p\}_7 = 5 = N-2,$
or $\{p\}_7 = 6 = N-1.$

$n = 6;\qquad \{p\}_8 = 3 = (N-2)/2,$
or $\{p\}_8 = 5 = (2N-1)/3,$
or $\{p\}_8 = 7 = N-1.$

In $N \geqslant n+2$ with $p \nmid N$, if $\{p\}_N = 1$ then

$$p\cdot w = (1, \ldots, 1, N-(n+1)) \in W_0 \quad \text{for } w = (1, \ldots, 1, N-(n+1)) \in W_0$$

with 1 repeated $n+1$ times.

In the following, we put up examples with "$N > n+2$" in which there exists w in W_0 with $p\cdot w \in W_0$, in two cases (II.1), (II.2), respectively.

(II.1) (i): $n = 5$; $N = 19$, $\{p\} = 7$. We have

$$B/A = 1/3 < \{p\}/N = 7/19 < D/C = 2/5,$$

$$((n+1)B+1)/((n+1)A) = 7/18 > 7/19 = \{p\}/N$$

and

$$a = [C/A] = [5/3] = 1,$$

$$D/C = F_1/E_1 < F_0/E_0 = F/E = 1/2 \quad \text{in } \mathscr{F}_5.$$

Moreover, $M = n-E+1 = 4$, $X = NF-\{p\}E = 5 > M$, and

(4.5.1) $\xi_0 = (MB+F)/(MA+E) = 5/14 < \{p\}/N < 6/16$
$= (MB+2F)/(MA+2E) = \xi_1.$

Hence $j = 1$ in (4.3.4). Applying Lemma 7 (2) to (4.5.1), we have

$$\xi_0 < \alpha_{3,1} = 4/11 < \{p\}/N < 3/8 = \alpha_{2,1} = \bar{\xi}_1.$$

In Lemma (II.1) (3),

$$A-j-1 = 1, \quad n+2-A+j = 5, \quad A_{-1} = 1 \quad (B_{-1} = 0), \quad A = 3 \quad (B = 1),$$

$$\{pA_{-1}\}_N = 7, \quad \{pA\}_N = 2.$$

Thus

$$w = (1, 3, 3, 3, 3, 3, 3) \in W_0 \quad \text{in (4.4.1)},$$

$$p\cdot w = (7, 2, 2, 2, 2, 2, 2) \in W_0 \quad \text{in (4.4.2)}.$$

(II.1) (ii): $n = 5$; $N = 15$, $\{p\} = 8$. We have

$$B/A = 1/2 < \{p\}/N = 8/15 < D/C = 3/5,$$

$$((n+1)B+1)/((n+1)A) = 7/12 > 8/15 = \{p\}/N$$

and

$$a = [C/A] = [5/2] = 2,$$

$$D/C = F_2/E_2 < F_1/E_1 = 2/3 < F_0/E_0 = F/E = 1/1 \quad \text{in } \mathscr{F}_5.$$

Moreover, $M = n-E+1 = 5$, $X = NF-\{p\}E = 7 > M$, and

(4.5.2) $\xi_0 = (MB)/(MA) = 5/10 < \{p\}/N < 6/11$
$= (MB+F)/(MA+E) = \xi_1.$

Hence $j = 0$ in (4.3.4). Applying Lemma 7 (2) to (4.5.2), we have

$$\bar{\xi}_0 = \alpha_{1,0} = 1/2 < \{p\}/N < 6/11 = \alpha_{M,1} = \xi_1.$$

In Lemma (II.1) (3),

$$A-j-1 = 1, \quad n+2-A+j = 5, \quad A_{-1} = 1 \quad (B_{-1} = 0), \quad A = 2 \quad (B = 1),$$

$$\{pA_{-1}\}_N = 8, \quad \{pA\}_N = 1.$$

Thus

$$w = (1, 2, 2, 2, 2, 2, 4) \in W_0 \quad \text{in (4.4.1)},$$

$$p\cdot w = (8, 1, 1, 1, 1, 1, 2) \in W_0 \quad \text{in (4.4.2)}.$$

(II.1) (iii): $n = 6$; $N = 51$, $\{p\} = 32$. We have

$$B/A = 3/5 < \{p\}/N = 32/51 < D/C = 2/3,$$

$$((n+1)B+1)/((n+1)A) = 22/35 > 32/51 = \{p\}/N$$

and

$$a = [C/A] = [3/5] = 0,$$

$$D/C = F/E = 2/3 \quad \text{in } \mathscr{F}_6.$$

Moreover, $M = n-E+1 = 4$, $X = NF-\{p\}E = 6 > M$, and

(4.5.3) $\xi_0 = (MB+4F)/(MA+4E) = 20/32 < \{p\}/N$
$< 22/35 = (MB+5F)/(MA+5E) = \xi_1.$

Hence $j = 4$ in (4.3.4). Applying Lemma 7 (2) to (4.5.3), we have

$$\bar{\xi}_0 = \alpha_{1,1} = 5/8 < \{p\}/N < 22/35 = \alpha_{4,5} = \xi_1.$$

In Lemma (II.1) (3),

$$A-j-1=0, \quad n+2-A+j=7, \quad A=5 \quad (B=3), \quad \{pA\}_N=7.$$

Thus

$$w=(5,5,5,5,5,5,5,16)\in W_0 \quad \text{in (4.4.1)},$$
$$p\cdot w=(7,7,7,7,7,7,7,2)\in W_0 \quad \text{in (4.4.2)}.$$

(II.1) (iv): $n=7$; $N=45$, $\{p\}=26$. We have

$$B/A=4/7<\{p\}/N=26/45<D/C=3/5,$$
$$((n+1)B+1)/((n+1)A)=33/56>26/45=\{p\}/N$$

and

$$a=[C/A]=[5/7]=0,$$
$$D/C=F/E=3/5 \quad \text{in } \mathscr{F}_7.,$$

Moreover, $M=n-E+1=3$, $X=NF-\{p\}E=5>M$, and

(4.5.4) $$\xi_0=(MB+F)/(MA+E)=15/26<\{p\}/N<18/31 =(MB+2F)/(MA+2E)=\xi_1.$$

Hence $j=1$ in (4.3.4). Applying Lemma 7 (2) to (4.5.4), we have

$$\xi_0=\alpha_{3,1}=15/26<\{p\}/N<11/19=\alpha_{2,1}<\xi_1.$$

In Lemma (II.1) (3),

$$A-j-1=5, \quad n+2-A+j=3, \quad A_{-1}=2 \quad (B_{-1}=1), \quad A=7 \quad (B=4),$$
$$\{pA_{-1}\}_N=7, \quad \{pA\}_N=2.$$

Thus

$$w=(2,2,2,2,2,7,7,7,14)\in W_0 \quad \text{in (4.4.1)},$$
$$p\cdot w=(7,7,7,7,7,2,2,2,4)\in W_0 \quad \text{in (4.4.2)}.$$

(II.2) (i): $n=5$; $N=29$, $\{p\}=17$. We have

$$B/A=1/2<\{p\}/N=17/29<D/C=3/5,$$
$$((n+1)B+1)/((n+1)A)=7/12<17/29=\{p\}/N.$$

Moreover, $K=n-C+1=1$, $X=ND-\{p\}C=2>K$, and

(4.5.5) $$\xi_0=(KB+2D)/(KA+2C)=7/12<\{p\}/N<10/17 =(KB+3D)/(KA+3C)=\xi_1.$$

Hence $l_0=2=A$ in (4.3.8). By $l_0+1-A=1\not\equiv 0 \pmod{n+1}$, we have

$$i_0=[(l_0+1-A)/(n+1)]+1=1, \quad j_0=l_0-i_0K=1.$$

Applying Lemma 7 (2) to (4.5.5), we have

$$\xi_0=(KB_1+D)/(KA_1+C)=\alpha_{1,1}<\{p\}/N<\alpha_{1,2} =(KB_1+2D)/(KA_1+2C)=\xi_1.$$

In Lemma (II.2) (3),

$$A_{i_0}-j_0-1=5, \quad n+2-A_{i_0}+j_0=1, \quad A_{i_0-1}=2 \quad (B_{i_0-1}=1),$$
$$A_{i_0}=7 \quad (B_{i_0}=4), \quad \{pA_{i_0-1}\}_N=5, \quad \{pA_{i_0}\}_N=3.$$

Thus

$$w=(2,2,2,2,2,7,12)\in W_0 \quad \text{in (4.4.3)},$$
$$p\cdot w=(5,5,5,5,5,3,1)\in W_0 \quad \text{in (4.4.4)}.$$

(II.2) (ii): $n=6$; $N=97$, $\{p\}=63$. We have

$$B/A=3/5<\{p\}/N=63/97<D/C=2/3,$$
$$((n+1)B+1)/((n+1)A)=22/35<63/97=\{p\}/N.$$

Moreover, $K=n-C+1=4$, $X=ND-\{p\}C=5>K$, and

(4.5.6) $$\xi_0=(KB+19D)/(KA+19C)=50/77<\{p\}/N<52/80 =(KB+20D)/(KA+20C)=\xi_1.$$

Hence $l_0=19>A$ in (4.3.8). By $l_0+1-A=15\not\equiv 0 \pmod{n+1}$,

$$i_0=[(l_0+1-A)/(n+1)]+1=3, \quad j_0=l_0-i_0K=7.$$

Applying Lemma 7 (2) to (4.5.6), we have

$$\xi_0=(KB_3+7D)/(KA_3+7C)=\alpha_{4,7}<\{p\}/N<\alpha_{1,2} =(B_3+2D)/(A_3+2C)=\xi_1.$$

In Lemma (II.2) (3),

$$A_{i_0}-j_0-1=6, \quad n+2-A_{i_0}+j_0=1, \quad A_{i_0-1}=11 \quad (B_{i_0-1}=7),$$
$$A_{i_0}=14 \quad (B_{i_0}=9), \quad \{pA_{i_0-1}\}_N=14, \quad \{pA_{i_0}\}_N=9.$$

Thus

$$w=(11,11,11,11,11,11,14,17)\in W_0 \quad \text{in (4.4.3)},$$
$$p\cdot w=(14,14,14,14,14,14,9,4)\in W_0 \quad \text{in (4.4.4)}.$$

(II.2) (iii): $n=7$; $N=80$, $\{p\}=31$. We have

$$B/A=1/3<\{p\}/N=31/80<D/C=2/5,$$
$$((n+1)B+1)/((n+1)A)=9/24<31/80=\{p\}/N.$$

Moreover, $K = n-C+1 = 3$, $X = ND-\{p\}C = 5 > K$, and

(4.5.7) $$\xi_0 = (KB+7D)/(KA+7C) = 17/44 < \{p\}/N < 19/49 = (KB+8D)/(KA+8C) = \xi_1.$$

Hence $l_0 = 7 > A$ in (4.3.8). By $l_0+1-A = 5 \not\equiv 0 \pmod{n+1}$,

$$i_0 = [(l_0+1-A)/(n+1)]+1 = 1, \quad j_0 = l_0 - i_0 K = 4.$$

Applying Lemma 7 (2) to (4.5.7), we have

$$\xi_0 = (KB_1+4D)/(KA_1+4C) < \alpha_{2,3} < \{p\}/N < \alpha_{3,5} = (KB_1+5D)/(KA_1+5C) = \xi_1.$$

In Lemma (II.2) (3),

$$A_{i_0} - j_0 - 1 = 3, \quad n+2-A_{i_0}+j_0 = 5, \quad A_{i_0-1} = 3 \quad (B_{i_0-1} = 1),$$

$$A_{i_0} = 8 \quad (B_{i_0} = 3), \quad \{pA_{i_0-1}\}_N = 13, \quad \{pA_{i_0}\}_N = 8.$$

Thus

$$w = (3, 3, 3, 8, 8, 8, 8, 8, 31) \in W_0 \quad \text{in (4.4.3)},$$

$$p \cdot w = (13, 13, 13, 8, 8, 8, 8, 8, 1) \in W_0 \quad \text{in (4.4.4)}.$$

(II.2) (iv): $n = 7$; $N = 189$, $\{p\} = 124$. We have

$$B/A = 3/5 < \{p\}/N = 124/189 < D/C = 2/3,$$

$$((n+1)B+1)/((n+1)A) = 25/40 < 124/189 = \{p\}/N.$$

Moreover, $K = n-C+1 = 5$, $X = ND-\{p\}C = 6 > K$, and

(4.5.8) $$\xi_0 = (KB+44D)/(KA+44C) = 103/157 < \{p\}/N < 105/160 = (KB+45D)/(KA+45C) = \xi_1.$$

Hence $l_0 = 44 > A$ in (4.3.8). By $l_0+1-A = 40 \equiv 0 \pmod{n+1}$,

$$i_0 = (l_0+1-A)/(n+1) = 5, \quad j_0 = l_0 - i_0 K = 19.$$

Applying Lemma 7 (2) to (4.5.8), we have

$$\xi_0 = (KB_5+19D)/(KA_5+19C) = \alpha_{5,19} < \{p\}/N < \alpha_{1,4} = (B_5+4D)/(A_5+4C) = \xi_1.$$

In Lemma (II.2) (3),

$$A_{i_0} - j_0 - 1 = 0, \quad n+2-A_{i_0}+j_0 = 8, \quad A_{i_0} = 20 \quad (B_{i_0} = 13),$$

$$\{pA_{i_0}\}_N = 23.$$

Thus

$$w = (20, 20, 20, 20, 20, 20, 20, 20, 29) \in W_0 \quad \text{in (4.4.3)},$$

$$p \cdot w = (23, 23, 23, 23, 23, 23, 23, 23, 5) \in W_0 \quad \text{in (4.4.4)}.$$

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