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On some sums involving the largest prime divisor of n

by

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1. Introduction. Using analytic methods, R. Balasubramanian and K. Ramachandra proved in [1] that

$$\Sigma_g(x) = \sum_{ng(n) \leqslant x} 1 \sim Cx(\log x)^{\lambda-1} \quad \text{as } x \to \infty \tag{1.1}$$

for a class of positive multiplicative functions g satisfying

$$\begin{cases} g(p) = 1/\lambda & \text{for all primes } p, \\ g(n) \gg n^{-1/16} & \text{for all positive integers } n. \end{cases} \tag{1.2}$$

In fact they obtained an asymptotic expansion of the form

$$\Sigma_g(x) = x(\log x)^{\lambda-1} \sum_{n \leqslant m \leqslant (\log x)^{4/5}} A_{m,n}(\log x)^{-m}(\log\log x)^n + O\big(x\exp(-A(\log x)^{3/5}(\log\log x)^{-1/5})\big). \tag{1.3}$$

This class of functions g includes the divisor function $d(n)$, when $\lambda = 1/2$, and its reciprocal, when $\lambda = 2$. In the final section of their paper, they remark that a similar result, but with a weaker exponential error term in some cases, can be obtained when the first condition in (1.2) is relaxed to

$$g(p) = 1/\lambda + O\big(\exp(-c(\log p)^a)\big),$$

$c > 0$ and $a \geqslant 1$ being constants. They asserted that, to establish this when $1 \leqslant a \leqslant 3/2$, the contour used to derive (1.3) should be replaced by a modification of the one used by P. T. Bateman, in his method C of [3], to prove that for any fixed $\varepsilon > 0$

$$\sum_{\varphi(n) \leqslant x} 1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} x + O\big(x\exp(-(1-\varepsilon)(\tfrac{1}{2}\log x \log\log x)^{1/2})\big), \tag{1.4}$$

where φ denotes Euler's function; an elementary proof of (1.4) has been given recently in [2], and similar sums for other multiplicative functions in a certain class are considered in [17]. When $\lambda = 1$, method C in [3] can be applied directly to estimate $\Sigma_g(x)$; see Theorem 7 in Section 8 below.

The functions g above can be regarded as 'small' multiplicative functions, in the sense that neither $g(n)$ nor $1/g(n)$ is very large too often. It is of interest to ask what happens when g is a 'large' multiplicative function (in a similar sense), such as a positive or negative fixed power of $\varphi(n)$ or $\sigma_\nu(n)$, where $\sigma_\nu(n) = \sum_{d|n} d^\nu$ and $\nu > 0$. In Section 8, we describe a procedure that enables us to deduce an estimate for $\Sigma_f(x)$ for certain 'large' positive multiplicative functions f from results analogous to (1.4), (1.1) or (1.3). In particular, we consider the functions

$$f(n) = (\varphi(n))^\gamma \quad \text{for } \gamma > -1, \qquad f(n) = (\sigma_\nu(n))^\gamma \quad \text{for } \gamma > -1/\nu$$

where $\nu > 0$, $\gamma \neq 0$.

The primary objective of this paper, however, is to investigate the sum $\Sigma_g(x)$ for positive functions $g(n)$ that are not multiplicative but which depend in some way on the factorization of n. We concentrate on some functions defined in terms of

$$P(n) = \max_{p|n} p, \tag{1.5}$$

but we also consider certain familiar additive functions. Obtaining an estimate for the sum $\Sigma_g(x)$ can be regarded as a way of quantifying how often $g(n)$, or $1/g(n)$ if $g(n) < 1$, is large for large n.

Our main results provide an asymptotic formula for $\Sigma_g(x)$ when $g(n)$ is a positive or negative power of $P(n)$ or of $\log P(n)$; these are 'large' and 'small' functions, respectively, in the sense described in the second paragraph. Let $\varrho(u)$ denote the well known de Bruijn–Dickman function defined by the differential-difference equation

$$\varrho(u) = 1 \quad (0 < u \leqslant 1), \qquad \varrho(u-1) = -u\varrho'(u) \quad (u > 1). \tag{1.6}$$

We prove the following results:

THEOREM 1. *Let γ be a fixed positive or negative real number. Then*

$$S_\gamma(x) = \sum_{\substack{n>1 \\ n(\log P(n))^\gamma \leqslant x}} 1 = A_\gamma x (\log x)^{-\gamma}\left(1 + \frac{\gamma^2 \log\log x + O(1)}{\log x}\right) \tag{1.7}$$

where

$$A_\gamma = \int_1^\infty u^{\gamma-1}\varrho(u-1)\,du. \tag{1.8}$$

THEOREM 2. *Let $\gamma > 0$ be fixed, and define*

$$T_\gamma(x) = \sum_{\substack{n>1 \\ n(P(n))^\gamma \leqslant x}} 1, \qquad I_\gamma(x) = \gamma \int_2^{x^{1/\gamma}} t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t} - \gamma\right)dt. \tag{1.9}$$

Then $T_\gamma(x)$ can be expressed in either of the forms

$$\text{(i)} \quad T_\gamma(x) = x\exp\left\{-(2\gamma \log x \log_2 x)^{1/2} \times \left(1 + k_\gamma(x) + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right)\right)\right\} \tag{1.10}$$

where $\log_2 x = \log(\log x)$, $\log_3 x = \log(\log_2 x)$ and

$$k_\gamma(x) = \frac{(\log_3 x + \log(\gamma/2) - 2)}{2\log_2 x}\left(1 + \frac{2}{\log_2 x}\right) - \frac{1}{8}\left(\frac{\log_3 x + \log(\gamma/2)}{\log_2 x}\right)^2; \tag{1.11}$$

$$\text{(ii)} \quad T_\gamma(x) = x I_\gamma(x)\left(1 + O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right). \tag{1.12}$$

The asymptotic formula (1.12) is more precise but less illuminative than (1.10).

THEOREM 3. *Let γ satisfying $-1 < \gamma < 0$ and the positive integer R be fixed. Then the sum $T_\gamma(x)$ defined by (1.9) satisfies*

$$T_\gamma(x) = \frac{x^{1/(1+\gamma)}}{\log x}\left\{(1+\gamma)\zeta\left(\frac{1}{1+\gamma}\right) + \sum_{r=1}^{R-1} A_r (\log x)^{-r} + O((\log x)^{-R})\right\}$$

where A_r, $r = 1, 2, \ldots, R-1$, are certain constants.

The restriction $\gamma > -1$ in Theorem 3 is a natural one, for the sum $T_\gamma(x)$ is undefined when $\gamma = -1$.

The shape of the results of Theorems 2 and 3 are very different, as are the details of their proofs. A key step in establishing all three theorems is to pick out the range of values of $P(n)$ that yields the dominant term in the result; for Theorems 1 and 3, this range contains the largest possible values of $P(n)$ whereas, for Theorem 2, the salient range for $P(n)$ is a relatively small interval containing the point $\exp((1/2\gamma \log x \log_2 x)^{1/2})$. The methods used to prove the two parts of Theorem 2 are similar to those used to estimate the sum $\sum_{1<n\leqslant x}(P(n))^{-\gamma}$ in [12] (for any $\gamma > 0$) and in [7] (for $\gamma = 1$), respectively, so we just give the key steps (see Sections 4, 5) and omit some of the computational details. The proof of Theorem 1 is given in Section 3; by the same method, one can prove that

$$\sum_{1<n\leqslant x} (\log P(n))^{-\gamma} = A_\gamma x (\log x)^{-\gamma}(1 + O(1/\log x)) \tag{1.13}$$

where A_γ is given by (1.8). This result is the case $u = 1$ of Theorem 9 in [18], where it is shown that

$$\sum_{\substack{1<n\leqslant x \\ P(n/P(n)) \leqslant (P(n))^{1/u}}} (\log P(n))^{-\gamma} = u^{-\gamma} f(u) x (\log x)^{-\gamma}(1 + O(1/\log x));$$

here $P(1) = 1$, $u \geq 1$, and $f(u)$ satisfies the differential-difference equation

$$(uf(u))' = (1+\gamma)f(u) - f(u+1).$$

The case $\gamma = -1$ was considered much earlier in [4].

In [1] and Section 8 of this paper, the sum $\Sigma_g(x)$ is investigated for certain multiplicative functions g. An obvious associated question to pose is that of estimating $\Sigma_g(x)$ when g is a 'small' or 'large' additive function. In Section 7, we consider the following specific additive functions, and also the reciprocals of the functions in (a):

(a) $\omega(n)$, $\Omega(n)$; (b) $\beta(n)$, $B(n)$

where, as usual, $\omega(n)$, $\Omega(n)$ denote the number of prime divisors of n counted without and with multiplicity, respectively, and $\beta(n) = \sum_{p|n} p$, $B(n) = \sum_{p^\alpha \| n} \alpha p$ are their 'large' analogues, studied, for example, in [12].

The authors of [1] remark that for positive multiplicative functions g satisfying (1.2)

$$\sum_{ng(n) \leq x} 1 \sim \sum_{n \leq x} 1/g(n) \quad \text{as } x \to \infty. \tag{1.14}$$

It is interesting to note that (1.14) holds also for the 'small' functions considered in this paper but not for the 'large' ones involving $P(n)$, $\varphi(n)$, $\sigma_\nu(n)$. For the function $(\log P(n))^\gamma$, we have only to compare (1.13) with (1.7) to obtain the result

$$\sum_{\substack{n>1 \\ n(\log P(n))^\gamma \leq x}} 1 - \sum_{1<n\leq x} (\log P(n))^{-\gamma} = \gamma^2 A_\gamma x \frac{(\log\log x + O(1))}{(\log x)^{\gamma+1}},$$

which is stronger than (1.14). For the additive functions ω, Ω and their reciprocals, (1.14) follows from Theorem 6 in Section 7 together with [13] and Theorem 430 of [8]. However, for the 'large' function $g(n) = (P(n))^\gamma$, we are able to obtain an asymptotic formula for the ratio

$$T_\gamma(x) / \sum_{1<n\leq x} P(n)^{-\gamma} \tag{1.15}$$

as $x \to \infty$ which shows that (1.14) is false; see Theorem 4 in Section 5 when $\gamma > 0$ and (6.1) for $-1 < \gamma < 0$. This result is particularly interesting when $\gamma > 0$; for the approximate formulae in Theorem 2 (i) and (4.2) take the same form, and the ratio (1.15) is quite small, especially when compared with $T_\gamma(x)$, although it tends to infinity as $x \to \infty$. For the 'large' multiplicative functions in (8.1), one could also easily obtain an asymptotic formula for the ratio corresponding to (1.15), using the results of Corollaries 3 and 4 of Theorem 7. However, one can deduce that (1.14) is false for these functions simply by noting that the pairs of sums

$$\sum_{n\leq x} (\varphi(n))^{-\gamma} \text{ and } \sum_{n\leq x} n^{-\gamma}; \quad \sum_{n\leq x} (\sigma_\nu(n))^{-\gamma} \text{ and } \sum_{n\leq x} n^{-\nu\gamma}$$

each differ by a factor that is $O(x^\varepsilon)$ for every $\varepsilon > 0$, and appealing to (8.5). The formula (1.14) does hold for the 'small' function $(\sigma_\nu(n))^\gamma$, $\nu < 0$, for each $\gamma \neq 0$.

The discussion in the last paragraph raises the interesting question of establishing some general conditions on g under which (1.14) holds for functions g that are not necessarily of the types considered in this paper. When $g(n)$ is a suitable function of $P(n)$, such conditions might be derived using recent results, such as those in [10], for the function $\psi(x, y)$ defined in (2.1). However, for a general function g, there seems no obvious way to attack this problem.

The author thanks the Referee for the observation that the method used to prove Theorem 6 is applicable not only to the specific functions ω and Ω but also to a general class of additive functions.

2. Some preliminary results. Throughout this paper, p denotes a prime, and $P(n)$ is given by (1.5).

The proofs of Theorems 1, 2, 3 depend on properties of the well known function

$$\psi(x, y) = \sum_{\substack{n\leq x \\ P(n)\leq y}} 1 \tag{2.1}$$

whose asymptotic behaviour for y large enough in terms of x is expressed in terms of the function $\varrho(u)$ defined in (1.6).

LEMMA 1. *Let $\varepsilon > 0$ be fixed. Uniformly for y satisfying $(\log_2 x)^{5/3+\varepsilon} \leq \log y \leq \log x$ $(x \geq 3)$,*

$$\psi(x, y) = x\varrho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right) \quad \textit{where } u = \frac{\log x}{\log y}. \tag{2.2}$$

For this range of y, (2.2) is established in [9]. The asymptotic behaviour of $\varrho(u)$ was derived in [5]:

LEMMA 2. *As $u \to \infty$,*

$$\varrho(u) = \exp\left\{-u\left(\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\left(\left(\frac{\log_2 u}{\log u}\right)^2\right)\right)\right\}. \tag{2.3}$$

From Lemma 2, we immediately deduce:

LEMMA 3. *For each fixed γ, δ $(\delta \geq 0)$, the integral*

$$\int_1^\infty u^{\gamma-1}(\log u)^\delta \varrho(u-1)\,du$$

converges to a positive constant $A_{\gamma,\delta}$, and moreover as $U \to \infty$,

$$\int_U^\infty u^{\gamma-1}(\log u)^\delta \varrho(u-1)\,du < \exp\left\{-U\log U\left(1 + O\left(\frac{\log_2 U}{\log U}\right)\right)\right\}.$$

The next two lemmas follow from the Prime Number Theorem in the form

$$\theta(y) = \sum_{p \le y} \log p = y + O(y\exp(-c\sqrt{\log y}))$$

on using partial summation. In Lemma 4, γ may be positive or negative, but $\gamma \ne 0$.

LEMMA 4. *For a suitable constant $c > 0$ and some constant C_γ,*

$$\sum_{p \le y} \frac{1}{p(\log p)^\gamma} = C_\gamma - \frac{1}{\gamma(\log y)^\gamma} + O(\exp(-c\sqrt{\log y})).$$

LEMMA 5. *Let $\gamma > 0$. For a suitable constant $c > 0$ and some constant C_γ,*

$$\sum_{p \le y} p^{-\gamma-1} = \int_2^y \frac{1}{t^{\gamma+1}\log t}\,dt + C_\gamma + O\left(\frac{\exp(-c\sqrt{\log y})}{y^\gamma \log y}\right).$$

As usual, let $\pi(y) = \sum_{p \le y} 1$.

LEMMA 6. *Let $\beta > 1$ be a constant, and suppose that m is an arbitrary positive integer satisfying $m \le x^{\beta/(\beta+1)}$. For any fixed positive integer R,*

$$\pi\left(\left(\frac{x}{m}\right)^\beta\right) = \frac{(x/m)^\beta}{\beta\log x}\left\{1 + \sum_{k=1}^{R-1} (\log x)^{-k} \sum_{l=0}^{k} C_{k,l}(\log m)^l + O\left(\left(\frac{\log m}{\log x}\right)^R\right)\right\} \tag{2.4}$$

for certain constants $C_{k,l}$.

Proof. By the Prime Number Theorem, there are constants B_r, with $B_0 = 1$, such that

$$\pi(y) = y\left\{\sum_{r=0}^{R-1} B_r(\log y)^{-r-1} + O((\log y)^{-R-1})\right\}. \tag{2.5}$$

Since $0 \le \log m/\log x \le \beta/(\beta+1) < 1$, we can expand

$$(\log(x/m)^\beta)^{-r-1} = (\beta\log x)^{-r-1}(1 - \log m/\log x)^{-r-1}$$

by the Binomial Theorem to deduce that

$$\sum_{r=0}^{R-1} B_r(\log(x/m)^\beta)^{-r-1} + O((\log(x/m)^\beta)^{-R-1}) \tag{2.6}$$

$$= \sum_{k=0}^{R-1} (\beta\log x)^{-k-1} \sum_{l=0}^{k} \binom{k}{l} B_{k-l}(\beta\log m)^l + O((\log m)^R(\log x)^{-R-1}).$$

The result of (2.4) now follows from (2.5) and (2.6) since $B_0 = 1$.

LEMMA 7. *Let β, R be as in Lemma 6. For each integer l, $0 \le l < R$, and a sufficiently small $\varepsilon > 0$,*

$$\sum_{m \le x^{\beta/(\beta+1)}} m^{-\beta}(\log m)^l = D_l + O(x^{-\beta(\beta-1)/(\beta+1)+\varepsilon})$$

where $D_0 = \zeta(\beta)$ and, for each l, D_l is a constant.

Proof. Since $\beta > 1$, the infinite series $\sum_{m=1}^{\infty} m^{-\beta}(\log m)^l$ converges, to D_l (say), and, for $0 \le l < R$,

$$\sum_{m > x^{\beta/(\beta+1)}} m^{-\beta}(\log m)^l \ll \int_{x^{\beta/(\beta+1)}}^{\infty} u^{-\beta}(\log u)^l\,du \ll x^{-\beta(\beta-1)/(\beta+1)+\varepsilon}$$

for any sufficiently small $\varepsilon > 0$ and x sufficiently large.

3. Proof of Theorem 1. Using the definition (2.1) of $\psi(x, y)$, we have (on writing $n = mp$, $p = P(n)$)

$$S_\gamma(x) = \sum_{\substack{n>1 \\ n(\log P(n))^\gamma \le x}} 1 = \sum_{p\log^\gamma p \le x} \sum_{\substack{m \le x/p\log^\gamma p \\ P(m) \le p}} 1 \tag{3.1}$$

$$= \sum_{p\log^\gamma p \le x} \psi\left(\frac{x}{p\log^\gamma p}, p\right) = \sum_{p \le X} \psi\left(\frac{x}{p\log^\gamma p}, p\right)$$

where $X = X(x)$ is defined for large x by

$$X\log^\gamma X = x. \tag{3.2}$$

We split the range for p at the point Y given by

$$\log Y = (\log x)^{2/3}, \quad \text{so} \quad \log\frac{\log x}{\log Y} = \tfrac{1}{3}\log_2 x, \tag{3.3}$$

and use Lemmas 1 and 2 to estimate the resulting two parts of the sum in (3.1).

LEMMA 8. *As $x \to \infty$*

$$\sum_{p \le Y} \psi\left(\frac{x}{p\log^\gamma p}, p\right) \ll x\exp(-(\log x)^{1/3}).$$

Proof. For $p \le Y$, $\psi(x/p\log^\gamma p, p) \le \psi(x/p\log^\gamma p, Y)$, and by Lemma 1

$$\psi\left(\frac{x}{p\log^\gamma p}, Y\right) = \frac{x}{p\log^\gamma p}\varrho\left(\frac{\log(x/p\log^\gamma p)}{\log Y}\right)\left(1 + O\left((\log Y)^{-1}\log\frac{\log x}{\log Y}\right)\right)$$

$$= \frac{x}{p\log^\gamma p}\varrho((\log x)^{1/3} + O(1))(1 + O((\log x)^{-2/3}\log_2 x))$$

by (3.3). Hence by Lemmas 2 and 4

$$\sum_{p \le Y} \psi\left(\frac{x}{p\log^\gamma p}, p\right) \ll x\exp\left\{-\tfrac{1}{3}\log^{1/3} x\log_2 x\left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\right\},$$

from which Lemma 8 follows.

LEMMA 9. *Let $\lambda(u) = \varrho(u-1)(\log u)^\delta$; for any fixed γ and non-negative δ,*

$$\sum_{Y < p \le X} \frac{1}{p\log^\gamma p}\lambda\left(\frac{\log x - \gamma\log_2 p}{\log p}\right) = A_{\gamma,\delta}\left(1 + \frac{\gamma^2\log_2 x}{\log x} + O\left(\frac{1}{\log x}\right)\right)\log^{-\gamma} x$$

where $A_{\gamma,\delta}$ is defined in Lemma 3.

Proof. Using partial summation and Lemma 4, we obtain, since $\varrho(u)$ is differentiable, decreasing and bounded,

$$\sum_{Y<p\leqslant X}\frac{1}{p\log^{\gamma}p}\lambda\left(\frac{\log x-\gamma\log_2 p}{\log p}\right) = \int_Y^X \frac{1}{t\log^{\gamma+1}t}\lambda\left(\frac{\log x-\gamma\log_2 t}{\log t}\right)dt + O(\exp(-c\sqrt{\log Y})).$$

In the integral, we use the substitution $u=(\log x-\gamma\log_2 t)/\log t$; note that $u=1$ when $t=X$, by (3.2), and when $t=Y$,

$$u=\log^{1/3}x-\tfrac{2}{3}\gamma\log_2 x(\log x)^{-2/3}=U \quad \text{(say)}. \tag{3.4}$$

We also find that

$$\frac{du}{dt}=-\frac{u\log t+\gamma}{t\log^2 t}$$

and, for $Y\leqslant t\leqslant X$, that

$$u\log t=\left\{1-\frac{\gamma(\log_2 x-\log u)}{\log x}+O\left(\frac{\log_2 x}{\log^2 x}\right)\right\}\log x.$$

Hence

$$\int_Y^X \frac{1}{t\log^{\gamma+1}t}\lambda\left(\frac{\log x-\gamma\log_2 t}{\log t}\right)dt$$
$$=\int_1^U \lambda(u)\left(\frac{u}{\log x}\right)^{\gamma-1}\left\{1-\frac{\gamma(\log_2 x-\log u)}{\log x}+O\left(\frac{\log_2 x}{\log^2 x}\right)\right\}^{-\gamma+1}$$
$$\times\frac{1}{\log x}\left\{1-\gamma\frac{(\log_2 x-\log u-1)}{\log x}+O\left(\frac{\log_2 x}{\log^2 x}\right)\right\}^{-1}du$$
$$=\log^{-\gamma}x\int_1^U \lambda(u)u^{\gamma-1}\left\{1+\frac{\gamma^2(\log_2 x-\log u)-\gamma}{\log x}+O\left(\left(\frac{\log_2 x}{\log x}\right)^2\right)\right\}du$$
$$=(\log x)^{-\gamma}\left\{1+\frac{\gamma^2\log_2 x-\gamma}{\log x}+O\left(\left(\frac{\log_2 x}{\log x}\right)^2\right)\right\}A_{\gamma,\delta}-(\log x)^{-\gamma-1}\gamma^2A_{\gamma,\delta+1}$$
$$+O\left((\log x)^{-\gamma}\exp\left\{-U\log U\left(1+O\left(\frac{\log_2 U}{\log U}\right)\right)\right\}\right)$$

by Lemma 3 and since $\log_2 x-\log u=O(\log_2 x)$ for $1\leqslant u\leqslant U$. By (3.4),

$$U\log U\left(1+O\left(\frac{\log_2 U}{\log U}\right)\right)=\frac{1}{3}\log^{1/3}x\log_2 x\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right).$$

Hence it follows from above that

$$\sum_{Y<p\leqslant X}\frac{1}{p\log^{\gamma}p}\lambda\left(\frac{\log x-\gamma\log_2 p}{\log p}\right)$$
$$=A_{\gamma,\delta}(\log x)^{-\gamma}\left(1+\frac{\gamma^2\log_2 x-\gamma}{\log x}\right)-\gamma^2A_{\gamma,\delta+1}(\log x)^{-\gamma-1}$$
$$+O\left(\frac{(\log_2 x)^2}{(\log x)^{\gamma+2}}+\exp(-c(\log x)^{1/3})\right)$$

which yields a result that is slightly sharper than the one stated in Lemma 9, but we do not need to use this stronger result.

LEMMA 10.

$$\sum_{Y<p\leqslant X}\psi\left(\frac{x}{p\log^{\gamma}p},\,p\right)=A_{\gamma,0}x\log^{-\gamma}x\left(1+\frac{\gamma^2\log_2 x+O(1)}{\log x}\right).$$

Proof. We deduce from Lemma 1 that

$$\sum_{Y<p\leqslant X}\psi\left(\frac{x}{p\log^{\gamma}p},\,p\right)$$
$$=x\sum_{Y<p\leqslant X}\frac{1}{p\log^{\gamma}p}\varrho\left(\frac{\log x-\gamma\log_2 p}{\log p}-1\right)\left(1+O\left(\frac{\log\left(\dfrac{\log x-\gamma\log_2 p}{\log p}\right)}{\log p}\right)\right).$$

For note that when $x/p\log^{\gamma}p<p\leqslant X$, Lemma 1 does not apply but $\psi(x/p\log^{\gamma}p,\,p)=[x/p\log^{\gamma}p]$, and moreover this range for p makes essentially the same contribution to the sums on the left and the right since the discrepancy $O(\pi(X))$ can be absorbed into the error term on the right. Estimates for the main term and for the error term are given by Lemma 9 with $\delta=0,\,1$, and the result of Lemma 10 follows.

From (3.1) and Lemmas 8 and 10, we deduce that

$$S_{\gamma}(x)=A_{\gamma,0}x\log^{-\gamma}x\left(1+\frac{\gamma^2\log_2 x+O(1)}{\log x}\right)$$

where

$$A_{\gamma,0}=\int_1^{\infty}u^{\gamma-1}\varrho(u-1)du,$$

which completes the proof of Theorem 1.

We observe that the error term in Lemma 10 arises directly from the error term in Lemma 1, which consequently imposes a limitation on the sharpness of the error term in Theorem 1. Some other approximation for $\psi(x,y)$, such as that used in [10] or in [16], might lead to an improvement in the error term of Lemma 10 and hence of Theorem 1.

4. Proof of Theorem 2(i). For this section and the next, assume that $\gamma > 0$. Using (2.1) and the argument in (3.1),

$$T_\gamma(x) = \sum_{\substack{n>1 \\ n(P(n))^\gamma \leqslant x}} 1 = \sum_{p \leqslant x^{1/(\gamma+1)}} \psi(x/p^{\gamma+1}, p). \tag{4.1}$$

We establish Theorem 2(i) by adopting the method used in [12] to prove that

$$\sum_{2 \leqslant n \leqslant x} (P(n))^{-\gamma} = x\exp\left\{-(2\gamma\log x\log_2 x)^{1/2}\left(1+k_\gamma(x)+O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right)\right)\right\} \tag{4.2}$$

where $k_\gamma(x)$ is given in (1.11). Note that the sum on the left of (4.2) may be written as

$$\sum_{p \leqslant x} p^{-\gamma}\psi(x/p, p).$$

Earlier papers on estimating the sum in (4.2), particularly in the case $\gamma = 1$, are cited in [11], where in §4 an approach different from that used in [12] is described. The estimates on the right of (1.10) and (4.2) take the same form, but (5.12) shows that the two sums are not quite of the same order of magnitude.

Define L by

$$L = L(x) = \exp\left(\left(\frac{1}{2\gamma}\log x\log_2 x\right)^{1/2}\right). \tag{4.3}$$

The crucial dominating interval for p in the sum on the right of (4.1) turns out to be one containing L, and the contribution from the primes outside the interval $L^{1/3} \leqslant p \leqslant L^3$ is negligible. (We could replace these exponents 1/3 and 3 by $1/a$, a for any $a > 2$.)

LEMMA 11.

$$\sum_{p<L^{1/3}} \psi(x/p^{\gamma+1}, p) + \sum_{p>L^3} \psi(x/p^{\gamma+1}, p) \ll x\exp(-\tfrac{3}{2}(2\gamma\log x\log_2 x)^{1/2}). \tag{4.4}$$

Proof.

$$\sum_{p>L^3} \psi(x/p^{\gamma+1}, p) \leqslant \sum_{p>L^3} x/p^{\gamma+1} \ll xL^{-3\gamma}$$

as required. When $\exp((\log_2 x)^2) \leqslant p < L^{1/3}$, Lemma 1 is applicable, and so, since $\varrho(\log x/\log p-\gamma-1)$ increases as p increases,

$$\begin{aligned}\psi(x/p^{\gamma+1}, p) &\leqslant \frac{x}{p^{\gamma+1}}\varrho\left(\frac{3\log x}{\log L}-\gamma-1\right)\left(1+O\left(\frac{1}{\log_2 x}\right)\right)\\ &\leqslant \frac{x}{p^{\gamma+1}}\exp\left\{-\frac{3}{2}\left(\frac{2\gamma\log x}{\log_2 x}\right)^{1/2}(\log_2 x+\log_3 x+O(1))\right\}\\ &\quad\times\left(1+O\left(\frac{1}{\log_2 x}\right)\right),\end{aligned}$$

by (2.3). When $p < \exp((\log_2 x)^2)$,

$$\begin{aligned}\psi(x/p^{\gamma+1}, p) &\leqslant \psi(x, \exp((\log_2 x)^2)) \ll x\varrho(\log x/(\log_2 x)^2)\\ &\ll x\exp\left\{-\frac{\log x}{\log_2 x}\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)\right\}.\end{aligned}$$

Using these inequalities, we obtain the required estimate for the first sum on the left of (4.4).

Our first step in dealing with the main sum

$$\sum_{L^{1/3} \leqslant p \leqslant L^3} \psi(x/p^{\gamma+1}, p)$$

is to choose c (in terms of x) in the interval $1/3 \leqslant c \leqslant 3$ to maximize the sum

$$U_{\gamma,c}(x) = \sum_{e^{-1}L^c \leqslant p \leqslant L^c} \frac{1}{p^{\gamma+1}}\varrho\left(\frac{\log x}{\log p}-\gamma-1\right).$$

LEMMA 12. *$U_{\gamma,c}(x)$ is maximal when $c = c_0$, where*

$$c_0 = (1+d_0)^{1/2} \quad \textit{and} \quad d_0 = \frac{(\log_3 x+\log(\gamma/2))}{\log_2 x}\left(1+\frac{2}{\log_2 x}\right), \tag{4.5}$$

and then

$$U_{\gamma,c_0}(x) = \exp\left\{-(2\gamma\log x\log_2 x)^{1/2}\left(1+k_\gamma(x)+O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right)\right)\right\}$$

where $k_\gamma(x)$ is given by (1.11).

Proof. This is proved in the same way as the corresponding result in [12] (see pp. 779–783), so we omit some of the lengthy computational details. For $e^{-1}L^c \leqslant p \leqslant L^c$,

$$u = \frac{\log x}{\log p}-\gamma-1 = \frac{1}{c}\left(\frac{2\gamma\log x}{\log_2 x}\right)^{1/2}+O(1)$$

and hence by Lemma 2

$$\log \varrho(u) = -\frac{1}{2c}(2\gamma \log x \log_2 x)^{1/2}(1+\varepsilon_c(x)) \tag{4.6}$$

where

$$\varepsilon_c(x) = \frac{(\log_3 x + \log(\gamma/2) - 2\log c - 2)}{\log_2 x}\left(1+\frac{2}{\log_2 x}\right) + O\left(\frac{\log_3^2 x}{\log_2^3 x}\right).$$

It is easily seen that

$$\sum_{e^{-1}L^c \leqslant p \leqslant L^c} \frac{1}{p^{\gamma+1}} = \exp\left\{-\frac{c}{2}(2\gamma \log x \log_2 x)^{1/2}\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\right\}.$$

Hence, in order to maximize $U_{\gamma,c}(x)$, we need to minimize

$$\frac{c}{2}+\frac{1}{2c}(1+\varepsilon_c(x)).$$

Since $\varepsilon_c(x) = o(1)$, this is clearly achieved when $c = 1+o(1)$, so write

$$c = (1+d)^{1/2}$$

where we find we can assume that $d = O(\log_3 x/\log_2 x)$.

Expanding by the Binomial Theorem, we obtain after some computation

$$\frac{c}{2}+\frac{1}{2c}(1+\varepsilon_c(x)) = \frac{1}{8}(d-d_0)^2 + 1 + k_\gamma(x) + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right) \tag{4.7}$$

where $k_\gamma(x)$ is given by (1.11) and d_0 by (4.5). Clearly the right side of (4.7) has its minimum value when $d = d_0$, and the result of the lemma now follows.

Lemma 13.

$$\sum_{L^{1/3} \leqslant p \leqslant L^3} \psi\left(\frac{x}{p^{\gamma+1}}, p\right)$$
$$= x\exp\left\{-(2\gamma \log x \log_2 x)^{1/2}\left(1+k_\gamma(x)+O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right)\right)\right\}.$$

Proof. For $L^{1/3} \leqslant p \leqslant L^3$,

$$\psi\left(\frac{x}{p^{\gamma+1}}, p\right) = \frac{x}{p^{\gamma+1}}\varrho\left(\frac{\log x}{\log p}-\gamma-1\right)\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)$$

by (2.2). We can split the interval $L^{1/3} \leqslant p \leqslant L^3$ into subintervals $L^c e^{m-1} < p \leqslant L^c e^m$ where m runs through a set of $(3-1/3)\log L + O(1)$ consecutive integers, and for each such m

$$\sum_{L^c e^{m-1} < p \leqslant L^c e^m} \frac{1}{p^{\gamma+1}}\varrho\left(\frac{\log x}{\log p}-\gamma-1\right) \leqslant U_{\gamma,c_0}(x)$$

by Lemma 12. Hence

$$xU_{\gamma,c_0}(x)\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)$$
$$\leqslant \sum_{L^{1/3} \leqslant p \leqslant L^3} \psi\left(\frac{x}{p^{\gamma+1}}, p\right)$$
$$\leqslant x((3-\tfrac{1}{3})\log L + O(1))\,U_{\gamma,c_0}(x)\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right).$$

The result of Lemma 13 now follows from Lemma 12 and (4.3) since the other factors can be absorbed into the error term inside the exponential.

From (4.1) and Lemmas 11 and 13, we obtain Theorem 2(i).

5. Proofs of Theorem 2(ii) and Theorem 4. The argument used to prove Theorem 2(ii) is similar to that used in [7] to establish (5.11) below when $\gamma = 1$.

From (4.1) and Lemmas 11 and 13, it is clear that

$$T_\gamma(x) = x\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\sum_{L^{1/3} \leqslant p \leqslant L^3} \frac{1}{p^{\gamma+1}}\varrho\left(\frac{\log x}{\log p}-\gamma-1\right) \tag{5.1}$$

where, by Theorem 2(i), the sum on the right equals

$$\exp\left\{-(2\gamma \log x \log_2 x)^{1/2}\left(1+\frac{\log_3 x + O(1)}{2\log_2 x}\right)\right\}. \tag{5.2}$$

From Lemma 2, we have for $L^{1/3} \leqslant p \leqslant L^3$,

$$\varrho\left(\frac{\log x}{\log p}-\gamma-1\right) = \exp\left\{-\frac{\log x \log_2 x}{2\log p}\left(1+\frac{\log_3 x + O(1)}{\log_2 x}\right)\right\}. \tag{5.3}$$

Lemma 14. *For a suitable constant $B > 0$,*

$$\sum_{L^{1/3} \leqslant p \leqslant L^3} \frac{1}{p^{\gamma+1}}\varrho\left(\frac{\log x}{\log p}-\gamma-1\right) \tag{5.4}$$
$$= (1+O(\exp(-B(2\gamma \log x \log_2 x)^{1/4})))\int_{L^{1/3}}^{L^3} \frac{1}{t^{\gamma+1}\log t}\varrho\left(\frac{\log x}{\log t}-\gamma-1\right)dt$$
$$+O(\exp(-\tfrac{5}{3}(2\gamma \log x \log_2 x)^{1/2})).$$

Proof. We use partial summation and integration by parts, and we recall that $\varrho(\log x/\log t-\gamma-1)$ increases with t. By Lemma 5,

$$\sum_{p\leqslant y}\frac{1}{p^{\gamma+1}}=J(y)+C_\gamma+R(y)$$

where

$$(5.5)\qquad \begin{cases} J(y)=\int\limits_2^y\dfrac{dt}{t^{\gamma+1}\log t}, & J'(y)=\dfrac{1}{y^{\gamma+1}\log y},\\ R(y)=O(\exp(-c\sqrt{\log y})/y^\gamma\log y). \end{cases}$$

Hence we obtain

$$\Sigma=\sum_{L^{1/3}\leqslant p\leqslant L^3}\frac{1}{p^{\gamma+1}}\varrho\left(\frac{\log x}{\log p}-\gamma-1\right)$$
$$=\left[R(t)\varrho\left(\frac{\log x}{\log t}-\gamma-1\right)\right]_{L^{1/3}}^{L^3}$$
$$+\int\limits_{L^{1/3}}^{L^3}\frac{1}{t^{\gamma+1}\log t}\varrho\left(\frac{\log x}{\log t}-\gamma-1\right)dt-\int\limits_{L^{1/3}}^{L^3}R(t)\frac{d}{dt}\varrho\left(\frac{\log x}{\log t}-\gamma-1\right)dt.$$

By (5.5), the last integral is

$$\ll\frac{\exp(-c(\log L^{1/3})^{1/2})}{\log L}\int\limits_{L^{1/3}}^{L^3}\frac{1}{t^\gamma}\frac{d}{dt}\varrho\left(\frac{\log x}{\log t}-\gamma-1\right)dt,$$

where the integrand is positive. On integrating by parts and substituting back in the expression for Σ, we deduce that

$$(5.6)\qquad \Sigma=\left(1+O\left(\exp(-c(\tfrac{1}{3}\log L)^{1/2})\right)\right)\int\limits_{L^{1/3}}^{L^3}\frac{1}{t^{\gamma+1}\log t}\varrho\left(\frac{\log x}{\log t}-\gamma-1\right)dt$$
$$+O\left(\exp(-c(\tfrac{1}{3}\log L)^{1/2})\left\{L^{-3\gamma}\varrho\left(\frac{\log x}{3\log L}-\gamma-1\right)+L^{-\gamma/3}\varrho\left(\frac{3\log x}{\log L}-\gamma-1\right)\right\}\right).$$

By (5.3) and (4.3),

$$(5.7)\qquad L^{-\gamma/3}\varrho\left(\frac{3\log x}{\log L}-\gamma-1\right)<L^{-3\gamma}\varrho\left(\frac{\log x}{3\log L}-\gamma-1\right)$$
$$=\exp\left\{-\left(\frac{3}{2}+\frac{1}{6}\left(1+\frac{\log_3 x+O(1)}{\log_2 x}\right)\right)(2\gamma\log x\log_2 x)^{1/2}\right\}.$$

If x is sufficiently large and we choose $B<c/\sqrt{6\gamma}$, (5.4) follows from (5.6) and (5.7).

LEMMA 15.

$$(5.8)\qquad \int\limits_{L^{1/3}}^{L^3}\frac{1}{t^{\gamma+1}\log t}\varrho\left(\frac{\log x}{\log t}-\gamma-1\right)dt$$
$$=\gamma\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\int\limits_{L^{1/3}}^{L^3}t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}-\gamma\right)dt$$
$$+O\left(\exp(-\tfrac{5}{3}(2\gamma\log x\log_2 x)^{1/2})\right).$$

Proof. From (1.6),

$$\varrho\left(\frac{\log x}{\log t}-\gamma-1\right)=t\log t\left(1-\frac{\gamma\log t}{\log x}\right)\frac{d}{dt}\varrho\left(\frac{\log x}{\log t}-\gamma\right)$$
$$=t\log t\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\frac{d}{dt}\varrho\left(\frac{\log x}{\log t}-\gamma\right)$$

for $L^{1/3}\leqslant t\leqslant L^3$. On substituting this in the left side of (5.8), integrating by parts and using an estimate analogous to (5.7), we obtain (5.8).

From (5.1), (5.2) and Lemmas 14 and 15, we deduce the

COROLLARY.

$$(5.9)\qquad T_\gamma(x)=\gamma x\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\int\limits_{L^{1/3}}^{L^3}t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}-\gamma\right)dt.$$

It remains to show that we can extend the range of integration on the right side of (5.9) to the interval $[2, x^{1/\gamma}]$ without incurring terms that cannot be absorbed into the error term.

LEMMA 16.

$$(5.10)\qquad \left(\int\limits_2^{L^{1/3}}+\int\limits_{L^3}^{x^{1/\gamma}}\right)t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}-\gamma\right)dt\ll\exp(-\tfrac{3}{2}(2\gamma\log x\log_2 x)^{1/2}).$$

Proof. Since $\varrho(\log x/\log t-\gamma)$ increases with t and is bounded, and $\gamma>0$, the left side of (5.10) is

$$\ll\varrho\left(\frac{3\log x}{\log L}-\gamma\right)+L^{-3\gamma}\ll\exp(-\tfrac{3}{2}(2\gamma\log x\log_2 x)^{1/2}).$$

The result of Theorem 2(ii) now follows from (5.9), (5.10) and (5.2).

By an argument similar to the one in this section, it can be shown that for $\gamma>0$

$$(5.11)\qquad \sum_{2\leqslant n\leqslant x}(P(n))^{-\gamma}=\gamma x\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\int\limits_{L^{1/3}}^{L^3}t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}\right)dt$$
$$=\gamma x\left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right)\int\limits_2^x t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}\right)dt;$$

for the case $\gamma=1$, see [7].

The sum on the left of (5.11) is of a slightly smaller order of magnitude than the sum $T_\gamma(x)$ of Theorem 2, as shown by the following result:

THEOREM 4. *For* $\gamma > 0$, *as* $x \to \infty$,

$$\sum_{\substack{n>1 \\ n(P(n))^\gamma \leqslant x}} 1 \sim (\tfrac{1}{2}\gamma \log x \log_2 x)^{\gamma/2} \sum_{2 \leqslant n \leqslant x} (P(n))^{-\gamma}. \tag{5.12}$$

Proof. By Theorem 2, (4.2), (5.9) and (5.11), it is sufficient to show that, as $x \to \infty$,

$$\int_{L^{1/3}}^{L^3} t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}-\gamma\right)dt \sim (\tfrac{1}{2}\gamma \log x \log_2 x)^{\gamma/2} \int_{L^{1/3}}^{L^3} t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}\right)dt. \tag{5.13}$$

Our first step is to justify reducing the range of integration on both sides of (5.13) by using some of the ideas in the proof of Lemma 12. Let $\delta = \gamma$ or 0 and write $t = L^c$ where $1/3 \leqslant c \leqslant 3$. Using (4.5), (4.6) and (4.7), we see that

$$t^{-\gamma}\varrho\left(\frac{\log x}{\log t}-\delta\right) = \exp\left\{-(2\gamma\log x\log_2 x)^{1/2}\left(\frac{c}{2}+\frac{1}{2c}(1+\varepsilon_c(x))\right)\right\}$$

where the right side is greatest when $c = c_0$. Moreover, when $1/3 \leqslant c \leqslant c_1$ or $c_2 \leqslant c \leqslant 3$, where

$$c_j = 1+\frac{\log_3 x+\log(\gamma/2)}{2\log_2 x}+\frac{(-1)^j}{2\log_2 x\log_3 x} \quad (j = 1, 2),$$

we have, by (4.5) and (4.7),

$$\frac{c}{2}+\frac{1}{2c}(1+\varepsilon_c(x)) \geqslant \frac{1}{8}\left(\frac{1}{\log_2 x\log_3 x}+O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^2\right)\right)^2+1+k_\gamma(x)+O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right),$$

and hence

$$t^{-\gamma}\varrho\left(\frac{\log x}{\log t}-\delta\right) \tag{5.14}$$

$$\leqslant \exp\left\{-(2\gamma\log x\log_2 x)^{1/2}\left(1+k_\gamma(x)+\frac{1}{8(\log_2 x\log_3 x)^2}+O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^3\right)\right)\right\}.$$

Let $L_j = L^{c_j}$ $(j = 1, 2)$; then it follows from Theorem 2, (4.2), (5.9), (5.11) and (5.14) that

$$\left(\int_{L^{1/3}}^{L_1}+\int_{L_2}^{L^3}\right)t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}-\delta\right)dt \tag{5.15}$$

$$\ll \exp\left\{\frac{-(2\gamma\log x\log_2 x)^{1/2}(1+o(1))}{8(\log_2 x\log_3 x)^2}\right\}\int_{L^{1/3}}^{L^3} t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}-\delta\right)dt.$$

In the range $L_1 \leqslant t \leqslant L_2$, we change the variable to $v = \log x/\log t-\delta$. For the case $\delta = 0$,

$$\int_{L_1}^{L_2} t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}\right)dt = \log x\int_{V_1}^{V_2}\exp\left(-\frac{\gamma\log x}{v}\right)\frac{\varrho(v)}{v^2}dv \tag{5.16}$$

where $V_1 = \log x/\log L_2$, $V_2 = \log x/\log L_1$. For the case $\delta = \gamma$,

$$\int_{L_1}^{L_2} t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}-\gamma\right)dt = \log x\int_{V_1-\gamma}^{V_2-\gamma}\exp\left(-\frac{\gamma\log x}{v}\right)\frac{\varrho(v)}{v^2}h_x(v)dv \tag{5.17}$$

where

$$h_x(v) = \left(\frac{v}{v+\gamma}\right)^2\exp\left(\frac{\gamma^2\log x}{v(v+\gamma)}\right) = \left(1-\frac{\gamma}{v+\gamma}\right)^2\exp\left(\frac{\gamma^2\log x}{(v+\gamma)^2}\left(1-\frac{\gamma}{v+\gamma}\right)^{-1}\right).$$

For $V_1 \leqslant v+\gamma \leqslant V_2$,

$$\frac{1}{v+\gamma} = \left(\frac{\log_2 x}{2\gamma\log x}\right)^{1/2}\left(1+\frac{\log_3 x+\log(\gamma/2)}{2\log_2 x}+O\left(\frac{1}{\log_3 x\log_2 x}\right)\right),$$

and hence

$$h_x(v) = \left(1+O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right) \tag{5.18}$$

$$\times\exp\left\{\frac{\gamma}{2}\left(\log_2 x+\log_3 x+\log\frac{\gamma}{2}\right)+O\left(\frac{1}{\log_3 x}\right)\right\}$$

$$= \left(\frac{\gamma}{2}\log x\log_2 x\right)^{\gamma/2}\left(1+O\left(\frac{1}{\log_3 x}\right)\right).$$

The integrals

$$\log x\int_{V_j-\gamma}^{V_j}\exp\left(\frac{-\gamma\log x}{v}\right)\frac{\varrho(v)}{v^2}dv \quad (j = 1, 2)$$

are bounded by the right side of (5.15), for the argument leading to (5.14) remains valid when c_1 is replaced by a larger quantity of the form $c_1(1+O((\log_2 x/\log x)^{1/2}))$.

From (5.15), (5.16), (5.17) and (5.18), we obtain

$$\int_{L_1}^{L_2} t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}-\gamma\right)dt$$

$$= \left(\frac{\gamma}{2}\log x\log_2 x\right)^{\gamma/2}\left(1+O\left(\frac{1}{\log_3 x}\right)\right)\int_{L_1}^{L_2} t^{-\gamma-1}\varrho\left(\frac{\log x}{\log t}\right)dt,$$

and hence (5.13), from which the result of Theorem 4 follows.

6. Proof of Theorem 3. In this section, $-1 < \gamma < 0$, and so $1/2 < 1/(2+\gamma) < 1 < 1/(1+\gamma)$. When $p > x/p^{1+\gamma}$,

$$\psi(x/p^{1+\gamma}, p) = [x/p^{1+\gamma}] = \sum_{m \leqslant x/p^{1+\gamma}} 1.$$

Since (4.1) holds also for $-1 < \gamma < 0$, we have

$$\begin{aligned} T_\gamma(x) &= \sum_{p \leqslant x^{1/(1+\gamma)}} \psi(x/p^{1+\gamma}, p) \\ &= \sum_{x^{1/(2+\gamma)} < p \leqslant x^{1/(1+\gamma)}} \sum_{m \leqslant x/p^{1+\gamma}} 1 + \sum_{p \leqslant x^{1/(2+\gamma)}} \psi(x/p^{1+\gamma}, p) \\ &= \sum_{m < x^{1/(2+\gamma)}} \{\pi((x/m)^{1/(1+\gamma)}) - \pi(x^{1/(2+\gamma)})\} + O(x \sum_{p \leqslant x^{1/(2+\gamma)}} p^{-1-\gamma}) \\ &= \sum_{m \leqslant x^{1/(2+\gamma)}} \pi((x/m)^{1/(1+\gamma)}) + O(x^{2/(2+\gamma)}). \end{aligned}$$

Applying the results of Lemmas 6 and 7, we obtain, for R a fixed positive integer,

$$T_\gamma(x) = \frac{x^{1/(1+\gamma)}}{\log x} \Big\{ (1+\gamma)\zeta\Big(\frac{1}{1+\gamma}\Big) + \sum_{k=1}^{R-1} A_k (\log x)^{-k} + O((\log x)^{-R}) + O(x^{\gamma/(1+\gamma)(2+\gamma)+\varepsilon}) \Big\}$$

where

$$A_k = (1+\gamma) \sum_{l=0}^{k} C_{k,l} D_l.$$

Since $\gamma/(1+\gamma)(2+\gamma) < 0$ and ε is arbitrarily small, the result of Theorem 3 follows.

By an analogous argument, it can be shown that

$$\sum_{2 \leqslant n \leqslant x} (P(n))^{-\gamma} = \frac{x^{1-\gamma}}{(1-\gamma)\log x} \zeta(1-\gamma) \Big(1 + O\Big(\frac{1}{\log x}\Big)\Big)$$

for $-1 < \gamma < 0$, and hence it follows that for γ in this range

(6.1) $$T_\gamma(x) \sim (1-\gamma^2) \frac{\zeta(1/(1+\gamma))}{\zeta(1-\gamma)} x^{\gamma^2/(1+\gamma)} \sum_{2 \leqslant n \leqslant x} (P(n))^{-\gamma}$$

as $x \to \infty$.

7. Additive functions. We investigate next $\Sigma_g(x)$ for the following functions g, where g or $1/g$ is additive:

(a) $$g(n) = \omega(n),\ \Omega(n),\ 1/\omega(n),\ 1/\Omega(n)$$

where $\omega(n) = \sum_{p|n} 1$, $\Omega(n) = \sum_{p^\alpha \| n} \alpha$;

(b) $$g(n) = \beta(n),\ B(n)$$

where $\beta(n) = \sum_{p|n} p$, $B(n) = \sum_{p^\alpha \| n} \alpha p$.

The functions in (a) are 'small' and those in (b) are 'large'.

We consider (b) first. Notice that

(7.1) $$P(n) \leqslant \beta(n) \leqslant \omega(n) P(n), \quad P(n) \leqslant B(n) \leqslant \Omega(n) P(n),$$

where for $1 < n \leqslant x$, from §22.10 of [8],

(7.2) $$\omega(n) \leqslant \frac{\log x}{\log_2 x}\Big(1 + O\Big(\frac{1}{\log_2 x}\Big)\Big), \quad \Omega(n) \leqslant \frac{\log x}{\log 2}.$$

This suggests that the sum $\Sigma_g(x)$ for the functions in (b) should behave like the sum $T_1(x)$ in Theorem 2(i), which is indeed the case. For by (7.1) and (7.2),

$$T_1\Big(\frac{x \log_2 x}{\log x}\Big(1 + O\Big(\frac{1}{\log_2 x}\Big)\Big)\Big) \leqslant \Sigma_\beta(x) \leqslant T_1(x),$$

$$T_1\Big(\frac{x \log 2}{\log x}\Big) \leqslant \Sigma_B(x) \leqslant T_1(x).$$

Hence we deduce from Theorem 2(i):

THEOREM 5. *For $g = \beta$ or B,*

$$\Sigma_g(x) = x \exp\Big\{ -(2 \log x \log_2 x)^{1/2} \Big(1 + k_1(x) + O\Big(\Big(\frac{\log_3 x}{\log_2 x}\Big)^3\Big)\Big)\Big\}$$

where $k_1(x)$ is given by (1.11) with $\gamma = 1$.

We turn next to the functions in (a), and we shall consider the more general problem of estimating the sums $\Sigma_{1/f}(x)$ and $\Sigma_f(x)$ for a class of 'small' additive functions f containing ω and Ω. Let

$$G(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-\tfrac{1}{2} t^2)\, dt$$

for all real z. We observe that

(7.3) $$G(z) = \begin{cases} \dfrac{1}{\sqrt{2\pi}|z|}(1 + O(1/z^2)) \exp(-\tfrac{1}{2} z^2) & \text{for } z < 0, \\ 1 - G(-z) & \text{for } z \geqslant 0. \end{cases}$$

DEFINITION. Let $\mathscr{C}_1$ denote the class of all additive functions f satisfying properties (i), (ii), (iii) below, and $\mathscr{C}_2$ the corresponding class when property (iii) is replaced by (iv) below:

(i) for all $n > 1$,

$$1 \leqslant f(n) \leqslant C(\log n)^{\alpha} \tag{7.4}$$

where α and C are positive constants;

(ii) uniformly for all real z and for x sufficiently large (independently of z),

$$\#\{n \leqslant x\colon f(n) - \log_2 x \leqslant z(\log_2 x)^{1/2}\} = x\big(G(z) + O((\log_2 x)^{-1/2})\big); \tag{7.5}$$

(iii) for $0 < z \ll (\log_2 x)^{1/6}$,

$$\#\{n \leqslant x\colon f(n) < \log_2 x - z(\log_2 x)^{1/2}\} \ll xG(-z); \tag{7.6}$$

(iv) for $z > 0$,

$$\#\{n \leqslant x\colon f(n) > \log_2 x + z(\log_2 x)^{1/2}\} \ll xz^{-2}. \tag{7.7}$$

Using (7.3), we see that the estimates in (iii) and (iv) are sharper than those implied by (ii) when z is a sufficiently large function of x. There is no restriction in (iv) on how large z can be in terms of x. Results of the type assumed in properties (ii) and (iii) have been obtained for specific additive functions, or for certain classes of such functions, by various authors; for a discussion of the relevant literature and its background, see (for example) pp. 286–288 of [6].

LEMMA 17. *The classes $\mathscr{C}_1$ and $\mathscr{C}_2$ contain the functions ω and Ω.*

Proof. Property (i) holds for ω and Ω by (7.2), and (ii) was established in [15] for a class of additive functions that contains ω and Ω. A more precise result than (iii) was established in the larger range $0 < |z| = o((\log_2 x)^{1/2})$ for the function $f = \omega$ in Theorem 9.2, p. 161, of [14]; hence, since $\omega(n) \leqslant \Omega(n)$ for all n, (iii) holds also for $f = \Omega$. Property (iv) is essentially proved for $f = \omega$ in §22.11 of [8], and the proof for $f = \Omega$ is similar since

$$\sum_{n \leqslant x} (\Omega(n))^2 = x(\log_2 x)^2 + O(x \log_2 x).$$

We can now state and prove the main results of this section:

THEOREM 6. (i) *When $f \in \mathscr{C}_1$,*

$$\Sigma_f(x) = \sum_{\substack{n>1\\ nf(n) \leqslant x}} 1 = \frac{x}{\log_2 x}\left(1 + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^{1/2}\right)\right).$$

(ii) *When $f \in \mathscr{C}_2$,*

$$\Sigma_{1/f}(x) = \sum_{\substack{n>1\\ n/f(n) \leqslant x}} 1 = x \log_2 x\big(1 + O((\log_2 x)^{-1/3})\big).$$

In particular, (i) *and* (ii) *hold when $f = \omega$ or Ω.*

Proof. (i) Let $f \in \mathscr{C}_1$. Consider first those n contributing to $\Sigma_f(x)$ for which

$$|f(n) - \log_2 x| \leqslant z(\log_2 x)^{1/2} \tag{7.8}$$

for some $z > 0$ to be chosen later in terms of x, with $z \to \infty$ as $x \to \infty$ and $z \ll (\log_2 x)^{1/6}$. For these n, the condition $nf(n) \leqslant x$ implies that

$$n \leqslant \frac{x}{\log_2 x}\big(1 + O(z(\log_2 x)^{-1/2})\big). \tag{7.9}$$

The number of positive integers n satisfying (7.9) but not (7.8) is

$$\ll \frac{x}{\log_2 x}\left(\frac{1}{z}\exp(-\tfrac{1}{2}z^2) + (\log_2 x)^{-1/2}\right) \tag{7.10}$$

by (7.3) and (7.5). Hence, by (7.5) again, the number of n contributing to $\Sigma_f(x)$ and satisfying (7.8) equals

$$\frac{x}{\log_2 x}\left(1 + O\left((z+1)(\log_2 x)^{-1/2} + \frac{1}{z}\exp(-\tfrac{1}{2}z^2)\right)\right). \tag{7.11}$$

When n fails to satisfy (7.8), we have

(a) $f(n) > \log_2 x + z(\log_2 x)^{1/2}$ or (b) $f(n) < \log_2 x - z(\log_2 x)^{1/2}$.

In case (a), any n contributing to $\Sigma_f(x)$ must also satisfy (7.9), and hence the number of such n is estimated by (7.10). The number of n contributing to $\Sigma_f(x)$ and satisfying (b) is estimated by (7.6), for we are assuming that $z \ll (\log_2 x)^{1/6}$ and clearly $n \leqslant x$ must hold.

We now choose $z = (3\log_3 x)^{1/2}$, so that

$$\exp(-\tfrac{1}{2}z^2) = (\log_2 x)^{-3/2}, \quad \text{whence} \quad G(-z) \ll (\log_2 x)^{-3/2}$$

by (7.3). The result of Theorem 6(i) then follows from (7.11), (7.10), (7.6) and (7.3).

(ii) Let $f \in \mathscr{C}_2$. By (7.4), the condition $n/f(n) \leqslant x$ implies that

$$n \leqslant Cx(\log x)^{\alpha}\left(1 + O\left(\frac{\log_2 x}{\log x}\right)\right); \tag{7.12}$$

we note that

$$\log_2\left\{Cx(\log x)^{\alpha}\left(1 + O\left(\frac{\log_2 x}{\log x}\right)\right)\right\} = \left(1 + O\left(\frac{1}{\log x}\right)\right)\log_2 x. \tag{7.13}$$

We partition those n contributing to $\Sigma_{1/f}(x)$ according to the size of $f(n)$, which in turn influences the range that contains the relevant n. The main term in Theorem 6(ii) arises from the positive integers n for which

$$f(n) \leqslant \log_2 x + z(\log_2 x)^{1/2}, \quad \text{where } 0 < z = o((\log_2 x)^{1/2}), \tag{7.14}$$

in which case

$$n \leqslant x\big(\log_2 x + z(\log_2 x)^{1/2}\big) \tag{7.15}$$

since $n/f(n) \leqslant x$. The number of integers n satisfying (7.15) but not (7.14) (or satisfying (7.15) but not $|f(n)-\log_2 x| \leqslant z(\log_2 x)^{1/2}$) is

$$\ll x(\log_2 x + z(\log_2 x)^{1/2})\left(\frac{1}{z}\exp(-\tfrac{1}{2}z^2)+(\log_2 x)^{-1/2}\right)$$

by (7.5) and (7.3). Hence, by (7.5) again, the number of integers n satisfying $n/f(n) \leqslant x$, (7.14) and (7.15) equals

$$x\log_2 x\left(1+O\left((z+1)(\log_2 x)^{-1/2}+\frac{1}{z}\exp(-\tfrac{1}{2}z^2)\right)\right). \tag{7.16}$$

For the remaining integers n satisfying $n/f(n) \leqslant x$ that are not counted already in deriving the estimate (7.16), both the inequalities

$$f(n) > \log_2 x + z(\log_2 x)^{1/2} \quad \text{and} \quad n > x(\log_2 x + z(\log_2 x)^{1/2}) \tag{7.17}$$

hold. We partition the set of these integers n into R subsets by imposing the restriction

$$\log_2 x + z(\log_2 x)^{1/2}u^{r-1} < f(n) \leqslant \log_2 x + z(\log_2 x)^{1/2}u^r \tag{7.18}$$

for $r = 1, 2, \ldots, R$ where u satisfies $u \to \infty$ as $x \to \infty$ and is to be chosen later and R is the least integer such that

$$\log_2 x + z(\log_2 x)^{1/2}u^R > \sup\{f(n)\colon n/f(n) \leqslant x\},$$

the right side being at most $C(\log x)^{\alpha}(1+O((\log x)^{-1}\log_2 x))$ by (7.4) and (7.12). Let N_r denote the number of positive integers n satisfying

$$n \leqslant x(\log_2 x + z(\log_2 x)^{1/2}u^r) \quad \text{and} \quad f(n) > \log_2 x + z(\log_2 x)^{1/2}u^{r-1} \tag{7.19}$$

for $r = 1, 2, \ldots, R$; the first inequality of (7.19) is derived from the right inequality of (7.18) and the condition $n/f(n) \leqslant x$. The sum $N_1+N_2+\ldots+N_R$ provides a crude upper bound for the number of integers n satisfying $n/f(n) \leqslant x$ and (7.17).

By (7.7), (7.19) and (7.13) and since $u \to \infty$ as $x \to \infty$, we have, for $1 \leqslant r \leqslant R$,

$$N_r \ll x(\log_2 x + z(\log_2 x)^{1/2}u^r)(zu^{r-1})^{-2} \ll \frac{x\log_2 x}{z^2}\left(\frac{1}{u^{2r-2}}+\frac{uz}{(\log_2 x)^{1/2}}\frac{1}{u^{r-1}}\right).$$

Hence, since $u > 1$,

$$\sum_{r=1}^{R} N_r \ll \frac{x\log_2 x}{z^2}\left(1+\frac{uz}{(\log_2 x)^{1/2}}\right). \tag{7.20}$$

From (7.16) and (7.20), it follows that

$$\Sigma_{1/f}(x) = x\log_2 x\left\{1+O\left((z+1+u/z)(\log_2 x)^{-1/2}+\frac{1}{z^2}+\frac{1}{z}\exp(-\tfrac{1}{2}z^2)\right)\right\}. \tag{7.21}$$

Choosing $z = (\log_2 x)^{1/6}$ and $u = (\log_2 x)^{1/3}$, the result of Theorem 6(ii) follows from (7.21).

8. Multiplicative functions. Let φ denote Euler's function, σ_ν the divisor function with $\nu \neq 0$, and γ be a non-zero constant. Our aim is to obtain an asymptotic formula for $\Sigma_f(x)$ when f is one of the 'large' multiplicative functions given by

$$\text{(i)}\ f(n) = (\varphi(n))^{\gamma},\ \gamma > -1; \qquad \text{(ii)}\ f(n) = (\sigma_\nu(n))^{\gamma},\ \nu > 0,\ \gamma > -1/\nu; \tag{8.1}$$

these restrictions on γ are natural ones—see the comment in Section 1 after Theorem 3. We deduce our results from the corresponding result for certain 'small' multiplicative functions g.

Let g be a positive multiplicative function satisfying

$$\begin{cases} g(p) = 1+O(p^{-c}) & \text{for all primes } p, \\ g(n) \gg n^{-b} & \text{for all positive integers } n, \end{cases} \tag{8.2}$$

where c and b are constants with $c > 0$, $0 < b < 1/2$. Proceeding as in method C of [3], we obtain by standard analytic arguments (the details of which we omit here):

Theorem 7. *Let g satisfy (8.2). Then for any fixed $\varepsilon > 0$,*

$$\Sigma_g(x) = \sum_{ng(n) \leqslant x} 1 = Ax + O\left(x\exp\left(-(1-\varepsilon)\left(\frac{c}{2}\log x\log_2 x\right)^{1/2}\right)\right) \tag{8.3}$$

where

$$A = \prod_p \left\{1+\sum_{\alpha=1}^{\infty} p^{-\alpha}\left(\frac{1}{g(p^{\alpha})}-\frac{1}{g(p^{\alpha-1})}\right)\right\}.$$

Corollary 1. *(8.3) holds when $g(n) = (\sigma_\nu(n))^{\gamma}$ with $\nu < 0$, $\gamma \neq 0$, in which case $c = |\nu|$.*

Let f be a positive multiplicative function such that, for some $\beta > -1$,

$$f(n) = (g(n))^{1+\beta}n^{\beta}, \tag{8.4}$$

where g satisfies the conditions in (8.2). Then we have:

COROLLARY 2. *For each fixed* $\varepsilon > 0$,

$$(8.5)\quad \Sigma_f(x) = \sum_{nf(n) \leqslant x} 1$$
$$= Ax^{1/(1+\beta)} + O\left(x^{1/(1+\beta)} \exp\left\{-(1-\varepsilon)\left(\frac{c}{2(1+\beta)} \log x \log_2 x\right)^{1/2}\right\}\right).$$

Proof. By (8.4), $nf(n) = (ng(n))^{1+\beta}$, and so the condition $nf(n) \leqslant x$ is equivalent to $ng(n) \leqslant x^{1/(1+\beta)}$. Applying Theorem 7, with x replaced by $x^{1/(1+\beta)}$, and ε by $\varepsilon/2$ (say), we obtain the result of Corollary 2.

COROLLARY 3. *Let* $f(n) = (\varphi(n))^\gamma$ *where* $\gamma > -1$, $\gamma \neq 0$; *then* (8.5) *holds with* $\beta = \gamma$, $c = 1$.

For (8.4) holds with $g(n) = (\varphi(n)/n)^{\gamma/(\gamma+1)}$, and then (8.2) holds with $c = 1$ and for any $b > 0$.

COROLLARY 4. *Let* $f(n) = (\sigma_\nu(n))^\gamma$ *where* $\nu > 0$, $\gamma > -1/\nu$, $\gamma \neq 0$; *then* (8.5) *holds with* $\beta = \nu\gamma$, $c = \nu$.

For (8.4) holds with $g(n) = (n^{-\nu}\sigma_\nu(n))^{\gamma/(1+\nu\gamma)} = (\sigma_{-\nu}(n))^{\gamma/(1+\nu\gamma)}$, and then (8.2) holds with $c = \nu$ and for any $b > 0$.

Finally, we remark that if the first condition in (8.2) is replaced by

$$(8.6)\qquad g(p) = 1/\lambda + O(\exp(-c(\log p)^a)) \quad \text{for all primes } p,$$

where $\lambda > 0$, $\lambda \neq 1$, $c > 0$, $a \geqslant 1$, then the method described in Section 8 of [1] can be applied to obtain a result corresponding to that of Theorem 7 but of the shape of (1.3) with, however, a possibly weaker exponential error term. By the method used to derive Corollary 2 of Theorem 7, an asymptotic formula for $\Sigma_f(x)$ then follows for a positive multiplicative function f satisfying (8.4) for some g satisfying (8.6) and the second condition in (8.2), and the main term will be of the form

$$Ax^{1/(1+\beta)}\left(\frac{1}{1+\beta}\log x\right)^{\lambda-1}.$$

Added in proof. A. Smati has recently given an elementary proof of Theorem 7 (*Rèpartition des valeurs d'une classe de fonctions multiplicatives*, preprint).

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