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On the number of abelian groups of a given order

by

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1. Introduction. Let $A(x)$ denote the number of distinct abelian groups (up to isomorphism) of order not exceeding x. In view of the basic theorem about the structure of abelian groups of finite order (see, e.g., [9, Chap. 4]), we immediately deduce that

$$A(x) = \sum_{(n_1,\ldots,n_r,\ldots)}' 1,$$

where the summation is taken over all distinct lattice points $(n_1, \ldots, n_r, \ldots) \in \boldsymbol{N}^\infty$ such that

$$n_1 n_2^2 \ldots n_r^r \ldots \leqslant x$$

($\boldsymbol{N}$ is the set of all natural numbers). The simplest asymptotic property of the function $A(x)$, first derived in 1935 by P. Erdős and G. Szekeres [3], is

$$A(x) = C_1 x + O(x^{1/2}). \tag{1}$$

Later, in 1947, D. G. Kendal and R. A. Rankin [7] obtained:

$$A(x) = C_1 x + C_2 x^{1/2} + O\big(x^{1/3}(\log x)^2\big). \tag{2}$$

In 1952, by passing to the estimation of exponential sums (the van der Corput–Phillips exponent pair method), H. E. Richert [11] was able to show that

$$A(x) = C_1 x + C_2 x^{1/2} + C_3 x^{1/3} + \Delta(x) \tag{3}$$

where, for $i = 1, 2, 3$, the constants C_i in (1), (2) and (3) are given by

$$C_i = \prod_{\substack{v=1\\ v\neq i}}^{\infty} \zeta(v/i)$$

and, for $\Delta(x)$ in (3), the following estimate holds:

$$\Delta(x) \ll x^{3/10}(\log x)^{9/10}.$$

As we will see in Section 3, the estimation for $\Delta(x)$ is equivalent to estimating certain multiple exponential sums. Hence the sharper upper bound for $\Delta(x)$ one would like to get, the stronger estimates of exponential sums one must look for. Since the establishment of (3), R. A. Rankin [10], W. Schwarz [14], P. G. Schmidt [12], [13], B. R. Srinivasan [15], G. Kolesnik [8] have worked on this interesting topic, and the hitherto sharpest estimate given in [8] is

(4) $$\Delta(x) \ll x^{97/381}(\log x)^{35}.$$

We note that besides its complication, the proof of (4) given in [8] contains errors (as will be pointed out in the proof of our Lemma 9).

In this paper, we shall develop a technique which enables us to benefit from both Kolesnik's method and a kind of new method inspired by the work of E. Bombieri and H. Iwaniec [2]. Our method leads to the following new estimate for $\Delta(x)$.

THEOREM A. *For any* $\varepsilon > 0$, *we have*

$$\Delta(x) \ll x^{40/159+\varepsilon}.$$

Remark 1. For a comparison between Kolesnik's estimate (4) and our Theorem A, note that

$$97/381 = 0.254593\ldots \quad \text{and} \quad 40/159 = 0.251572\ldots$$

Remark 2. It should be noted that A. Ivić [6] proved, via the techniques from the theory of the Riemann zeta-function, that

$$A(x) = \sum_{i=1}^{5} C_i x^{1/i} + \Omega\big(x^{1/6}(\log x)^{1/2}\big).$$

Remark 3. We can, of course, as in (3) and (4), replace x^{ε} in Theorem A by a suitable power of $\log x$.

2. A bound for a kind of multiple exponential sums. In this section, we shall prove the following result, which is a sharpened form of Theorem 3 due to E. Fouvry and H. Iwaniec [4]. Our innovation is the use of Lemma 2 below, which leads to an optimal choice of the parameter.

LEMMA A. *Let* $H \geqslant 1$, $X \geqslant 1$, $Y \geqslant 1000$, $\alpha\gamma(\gamma-1)(\beta-1) \neq 0$, $A > C_0(\alpha, \beta, \gamma) > 0$, $f(h, x, y) = Ah^{\alpha}x^{\beta}y^{\gamma}$,

(5) $$S(H, X, Y) = \sum_{(h,x,y)\in D} C_1(h, x)C_2(y)e\big(f(h, x, y)\big)$$

where D *is a region contained in the rectangle*

$$\{(h, x, y)|\ h\sim H, x\sim X, y\sim Y\}$$

($h\sim H$ *means that* $H \leqslant h < 2H$) *such that for any fixed* h_0 *and* x_0 ($h_0\sim H$, $x_0 \sim X$), *the intersection* $D\cap\{(h_0, x_0, y)|\ y\sim Y\}$ *has at most* $O(1)$ *segments. Also,* $|C_1(h, x)| \leqslant 1$, $|C_2(y)| \leqslant 1$, $F \equiv AH^{\alpha}X^{\beta}Y^{\gamma} \gg Y$. *Then*

(6) $$L^{-3}S(H, X, Y) \ll \sqrt[14]{(HX)^{13}Y^9F} + HXY^{2/3}(1+Y^5F^{-3})^{1/12} + \sqrt[24]{(HX)^{23}F^{-4}Y^{24}Z^5} + \sqrt[4]{(HX)^3Y^4Z} \equiv E_1$$

where $L = \log(AHXY+2)$, $Z = \max(1, FY^{-2})$.

To give a detailed proof of Lemma A, we list a number of lemmas here. The most important and new ones are Lemmas 4 and 6.

LEMMA 1. *Let* $M \leqslant N < N_1 \leqslant M_1$, *and let* a_n *be any complex numbers. Then*

$$\Big|\sum_{N<n\leqslant N_1} a_n\Big| \leqslant \int_{-\infty}^{\infty} K(\theta)\Big|\sum_{M<m\leqslant M_1} a_m e(\theta m)\Big|d\theta$$

with $K(\theta) = \min\big(M_1 - M + 1, (\pi|\theta|)^{-1}, (\pi\theta)^{-2}\big)$ *and*

$$\int_{-\infty}^{\infty} K(\theta)d\theta \leqslant 3\log(2+M_1-M).$$

LEMMA 2. *Let* $M > 0$, $N > 0$, $u_m > 0$, $v_n > 0$, $A_m > 0$, $B_n > 0$ ($1 \leqslant m \leqslant M$, $1 \leqslant n \leqslant N$), *and let* Q_1 *and* Q_2 *be given non-negative numbers,* $Q_1 \leqslant Q_2$. *Then there is a* q *such that* $Q_1 \leqslant q \leqslant Q_2$ *and*

$$\sum_{m=1}^{M} A_m q^{u_m} + \sum_{n=1}^{N} B_n q^{-v_n} \ll \sum_{m=1}^{M}\sum_{n=1}^{N} (A_m^{v_n}B_n^{u_m})^{1/(u_m+v_n)} + \sum_{m=1}^{M} A_m Q_1^{u_m} + \sum_{n=1}^{N} B_n Q_2^{-v_n}.$$

LEMMA 3. *Let* I *be a subinterval of* $(Y, 2Y]$ *and let* J *be a positive integer. Then, for any complex* Z_n,

$$\Big|\sum_{n\in I} Z_n\Big|^2 \leqslant 2(1+YJ^{-1}) \sum_{1\leqslant 2|l|+1\leqslant J} \big(1-(2|l|+1)J^{-1}\big) \sum_{\substack{n+l\in I\\ n-l\in I}} \bar{Z}_{n+l}Z_{n-l}.$$

LEMMA 4. *Let*

$$\omega_{\phi\psi}(X, Y) = \sum_r\sum_s \phi_r\psi_s e(x_r y_s)$$

where $X = (x_r)$, $Y = (y_s)$ *are finite sequences of real numbers with*

$$|x_r| \leqslant P, \quad |y_s| \leqslant Q$$

and ϕ_r, ψ_s *are complex numbers. Then*

$$|\omega_{\phi\psi}(X, Y)|^2 \leqslant 20(1+PQ)\omega_{\phi}(X, Q)\omega_{\psi}(Y, P)$$

with

$$\omega_\phi(X, Q) = \sum_{|x_{r_1} - x_{r_2}| \leqslant Q^{-1}} |\phi_{r_1} \phi_{r_2}|$$

and $\omega_\psi(Y, P)$ *similarly defined.*

LEMMA 5. *Let* $H \geqslant 1$, $N \geqslant 1$, $\Delta > 0$, *and let* γ *be real. Then the number of solutions of the inequality* $|hn^\gamma - kr^\gamma| \leqslant \Delta$ *in lattice points* (h, k, n, r) *satisfying* $h, k \sim H$, $n, r \sim N$ *is*

$$< C(\gamma)(HN \log^2(2HN) + \Delta HN^{2-\gamma}).$$

LEMMA 6. *Let* $Q \geqslant 1$, $m \sim M$, $q \sim Q$, $3Q < M$, *let* α $(\neq 0, 1)$ *be a real number,* $t(m, q) = (m+q)^\alpha - (m-q)^\alpha$, $T = M^{\alpha-1}$, *and let* $B(M, Q, \Delta)$ *be the number of lattice points* $(m, \tilde{m}, q, \tilde{q})$ *such that*

$$|t(m, q) - t(\tilde{m}, \tilde{q})| < \Delta T.$$

Then, if $Q \leqslant M^{2/3}$,

$$B(M, Q, \Delta) \ll (MQ + \Delta M^2 Q^2 + Q^6 M^{-2})(\log M)^2$$

with the implied constant depending at most on α.

Lemmas 1 and 4 are, respectively, Lemmas 2.2 and 2.4 (with $k = 1$) of E. Bombieri and H. Iwaniec [2]. Lemma 3 can be proved similarly to Lemma 5 of D. R. Heath-Brown [5]. Lemma 5, essentially due to [5], was first formally stated as Lemma 8 of R. C. Baker [1]. Lemma 2 is Lemma 2 of B. R. Srinivasan [15]. Lemma 6 is Proposition 2 of E. Fouvry and H. Iwaniec [4].

Now we are ready to prove Lemma A. We have

$$S(H, X, Y) \ll \sum_{h \sim H} \sum_{x \sim X} \Big| \sum_{y \in I(h,x)} C(y) e(f(h, x, y)) \Big|$$

where $I(h, x)$ is some subinterval of $(Y, 2Y]$. From Lemma 1, we have

$$L^{-1} S(H, X, Y) \ll \sum_{h \sim H} \sum_{x \sim X} \Big| \sum_{y \sim Y} C(y, \theta) e(f(h, x, y)) \Big|$$

where $C(y, \theta) = C(y) e(\theta y)$ for some real θ (θ is independent of h, x, y).

We consider the expression

$$R(q) \equiv (HXY)^2 q^{-1} + \sqrt{(HX)^4 F^{-1} Y^5 Z q^{-1}} + \sqrt{(HX)^3 F Y^{-1} q^5} + (HX)^2 q^2 + (HX)^{3/2} Y^2 Z^{1/2}. \tag{7}$$

By Lemma 2, we can choose a $Q \in (0, Y^{2/3}]$ such that (see (6))

$$R(Q) \ll \sqrt[7]{(HX)^{13} Y^9 F} + (HX)^2 Y^{4/3} + \sqrt[12]{(HX)^{23} F^{-4} Y^{24} Z^5} + (HX)^2 (F^{-1} Z Y^5)^{2/5} + \sqrt{(HX)^3 Y^4 Z} + (HX)^2 (F^{-3} Z^3 Y^{13})^{1/6} \ll E_1^2. \tag{8}$$

If $Q \leqslant 100$, then we trivially have

$$L^{-1} S(H, X, Y) \ll HXYQ^{-1/2} \ll \sqrt{R(Q)} \ll E_1.$$

Now we assume that $Q \geqslant 100$. By Cauchy's inequality and Lemma 3, we have

$$L^{-3} |S(H, X, Y)|^2 \ll (HXY)^2 Q^{-1} + (HXY) Q^{-1} |S_1| \tag{9}$$

where

$$S_1 = \sum_{(q,y,h,x) \in D_1} C(y+q, \theta) \overline{C(y-q, \theta)} e(Ah^\alpha x^\beta t(y, q)),$$

$$t(y, q) = (y+q)^\gamma - (y-q)^\gamma,$$

$$D_1 = D_1(Q_1) = \{(q, y, h, x) |\ y+q,\ y-q \sim Y,\ q \sim Q_1,\ h \sim H,\ x \sim X\}$$

for some Q_1 with $1 \leqslant 2Q_1 \leqslant Q/2$. By Lemma 4 we have (note that $F \gg Y$ by our assumption)

$$|S_1|^2 \ll FY^{-1} Q_1 A_1 A_2 \tag{10}$$

where A_1 is the number of lattice points $(h, x, \tilde{h}, \tilde{x})$ such that

$$|h^\alpha x^\beta - \tilde{h}^\alpha \tilde{x}^\beta| \ll A^{-1} Q_1^{-1} Y^{1-\gamma} \tag{11}$$

with $h, \tilde{h} \sim H$, $x, \tilde{x} \sim X$. Notice that (11) is equivalent to

$$|hx^{\beta/\alpha} - \tilde{h}\tilde{x}^{\beta/\alpha}| \ll HYX^{\beta/\alpha} F^{-1} Q_1^{-1}$$

so that Lemma 5 gives

$$A_1 \ll (HX + H^2 X^2 Y Q_1^{-1} F^{-1}) L^2. \tag{12}$$

Let A_2 stand for the number of lattice points (q, y, q_1, y_1) such that

$$|t(y, q) - t(y_1, q_1)| \ll (AH^\alpha X^\beta)^{-1}$$

with $Y/2 < y,\ y_1 < 3Y$, $q, q_1 \sim Q_1$. Recall that $Q_1 \leqslant Q/4 < Y^{2/3}$. Lemma 6 gives (with $\Delta = Q_1^{-1} Y F^{-1}$)

$$A_2 \ll (Q_1 Y + Q_1 Y^3 F^{-1} + Q_1^6 Y^{-2}) L^2. \tag{13}$$

From (9), (10) and (12), (13), we deduce that (see (7), (8))

$$L^{-5} |S(H, X, Y)|^2 \ll (HXY)^2 Q^{-1} + HXYQ^{-1} [FHXQ(Q + HXYF^{-1})(1 + Y^2 F^{-1} + Q^5 Y^{-3})]^{1/2} \ll R(Q). \tag{14}$$

Now Lemma A follows from our choice of Q and (14).

3. Lemmas cited. Write

$$A_3(x) = \sum_{n_1 n_2^2 n_3^3 \leqslant x} 1 = \sum_{i=1}^{3} C_{3,i} x^{1/i} + \Delta_3(x)$$

where

$$C_{3,i} = \prod_{\substack{j=1\\ j\neq i}}^{3} \zeta(j/i).$$

By a standard argument, we easily see the connection between $\Delta_3(x)$ and $\Delta(x)$.

LEMMA 7. *If* $\beta > 1/4$, *then*

$$\Delta_3(x) \ll x^{\beta+\varepsilon} \Rightarrow \Delta(x) \ll x^{\beta+\varepsilon}.$$

Proof. It is Lemma 1 of P. G. Schmidt [12].

LEMMA 8. *Let* (α, β, γ) *be any permutation of* $(1, 2, 3)$, $\Psi(u) = u-[u]-1/2$, *and*

$$S_{\alpha,\beta,\gamma} \equiv \sum_{\substack{m^{\alpha+\beta}n^{\gamma} \leqslant x\\ m>n}} \Psi(\sqrt[\alpha]{xm^{-\beta}n^{-\gamma}}).$$

Then, as $x\to\infty$,

$$\Delta_3(x) = -\sum_{(\alpha,\beta,\gamma)} S_{\alpha,\beta,\gamma} + O(x^{1/6}).$$

Proof. It is Theorem 1 of P. G. Schmidt [12].

Obviously, we have

$$S_{\alpha,\beta,\gamma} = \sum_{(M,N)} S_{\alpha,\beta,\gamma}(M, N) + O(x^{1/4}\log^2 x) \tag{15}$$

where M and N run through sequences $\{2^{-j}x^{1/(\alpha+\beta)}: j = 0, 1, \ldots\}$ and $\{2^{-k}x^{1/\gamma}: k = 0, 1, \ldots\}$ respectively, such that

$$MN \geqslant x^{1/4}, \quad 2M \geqslant N, \quad M^{\alpha+\beta}N^{\gamma} \leqslant x \tag{16}$$

and

$$S_{\alpha,\beta,\gamma}(M, N) = \sum_{(m,n)\in D} \Psi(\sqrt[\alpha]{xm^{-\beta}n^{-\gamma}}),$$

$$D \equiv D(M, N) \equiv \{(m, n)|\ m\sim M, n\sim N, m^{\alpha+\beta}n^{\gamma} \leqslant x, m>n\}. \tag{17}$$

As in [8], for any K (viewed as a parameter), $K\in[100, MN]$, we have

$$S_{\alpha,\beta,\gamma}(M, N) \ll (\log K)MNK^{-1} + \sum_{1\leqslant h\leqslant K^2} \min(1/h, K/h^2)\Big|\sum_{(m,n)\in D} e(f(h, m, n))\Big|$$

where

$$f(h, m, n) \equiv f(h, m, n, \alpha, \beta, \gamma) \equiv h\sqrt[\alpha]{xm^{-\beta}n^{-\gamma}}.$$

Thus, for some $H\in[1, K^2]$, we have

$$x^{-\varepsilon}S_{\alpha,\beta,\gamma}(M, N) \ll MNK^{-1} + \min(1, K/H)\Phi_{\alpha,\beta,\gamma}(H, M, N) \tag{18}$$

where

$$\Phi_{\alpha,\beta,\gamma}(H, M, N) \equiv H^{-1}\sum_{h\sim H}\Big|\sum_{(m,n)\in D} e(f(h, m, n))\Big|. \tag{19}$$

LEMMA 9. *Let* α, β *be real numbers,* $\alpha\beta(\alpha+\beta-1)(\alpha+\beta-2)\neq 0$. *Let* $f(x, y) = Ax^{\alpha}y^{\beta}$, $D\subset\{(x, y)|\ x\sim X, y\sim Y\}$, $X\geqslant Y$, $F\equiv AX^{\alpha}Y^{\beta}$, $N\equiv XY$. *Then*

$$S \equiv (NF)^{-\varepsilon}\sum_{(x,y)\in D} e(f(x, y))$$
$$\ll \sqrt[6]{F^2N^3} + N^{5/6} + \sqrt[8]{N^8F^{-1}X^{-1}} + NF^{-1/4} + NY^{-1/2}.$$

Proof. This lemma is actually a special case (with "Δ" = 0) of Theorem 1 of G. Kolesnik [8]. But the proof given in [8] contains some computational errors, so the final expression given there does not include the sum

$$N^{5/6} + N\sqrt[8]{F^{-1}X^{-1}} + NF^{-1/4}.$$

(Thus the proof of the estimate (4) given in [8] needs revision.) In fact, from R_1 and R_2 of p. 167 of [8], we can, after making easy calculation, obtain

$$(S/N)^2 \ll q^{-1} + \sqrt{F^2qN^{-3}} + Y^{-1} + (F^2qN^{-1})^{-1/4}$$
$$+ \sqrt[4]{F^2q^3X^{-4}N^{-3}} + \sqrt[12]{F^2X^{-12}qN^{-1}}$$

(noticing that in the second line of p. 168 of [8], the term $(F^2qN^{-1})^{-1/4}$ is written as $(F^2qN^{-1})^{-1/2}$, but from the context, this is obviously due to a computation error.) Then, using our Lemma 2, we can choose a q in the range $0<q<N/(\log N)$ which minimizes the above expression and gives

$$S \ll \sqrt[6]{F^2N^3} + N^{5/6} + NF^{-1/4} + \sqrt[8]{N^8F^{-1}X^{-1}} + NY^{-1/2}$$
$$+ \sqrt[26]{Y^{12}F^2N^{13}} + \sqrt[14]{Y^4F^2N^7} + NX^{-3/8}.$$

Since we always have

$$\sqrt[26]{Y^{12}F^2N^{13}} + \sqrt[14]{Y^4F^2N^7} + NX^{-3/8} \ll \sqrt[6]{F^2N^3} + N^{5/6}$$

the lemma then follows.

LEMMA 10. *Let* $0<a<b\leqslant 2a$, *let* $f(z)$ *be analytic on a domain* R *containing the real segment* $[a, b]$, *and let* $R' = \{z|\ az\in R\}$ *be an open convex set. Moreover,* $|f''(z)|\leqslant M$ *for* $z\in R$, $f(x)$ *is real for* $x\in R$ *is real and* $f''(x)\leqslant -kM$, $k>0$. *Let* $f'(b)=\alpha$, $f'(a)=\beta$, *and define* x_v *for each integer* v *in the range* $\alpha<v\leqslant\beta$ *by* $f'(x_v)=v$. *Then*

$$\sum_{a<n\leqslant b} e(f(n)) = e(-1/8)\sum_{\alpha<v\leqslant\beta} |f''(x_v)|^{-1/2}e(f(x_v)-vx_v)$$
$$+ O(M^{-1/2}) + O(\log(2+(b-a)M)).$$

Proof. This is Lemma 6 of D. R. Heath-Brown [14].

4. The estimation of $S_{\alpha,\beta,\gamma}$, $(\alpha, \beta, \gamma) \neq (1, 2, 3)$ and $(2, 1, 3)$. In this section, we shall prove

THEOREM 1. *If $(\alpha, \beta, \gamma) \neq (1, 2, 3)$ and $(2, 1, 3)$, then*

$$S_{\alpha,\beta,\gamma} \ll x^{1/4+\varepsilon}.$$

This is an easy consequence of the following lemma.

LEMMA 11. *Let (α, β, γ) be a permutation of $(1, 2, 3)$. Then*

$$x^{-\varepsilon} S_{\alpha,\beta,\gamma}(M, N) \ll \sqrt[8]{x^2 M^{5\alpha-2\beta} N^{5\alpha-2\gamma}} + x^{1/4}.$$

Proof. By Lemma 10 and partial summation (using (16)), we get

$$\sum_{(m,n)\in D_1} e(f(h, m, n)) \ll MF^{-1/2} \sum_{(n,u)\in D_1} e(g(h, n, u)) + x^{1/4} \tag{20}$$

where D is given by (17), D_1 is a suitable subregion of

$$\left\{(n, u) \,\middle|\, u = \frac{\partial f}{\partial m}(h, m, n),\ (m, n)\in D\right\}$$

(D_1 satisfies the requirements of Lemma A), and

$$g(h, n, u) = C(\alpha, \beta, \gamma)(xh^\alpha u^\beta n^{-\gamma})^{1/(\alpha+\beta)},$$

$$F \equiv GH \equiv H(\sqrt[\alpha]{xM^{-\beta}N^{-\gamma}}). \tag{21}$$

Note that

$$(n, u)\in D_1 \Rightarrow u \simeq F/M$$

($b \simeq B$ means that $C_1 \leqslant b/B \leqslant C_2$ for two suitable constants C_1 and C_2). By Lemma 9 we have

$$x^{-\varepsilon} \sum_{(n,u)\in D_1} e(g(h, n, u)) \ll \sqrt[6]{F^5 N^3 M^{-3}} + NF^{3/4} M^{-1}$$
$$+ NFM^{-1}(F^{-1}N^{-1})^{1/8} + N(FM^{-1})^{1/2} + N^{1/2} FM^{-1}.$$

Again using (16), we see that

$$x^{-\varepsilon} MF^{-1/2} \sum_{(n,u)\in D_1} e(g(h, n, u)) \ll \sqrt[6]{F^2 M^3 N^3} + \sqrt{FN}. \tag{22}$$

From (18)–(22), we get

$$x^{-\varepsilon} S_{\alpha,\beta,\gamma}(M, N) \tag{23}$$
$$\ll MNK^{-1} + \sqrt[6\alpha]{K^{2\alpha} x^2 M^{3\alpha-2\beta} N^{3\alpha-2\gamma}} + \sqrt[2\alpha]{K^\alpha x M^{-\beta} N^{\alpha-\gamma}} \equiv E_1(K).$$

By Lemma 2, we can choose a $K_0 \in [0, MN]$ such that

$$E_1(K_0) \ll \sqrt[8\alpha]{x^2 M^{5\alpha-2\beta} N^{5\alpha-2\gamma}} + x^{1/4}. \tag{24}$$

If $K_0 \geqslant 100$, we specify $K = K_0$ in (23), and the lemma follows from (23) and (24). If $K_0 < 100$, we trivially have

$$S_{\alpha,\beta,\gamma}(M, N) \ll MNK_0^{-1} \ll E_1(K_0) \tag{25}$$

and the lemma follows from (24) and (25).

Proof of Theorem 1. Assume that $(\alpha, \beta, \gamma) \neq (1, 2, 3)$ and $(2, 1, 3)$. By (16), we see that

$$M^{5\alpha-2\beta} N^{5\alpha-2\gamma} \ll (M^{\alpha+\beta} N^\gamma)^{2\alpha-2} \ll x^{2\alpha-2}.$$

Hence Theorem 1 follows from Lemma 11.

5. The estimation of $S_{2,1,3}$. The main object of the present section is to prove the following theorem.

THEOREM 2.

$$S_{2,1,3} \ll x^{1/4+\varepsilon}.$$

LEMMA 12. *Let (α, β, γ) be a permutation of $(1, 2, 3)$, then*

$$x^{-\varepsilon} S_{\alpha,\beta,\gamma}(M, N) \ll \sqrt[20]{G^7 M^7 N^{15}} + \sqrt[9]{G^3 M^3 N^7} + \sqrt[15]{G^3 M^3 N^{16}} \tag{26}$$
$$+ \sqrt[30]{G^7 M^7 N^{30}} + \sqrt[35]{G^{12} M^{12} N^{25}} + \sqrt[4]{GMN^4} + \sqrt[4]{G^2 M^2 N^3} + x^{1/4} := E_2,$$

where G is given by (21).

Proof. From (19) and (20), we see that

$$x^{-\varepsilon/2} \Phi_{\alpha,\beta,\gamma}(H, M, N) \ll M(H^3 G)^{-1/2} \sum_{h\sim H} \Big| \sum_{(n,u)\in D_1} e(g(h, n, u))\Big| + x^{1/4}. \tag{27}$$

We apply Lemma A, with $(H, X, Y) \simeq (H, F/M, N)$. ($(X_1, X_2, X_3) \simeq (Y_1, Y_2, Y_3)$ means that $X_i \simeq Y_i$ for $1 \leqslant i \leqslant 3$.) We get

$$x^{-\varepsilon/2} \sum_{h\sim H} \Big| \sum_{(n,u)\in D_1} e(g(h, n, u))\Big| \ll \sqrt[14]{H^{27} G^{14} M^{-13} N^9} + H^2 GM^{-1} N^{2/3} \tag{28}$$
$$+ \sqrt[12]{H^{21} G^9 M^{-12} N^{13}} + \sqrt[24]{H^{42} G^{19} M^{-23} N^{24}} + \sqrt[24]{H^{47} G^{24} M^{-23} N^{14}}$$
$$+ \sqrt[4]{H^6 G^3 M^{-3} N^4} + \sqrt[4]{H^7 G^4 M^{-3} N^2}.$$

From (18), (19), (27) and (28), we have

$$x^{-\varepsilon} S_{\alpha,\beta,\gamma}(M, N) \ll MNK^{-1} + \sqrt[14]{K^6 G^7 MN^9} + \sqrt[6]{K^3 G^3 N^4} \tag{29}$$
$$+ \sqrt[12]{K^3 G^3 N^{13}} + \sqrt[24]{K^6 G^7 MN^{24}} + \sqrt[24]{K^{11} G^{12} MN^{14}} + \sqrt[4]{KG^2 MN^2}$$
$$+ \sqrt[4]{GMN^4} := E_2(K).$$

By Lemma 2, there is a $K_0\in[0, MN]$ such that

(30) $$E_2(K_0)\ll E_2$$

(see (26)). The lemma is proved in view of (29) and (30).

LEMMA 13. *Let (α, β, γ) be a permutation of* (1, 2, 3), *Then*

(31) $$x^{-\varepsilon}S_{\alpha,\beta,\gamma}(M, N)\ll\sqrt[16]{G^3M^7N^{15}}+\sqrt[7]{GM^3N^7}+\sqrt[16]{G^4M^3N^{16}}+\sqrt[31]{G^8M^7N^{30}}+\sqrt[26]{G^3M^{12}N^{25}}+\sqrt[5]{G^2MN^4}+\sqrt[4]{GM^2N^3}+x^{1/4}:=E_3$$

where G is given by (21).

Proof. Applying Lemma A with $(H, X, Y)\simeq(H, N, F/M)$, we get

(32) $$x^{-\varepsilon/2}\sum_{h\sim H}\Big|\sum_{(n,u)\in D_1}e(g(h, n, u))\Big|\ll\sqrt[14]{H^{23}G^{10}M^{-9}N^{13}}+\sqrt[3]{H^5G^2M^{-2}N^3}+\sqrt[12]{H^{22}G^{10}M^{-13}N^{12}}+\sqrt[24]{H^{43}G^{20}M^{-24}N^{23}}+\sqrt[4]{H^7G^4M^{-4}N^3}+\sqrt[4]{H^6G^3M^{-2}N^3}+x^{1/4}.$$

Therefore, from (18), (19), (27) and (32) we get

$$x^{-\varepsilon}S_{\alpha,\beta,\gamma}(M, N)\ll MNK^{-1}+\sqrt[14]{K^2G^3M^5N^{13}}+\sqrt[6]{KGM^2N^6}+\sqrt[12]{K^4G^4M^{-1}N^{12}}+\sqrt[24]{K^7G^8N^{23}}+\sqrt[24]{K^2G^3M^{10}N^{23}}+\sqrt[4]{KG^2N^3}+\sqrt[4]{GM^2N^3}+x^{1/4}:=E_3(K).$$

Now, by Lemma 2, there is a $K_0\in[0, MN]$ such that

$$E_3(K)\ll E_3$$

(see (31)). The lemma is proved.

Proof of Theorem 2. From (16), (21) and Lemma 12, we have

(33) $$x^{-\varepsilon}S_{2,1,3}(M, N)\ll\sqrt[40]{x^7M^7N^9}+\sqrt[18]{x^3M^3N^5}+\sqrt[30]{x^3M^3N^{23}}+\sqrt[60]{x^7M^7N^{39}}+\sqrt[35]{x^6M^6N^7}+\sqrt[5]{xM}+x^{1/4}\ll\sqrt[5]{xM}+x^{1/4}.$$

From (16), (21) and Lemma 13, we also have

(34) $$x^{-\varepsilon}S_{2,1,3}(M, N)\ll\sqrt[32]{x^3M^{11}N^{21}}+\sqrt[14]{xM^5N^{11}}+\sqrt[16]{x^2MN^{10}}+\sqrt[31]{x^4M^3N^{18}}+\sqrt[52]{x^3M^{21}N^{41}}+\sqrt[5]{xN}+\sqrt[8]{xM^3N^3}+x^{1/4}$$
$$\ll\sqrt[32]{x^3M^{11}N^{21}}+\sqrt[14]{xM^5N^{11}}+\sqrt[52]{x^3M^{21}N^{41}}+\sqrt[5]{xN}+x^{1/4}.$$

From (33) and (34), we conclude that

(35) $$x^{-\varepsilon}S_{2,1,3}(M, N)\ll\sum_{i=1}^{4}R_i(M, N)+x^{1/4}$$

where, by virtue of (16),

(36) $$R_1(M, N)=\min(\sqrt[5]{xM}, \sqrt[32]{x^3M^{11}N^{21}})\leqslant(\sqrt[5]{xM})^{\sigma_1}(\sqrt[32]{x^3M^{11}N^{21}})^{\delta_1}=x^{13/82}(MN)^{21/82}\leqslant x^{20/82}$$

with $(\sigma_1, \delta_1)=(50/82,\ 32/82)$;

(37) $$R_2(M, N)=\min(\sqrt[5]{xM}, \sqrt[14]{xM^5N^{11}})\leqslant(\sqrt[5]{xM})^{\sigma_2}(\sqrt[14]{xM^5N^{11}})^{\delta_2}=x^{7/44}(MN)^{1/4}\leqslant x^{32/132}$$

with $(\sigma_2, \delta_2)=(30/44,\ 14/44)$;

(38) $$R_3(M, N)=\min(\sqrt[5]{xM}, \sqrt[52]{x^3M^{21}N^{41}})\leqslant(\sqrt[5]{xM})^{\sigma_3}(\sqrt[52]{x^3M^{21}N^{41}})^{\delta_3}=x^{23/152}(MN)^{41/152}\leqslant x^{110/456}$$

with $(\sigma_3, \delta_3)=(25/38,\ 13/38)$;

(39) $$R_4(M, N)=\min(\sqrt[5]{xM}, \sqrt[5]{xN})\leqslant(\sqrt[5]{xM})^{1/2}(\sqrt[5]{xN})^{1/2}=x^{1/5}(MN)^{1/10}\leqslant x^{7/30}.$$

From (35)–(39), we have

(40) $$S_{2,1,3}(M, N)\ll x^{1/4+\varepsilon}.$$

Theorem 2 follows from (15) and (40).

6. The estimation of $S_{1,2,3}$ and the proof of Theorem A. Throughout this section, we assume that

$$\theta=40/159,\quad 100x^{\theta}\leqslant MN\leqslant x^{1/3}.$$

We proceed to estimate the crucial sum $S_{1,2,3}$. We have

THEOREM 3.

$$S_{1,2,3}\ll x^{\theta+\varepsilon}.$$

Theorem A follows from Lemmas 7, 8 and Theorems 1, 2, 3 explicitly.

LEMMA 14.

$$x^{-\varepsilon}\Phi_{1,2,3}(H, M, N)\ll\sqrt[14]{xM^7N^{10}}+NM^{2/3}+\sqrt[12]{H^{-3}x^{-3}M^{19}N^{21}}+\sqrt[24]{H^{-5}x^{-4}M^{32}N^{35}}+\sqrt[24]{xM^{12}N^{20}}+\sqrt[4]{H^{-1}M^4N^3}+x^{1/4}$$

where $\Phi_{1,2,3}(H, M, N)$ is given by (19).

Proof. Using Lemma A to the sum $H\cdot\Phi_{1,2,3}(H, M, N)$ directly, with $(H, X, Y)\simeq(H, N, M)$, we get the required estimate.

LEMMA 15. *For* $K = MNx^{-\theta}$, $1 \leqslant H \leqslant K^2$, *we have*

$$x^{-\varepsilon}\min(1, K/H)\cdot\Phi_{1,2,3}(H, M, N) \ll \sqrt[14]{x^{3-2\theta}MN^6} + \sqrt[6]{x^{1-\theta}MN^4}$$
$$+\sqrt[12]{x^{4-4\theta}M^{-5}N^4} + \sqrt[24]{x^{8-7\theta}M^{-9}N^6} + \sqrt[24]{x^{3-2\theta}M^6N^{16}}$$
$$+\sqrt[24]{x^3M^7N^{12}} + \sqrt[22]{x^3M^6N^{10}} + \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}}, \sqrt[14]{xM^7N^{10}})$$
$$+\min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}}, NM^{2/3}) + \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}}, \sqrt[24]{xM^{12}N^{20}}).$$

Proof. From (27), (32), we get (with $G \simeq xM^{-2}N^{-3}$)

$$x^{-\varepsilon/2}\Phi_{1,2,3}(H, M, N) \ll \sqrt[14]{H^2x^3M^{-1}N^4} + \sqrt[6]{HxN^3} + \sqrt[12]{H^4x^4M^{-9}}$$
$$+\sqrt[24]{H^7x^8M^{-16}N^{-1}} + \sqrt[24]{H^2x^3M^4N^{14}} + \sqrt[4]{x^2HM^{-4}N^{-3}} + x^{1/4}. \tag{41}$$

In view of Lemma 14 and (41), we have

$$x^{-\varepsilon/2}\Phi_{1,2,3}(H, M, N) \ll \sqrt[14]{H^2x^3M^{-1}N^4} + \sqrt[6]{HxN^3}$$
$$+\sqrt[12]{H^4x^4M^{-9}} + \sqrt[24]{H^2x^3M^4N^{14}} + \sqrt[24]{H^7x^8M^{-16}N^{-1}}$$
$$+\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[14]{xM^7N^{10}}) + \min(\sqrt[4]{x^2HM^{-4}N^{-3}}, NM^{2/3})$$
$$+\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[12]{H^{-3}x^{-3}M^{19}N^{21}})$$
$$+\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[24]{H^{-5}x^{-4}M^{32}N^{35}})$$
$$+\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[24]{xM^{12}N^{20}})$$
$$+\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[4]{H^{-1}M^4N^3}) + x^{1/4}. \tag{42}$$

We have

$$\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[12]{H^{-3}x^{-3}M^{19}N^{21}})$$
$$\leqslant (\sqrt[4]{x^2HM^{-4}N^{-3}})^{1/2}(\sqrt[12]{H^{-3}x^{-3}M^{19}N^{21}})^{1/2} = \sqrt[24]{x^3M^7N^{12}}; \tag{43}$$

similarly,

$$\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[24]{H^{-5}x^{-4}M^{32}N^{35}}) \leqslant \sqrt[22]{x^3M^6N^{10}}, \tag{44}$$

$$\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[4]{H^{-1}M^4N^3}) \leqslant x^{1/4} \tag{45}$$

and

$$\min(1, K/H)\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[14]{xM^7N^{10}})$$
$$\leqslant \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}}, \sqrt[14]{xM^7N^{10}}), \tag{46}$$

$$\min(1, K/H)\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, NM^{2/3})$$
$$\leqslant \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}}, NM^{2/3}), \tag{47}$$

$$\min(1, K/H)\min(\sqrt[4]{x^2HM^{-4}N^{-3}}, \sqrt[24]{xM^{12}N^{20}})$$
$$\leqslant \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}}, \sqrt[24]{xM^{12}N^{20}}). \tag{48}$$

The lemma follows from (42)–(48).

Proof of Theorem 3. From Lemma 11, we have

$$x^{-\varepsilon}S_{1,2,3}(M, N) \ll \sqrt[8]{x^2MN^{-1}} + x^{1/4}. \tag{49}$$

From (16), (18), Lemma 15 and (49), we have

$$x^{-\varepsilon}S_{1,2,3}(M, N) \ll \sqrt[14]{x^{3-2\theta}MN^6} + \sqrt[6]{x^{1-\theta}MN^4} + \sqrt[24]{x^{3-2\theta}M^6N^{16}}$$
$$+\sqrt[24]{x^3M^7N^{12}} + \sqrt[22]{x^3M^6N^{10}} + \sum_{i=5}^{7} R_i(M, N) + x^\theta \tag{50}$$

where

$$R_5(M, N) = \min(\sqrt[4]{x^{2-\theta}M^{-3}N^{-2}}, \sqrt[14]{xM^7N^{10}}, \sqrt[8]{x^2MN^{-1}})$$
$$\leqslant (\sqrt[4]{x^{7/4}M^{-3}N^{-2}})^{\alpha_1}(\sqrt[14]{xM^7N^{10}})^{\beta_1}(\sqrt[8]{x^2MN^{-1}})^{\gamma_1}$$
$$= x^{267/1064} \tag{51}$$

with $(\alpha_1, \beta_1, \gamma_1) = (272/1064, 280/1064, 512/1064)$;

$$R_6(M, N) = \min(\sqrt[4]{x^{7/4}M^{-3}N^{-2}}, NM^{2/3}, \sqrt[8]{x^2MN^{-1}})$$
$$\leqslant (\sqrt[4]{x^{7/4}M^{-3}N^{-2}})^{\alpha_2}(NM^{2/3})^{\beta_2}(\sqrt[8]{x^2MN^{-1}})^{\gamma_2} = x^{1/4} \tag{52}$$

with $(\alpha_2, \beta_2, \gamma_2) = (4/15, 3/15, 8/15)$;

$$R_7(M, N) = \min(\sqrt[4]{x^{7/4}M^{-3}N^{-2}}, \sqrt[24]{xM^{12}N^{20}}, \sqrt[8]{x^2MN^{-1}})$$
$$\leqslant (\sqrt[4]{x^{7/4}M^{-3}N^{-2}})^{\alpha_3}(\sqrt[24]{xM^{12}N^{20}})^{\beta_3}(\sqrt[8]{x^2MN^{-1}})^{\gamma_3}$$
$$= x^{133/536} \tag{53}$$

with $(\alpha_3, \beta_3, \gamma_3) = (16/67, 15/67, 36/67)$.

Hence from (50)–(53), we have

$$x^{-\varepsilon}S_{1,2,3}(M, N) \ll \sqrt[14]{x^{3-2\theta}MN^6} + \sqrt[6]{x^{1-\theta}MN^4} + \sqrt[24]{x^{3-2\theta}M^6N^{16}}$$
$$+\sqrt[24]{x^3M^7N^{12}} + \sqrt[22]{x^3M^6N^{10}} + x^\theta. \tag{54}$$

From Lemma 12 (with $G \simeq xM^{-2}N^{-3}$), using (16), we get

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sqrt[20]{x^7M^{-7}N^{-6}} + \sqrt[9]{x^3M^{-3}N^{-2}} + \sqrt[35]{x^{12}M^{-12}N^{-11}} + \sqrt[5]{x^2M^{-2}N^{-3}} + x^{1/4}. \tag{55}$$

If $MN \leqslant x^{0.3}$, from (54) we see (using (16)) that

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sqrt[14]{x^{3-2\theta}MN^6} + x^{\theta}. \tag{56}$$

From (49), (55) and (56), we have

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sum_{i=8}^{11} R_i(M,N) + x^{\theta} \tag{57}$$

where

$$\begin{aligned} R_8(M,N) &= \min\left(\sqrt[14]{x^{3-2\theta}MN^6}, \sqrt[20]{x^7M^{-7}N^{-6}}, \sqrt[8]{x^2MN^{-1}}\right) \\ &\leqslant \left(\sqrt[14]{x^{3-2\theta}MN^6}\right)^{\alpha_4}\left(\sqrt[20]{x^7M^{-7}N^{-6}}\right)^{\beta_4}\left(\sqrt[8]{x^2MN^{-1}}\right)^{\gamma_4} \\ &= x^{(160-26\theta)/610} = x^{\theta} \end{aligned} \tag{58}$$

with $(\alpha_4, \beta_4, \gamma_4) = (182/610,\ 140/610,\ 288/610)$;

$$\begin{aligned} R_9(M,N) &= \min\left(\sqrt[14]{x^{3-2\theta}MN^6}, \sqrt[9]{x^3M^{-3}N^{-2}}, \sqrt[8]{x^2MN^{-1}}\right) \\ &\leqslant \left(\sqrt[14]{x^{3-2\theta}MN^6}\right)^{\alpha_5}\left(\sqrt[9]{x^3M^{-3}N^{-2}}\right)^{\beta_5}\left(\sqrt[8]{x^2MN^{-1}}\right)^{\gamma_5} \\ &= x^{(68-10\theta)/261} < x^{\theta} \end{aligned} \tag{59}$$

with $(\alpha_5, \beta_5, \gamma_5) = (70/261,\ 63/261,\ 128/261)$;

$$\begin{aligned} R_{10}(M,N) &= \min\left(\sqrt[14]{x^{3-2\theta}MN^6}, \sqrt[35]{x^{12}M^{-12}N^{-11}}, \sqrt[8]{x^2MN^{-1}}\right) \\ &\leqslant \left(\sqrt[14]{x^{3-2\theta}MN^6}\right)^{\alpha_6}\left(\sqrt[35]{x^{12}M^{-12}N^{-11}}\right)^{\beta_6}\left(\sqrt[8]{x^2MN^{-1}}\right)^{\gamma_6} \\ &= x^{(550-92\theta)/2110} < x^{1/4} \end{aligned} \tag{60}$$

with $(\alpha_6, \beta_6, \gamma_6) = (644/2110,\ 490/2110,\ 976/2110)$;

$$\begin{aligned} R_{11}(M,N) &= \min\left(\sqrt[14]{x^{3-2\theta}MN^6}, \sqrt[5]{x^2M^{-2}N^{-3}}, \sqrt[8]{x^2MN^{-1}}\right) \\ &\leqslant \left(\sqrt[14]{x^{3-2\theta}MN^6}\right)^{\alpha_7}\left(\sqrt[5]{x^2M^{-2}N^{-3}}\right)^{\beta_7}\left(\sqrt[8]{x^2MN^{-1}}\right)^{\gamma_7} \\ &= x^{(47-10\theta)/177} < x^{\theta} \end{aligned} \tag{61}$$

with $(\alpha_7, \beta_7, \gamma_7) = (70/177,\ 35/177,\ 72/177)$.

From (57)–(61) we have

$$S_{1,2,3}(M,N) \ll x^{\theta+\varepsilon}. \tag{62}$$

If $MN > x^{0.3}$, from (55) we have

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sqrt[20]{x^7M^{-7}N^{-6}} + \sqrt[5]{x^2M^{-2}N^{-3}} + x^{1/4}. \tag{63}$$

From (49), (54) and (63), we see that

$$x^{-\varepsilon}S_{1,2,3}(M,N) \ll \sum_{i=12}^{21} R_i(M,N) + x^{\theta} \tag{64}$$

where

$$R_{12}(M,N) = R_8(M,N) \leqslant x^{\theta}, \tag{65}$$

$$\begin{aligned} R_{13}(M,N) &= \min\left(\sqrt[20]{x^7M^{-7}N^{-6}}, \sqrt[6]{x^{1-\theta}MN^4}, \sqrt[8]{x^2MN^{-1}}\right) \\ &\leqslant \left(\sqrt[20]{x^7M^{-7}N^{-6}}\right)^{\alpha_8}\left(\sqrt[6]{x^{1-\theta}MN^4}\right)^{\beta_8}\left(\sqrt[8]{x^2MN^{-1}}\right)^{\gamma_8} \\ &= x^{(92-13\theta)/354} < x^{\theta} \end{aligned} \tag{66}$$

with $(\alpha_8, \beta_8, \gamma_8) = (100/354,\ 78/354,\ 176/354)$;

$$\begin{aligned} R_{14}(M,N) &= \min\left(\sqrt[20]{x^7M^{-7}N^{-6}}, \sqrt[24]{x^{3-2\theta}M^6N^{16}}, \sqrt[8]{x^2MN^{-1}}\right) \\ &\leqslant \left(\sqrt[20]{x^7M^{-7}N^{-6}}\right)^{\alpha_9}\left(\sqrt[24]{x^{3-2\theta}M^6N^{16}}\right)^{\beta_9}\left(\sqrt[8]{x^2MN^{-1}}\right)^{\gamma_9} \\ &= x^{(345-26\theta)/1360} < x^{1/4} \end{aligned} \tag{67}$$

with $(\alpha_9, \beta_9, \gamma_9) = (110/340,\ 78/340,\ 152/340)$;

$$\begin{aligned} R_{15}(M,N) &= \min\left(\sqrt[20]{x^7M^{-7}N^{-6}}, \sqrt[24]{x^3M^7N^{12}}, \sqrt[8]{x^2MN^{-1}}\right) \\ &\leqslant \left(\sqrt[20]{x^7M^{-7}N^{-6}}\right)^{\alpha_{10}}\left(\sqrt[24]{x^3M^7N^{12}}\right)^{\beta_{10}}\left(\sqrt[8]{x^2MN^{-1}}\right)^{\gamma_{10}} \\ &= x^{64/257} \end{aligned} \tag{68}$$

with $(\alpha_{10}, \beta_{10}, \gamma_{10}) = (95/257,\ 78/257,\ 84/257)$;

$$\begin{aligned} R_{16}(M,N) &= \min\left(\sqrt[20]{x^7M^{-7}N^{-6}}, \sqrt[22]{x^3M^6N^{10}}, \sqrt[8]{x^2MN^{-1}}\right) \\ &\leqslant \left(\sqrt[20]{x^7M^{-7}N^{-6}}\right)^{\alpha_{11}}\left(\sqrt[22]{x^3M^6N^{10}}\right)^{\beta_{11}}\left(\sqrt[8]{x^2MN^{-1}}\right)^{\gamma_{11}} \\ &= x^{219/878} \end{aligned} \tag{69}$$

with $(\alpha_{11}, \beta_{11}, \gamma_{11}) = (160/439,\ 143/439,\ 136/439)$;

$$R_{17}(M,N) = R_{11}(M,N) < x^{\theta}, \tag{70}$$

$$R_{18}(M,N)=\min(\sqrt[5]{x^2M^{-2}N^{-3}},\ \sqrt[6]{x^{1-\theta}MN^4},\ \sqrt[8]{x^2MN^{-1}}) \tag{71}$$
$$\leqslant(\sqrt[5]{x^2M^{-2}N^{-3}})^{\alpha_{12}}(\sqrt[6]{x^{1-\theta}MN^4})^{\beta_{12}}(\sqrt[8]{x^2MN^{-1}})^{\gamma_{12}}$$
$$=x^{(5-\theta)/19}<x^{1/4}$$

with $(\alpha_{12},\beta_{12},\gamma_{12})=(5/19,\ 6/19,\ 8/19)$;

$$R_{19}(M,N)=\min(\sqrt[5]{x^2M^{-2}N^{-3}},\ \sqrt[24]{x^{3-2\theta}M^6N^{16}},\ \sqrt[8]{x^2MN^{-1}}) \tag{72}$$
$$\leqslant(\sqrt[5]{x^2M^{-2}N^{-3}})^{\alpha_{13}}(\sqrt[24]{x^{3-2\theta}M^6N^{16}})^{\beta_{13}}(\sqrt[8]{x^2MN^{-1}})^{\gamma_{13}}$$
$$=x^{(87-10\theta)/342}<x^{1/4}$$

with $(\alpha_{13},\beta_{13},\gamma_{13})=(55/171,\ 60/171,\ 56/171)$;

$$R_{20}(M,N)=\min(\sqrt[5]{x^2M^{-2}N^{-3}},\ \sqrt[24]{x^3M^7N^{12}},\ \sqrt[8]{x^2MN^{-1}}) \tag{73}$$
$$\leqslant(\sqrt[5]{x^2M^{-2}N^{-3}})^{\alpha_{14}}(\sqrt[24]{x^3M^7N^{12}})^{\beta_{14}}(\sqrt[8]{x^2MN^{-1}})^{\gamma_{14}}$$
$$=x^{59/239}$$

with $(\alpha_{14},\beta_{14},\gamma_{14})=(95/239,\ 120/239,\ 24/239)$;

$$R_{21}(M,N)=\min(\sqrt[5]{x^2M^{-2}N^{-3}},\ \sqrt[22]{x^3M^6N^{10}},\ \sqrt[8]{x^2MN^{-1}}) \tag{74}$$
$$\leqslant(\sqrt[5]{x^2M^{-2}N^{-3}})^{\alpha_{15}}(\sqrt[22]{x^3M^6N^{10}})^{\beta_{15}}(\sqrt[8]{x^2MN^{-1}})^{\gamma_{15}}$$
$$=x^{51/206}$$

with $(\alpha_{15},\beta_{15},\gamma_{15})=(80/206,\ 110/206,\ 16/206)$.

From (64)–(74), we have

$$S_{1,2,3}(M,N)\ll x^{\theta+\varepsilon}. \tag{75}$$

Theorem 3 follows from (15), (62) and (75).

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