$\varphi_1, \varphi_3, \varphi_4, \varphi_6$ proviennent de faces, φ_2, φ_5 proviennent de points extrémaux, φ_7 provient d'une unité formelle.

$$|\mathcal{E}_X/\mathcal{U}_X^{(1)}| = 3, \quad E = E_0 = \{n;\ \varepsilon(n) \text{ entier de } K_n\}, \quad \overline{\lim_{\substack{n\in E\\ n\to+\infty}}}\ |\mathcal{E}_n/\mathcal{U}_n| = 7.$$

Remarque sur le théorème 1. Pour montrer le théorème 1, on pourrait penser utiliser le résultat de H. C. Williams [6] qui dit intuitivement la chose suivante:

"Si φ est un élément d'un ordre $\mathcal{O}$ de petite norme alors φ est une meilleure approximation de G". Précisément:

Soit $\mathcal{O} = \mathbf{Z}+\mathbf{Z}\mu+\mathbf{Z}\nu$ un ordre de K (corps cubique pur) et D_0 son discriminant. Soit φ un élément non nul de $\mathcal{O}$. Si:

$$|N(\varphi)| < \sqrt[4]{D_0/27} \tag{40}$$

alors φ est une meilleure approximation de $\mathcal{O}$.

Malheureusement, l'approximation formelle obtenue de façon générale et le résultat de H.C.Williams ne sont pas suffisants pour démontrer le théorème 1. En effet, si φ est un point extrémal de $\mathcal{O}_X$ on a:

$$\deg N(\varphi) \leqslant 3d-2 \text{ (voir proposition 3)} \quad \text{et} \quad \deg(\operatorname{Disc} \Delta) = 6d,$$

l'inégalité (40) n'est donc pas satisfaite asymptotiquement.

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A Dirichlet series for modular forms of degree n

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1. Introduction. In a recent paper Kohnen and Skoruppa [6] introduced a new type of Dirichlet series, which is associated with the Fourier–Jacobi expansion of a pair f, g of Siegel cusp forms of the same weight and degree 2. The proof is based on the Rankin–Selberg method. If $f = g$ is a simultaneous Hecke eigenform in the so-called Maaß Spezialschar, this Dirichlet series is proportional to the spinor zeta function attached to f by Andrianov.

Then Yamazaki [17], [18] derived the analytic continuation and functional equation for this type of Dirichlet series in the case of arbitrary degree n. The main point of his proof is the analytic continuation of an Eisenstein series of Klingen type, which is derived from Langlands' theory. Recently Raghavan and Sengupta [13] obtained analogous results for Hermitian modular forms of degree 2 with respect to the Gaussian number field.

In this paper we give an elementary proof of Yamazaki's result [17], which only makes use of well-known properties of Epstein zeta functions. A modification of the arguments and another application of the Rankin–Selberg method lead to the corresponding result for Hermitian modular forms of degree n with respect to the Gaussian number field. Finally, we deal with modular forms of quaternions of degree n (cf. [11], [7]). We derive analytic continuation and functional equation of the attached Dirichlet series. If f and g belong to the Maaß space (cf. [8]) and if f is a simultaneous eigenform under all Hecke operators, then the Dirichlet series possesses an Euler product expansion and is proportional to the Andrianov zeta function attached to f.

2. Preliminaries. A good deal of work can be done for all the three types of modular forms simultaneously just as in [7]. Therefore let $\boldsymbol{F}$ stand for the field $\boldsymbol{R}$ of real numbers resp. $\boldsymbol{C}$ of complex numbers resp. for the skew-field $\boldsymbol{H}$ of Hamiltonian quaternions. Set $r = [\boldsymbol{F}:\boldsymbol{R}]$ and denote the standard basis of $\boldsymbol{F}$ over $\boldsymbol{R}$ by $e_1, \ldots, e_r$, where

$$e_1 = 1, \quad e_2^2 = e_3^2 = e_4^2 = -1, \quad e_2 e_3 = -e_3 e_2 = e_4.$$

Let $a \mapsto \bar{a}$ be the canonical involution on $\boldsymbol{F}$ and define

$$\operatorname{Re}(a) := \tfrac{1}{2}(a+\bar{a}), \quad \langle a, b\rangle := \operatorname{Re}(\bar{a}b), \quad N(a) := a\bar{a}.$$

The letter I is reserved for the $n\times n$ identity matrix and J for the $2n\times 2n$ matrix

$$J:=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}.$$

Given matrices A, B of appropriate size set

$$A[B]:=\bar{B}'AB,$$

where the prime stands for the transpose.

Let $\mathrm{Sym}(n;\boldsymbol{F})$ denote the $n+\frac{1}{2}rn(n-1)$-dimensional real vector space of all over $\boldsymbol{F}$ Hermitian $n\times n$ matrices and $\mathrm{Pos}(n;\boldsymbol{F}):=\{Y\in\mathrm{Sym}(n;\boldsymbol{F})\mid Y>0\}$ the open subset of all positive definite matrices. Then the *half-space of degree* n is defined to be

$$H(n;\boldsymbol{F}):=\mathrm{Sym}(n;\boldsymbol{F})+i\,\mathrm{Pos}(n;\boldsymbol{F})\subset\mathrm{Sym}(n;\boldsymbol{F})\otimes_{\boldsymbol{R}}\boldsymbol{C}.$$

Clearly $H(n;\boldsymbol{R})$ is the *Siegel half-space*, $H(n;\boldsymbol{C})$ the *Hermitian half-space*, and $H(n;\boldsymbol{H})$ the *half-space of quaternions*. Moreover, $H:=H(1;\boldsymbol{F})$ equals the upper half-plane in $\boldsymbol{C}$. The *symplectic group* (over $\boldsymbol{F}$)

$$\mathrm{Sp}(n;\boldsymbol{F}):=\{M\in\mathrm{Mat}(2n;\boldsymbol{F})\mid J[M]=J\}$$

acts on $H(n;\boldsymbol{F})$ in the usual way by

$$Z\mapsto M\langle Z\rangle:=(AZ+B)(CZ+D)^{-1},\qquad M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}(n;\boldsymbol{F}).$$

Given $Z=X+iY\in H(n;\boldsymbol{F})$ consider

$$P_Z:=\begin{pmatrix}Y & 0\\ 0 & Y^{-1}\end{pmatrix}\left[\begin{pmatrix}I & 0\\ X & I\end{pmatrix}\right]\in\mathrm{Sp}(n;\boldsymbol{F})\cap\mathrm{Pos}(2n;\boldsymbol{F}).$$

Given $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}(n;\boldsymbol{F})$ and $Z\in H(n;\boldsymbol{F})$ with $M\langle Z\rangle=X_M+iY_M$, we get

$$P_{M\langle Z\rangle}=P_Z[\bar{M}'],\tag{1}$$

$$Y_M^{-1}=P_Z\left[\begin{pmatrix}\bar{C}'\\ \bar{D}'\end{pmatrix}\right]\tag{2}$$

from [7, II.1.9 and II.1.7].

We choose a particular order $\mathcal{O}=\mathcal{O}(\boldsymbol{F})$ for each case of $\boldsymbol{F}$, namely

$$\mathcal{O}(\boldsymbol{R})=\boldsymbol{Z},\quad \mathcal{O}(\boldsymbol{C})=\boldsymbol{Z}e_1+\boldsymbol{Z}e_2,\quad \mathcal{O}(\boldsymbol{H})=\boldsymbol{Z}e_0+\boldsymbol{Z}e_1+\boldsymbol{Z}e_2+\boldsymbol{Z}e_3,$$

where $e_0=\frac{1}{2}(e_1+e_2+e_3+e_4)$. Then clearly $\mathcal{O}(\boldsymbol{C})$ equals the Gaussian integers and $\mathcal{O}(\boldsymbol{H})$ the quaternions of Hurwitz. The dual lattice of $\mathcal{O}$ with respect to $\langle,\rangle$ is denoted by $\mathcal{O}^{\#}$, i.e.,

$$\mathcal{O}^{\#}(\boldsymbol{F})=\mathcal{O}(\boldsymbol{F})\quad\text{for }\boldsymbol{F}=\boldsymbol{R},\boldsymbol{C},$$

$$\mathcal{O}^{\#}(\boldsymbol{H})=\boldsymbol{Z}2e_1+\boldsymbol{Z}(e_1+e_2)+\boldsymbol{Z}(e_1+e_3)+\boldsymbol{Z}(e_1+e_4).$$

Moreover, let

$$\Gamma_n:=\Gamma(n;\boldsymbol{F}):=\mathrm{Sp}(n;\boldsymbol{F})\cap\mathrm{Mat}(2n;\mathcal{O}(\boldsymbol{F}))$$

stand for the *modular group of degree n*.

3. Jacobi forms. If $n>1$ and k is an integer, which is even for $\boldsymbol{F}=\boldsymbol{H}$, the vector space $[\Gamma(n;\boldsymbol{F}),k]$ of *modular forms of degree n and weight k* consists of all holomorphic functions $f\colon H(n;\boldsymbol{F})\to\boldsymbol{C}$ such that

$$f(M\langle Z\rangle)=(\det(CZ+D))^k f(Z)$$

for all $M\in\Gamma(n;\boldsymbol{F})$. If $\boldsymbol{F}=\boldsymbol{H}$ we refer to [7, p. 78] for the definition of the determinant. Let τ denote the reduced trace form

$$\tau(A,B):=\tfrac{1}{2}\operatorname{trace}(A\bar{B}'+B\bar{A}')\qquad\text{for } n\times m \text{ matrices } A,B.$$

Then each $f\in[\Gamma(n;\boldsymbol{F}),k]$ possesses a Fourier expansion of the form

$$f(Z)=\sum_{T\in\mathrm{Sym}^{\tau}(n;\mathcal{O}),\,T\geq 0}\alpha_f(T)e^{2\pi i\tau(T,Z)},\qquad Z\in H(n;\boldsymbol{F}),\tag{3}$$

where $\mathrm{Sym}^{\tau}(n;\mathcal{O})$ denotes the dual lattice of $\mathrm{Sym}(n;\mathcal{O})$ with respect to τ. The subspace $[\Gamma(n;\boldsymbol{F}),k]_0$ of all cusp forms consists of those $f\in[\Gamma(n;\boldsymbol{F}),k]$ with the Fourier expansion (3), where $\alpha_f(T)\neq 0$ implies $T\in\mathrm{Pos}(n;\boldsymbol{F})$ (cf. [7] for details). Now consider a decomposition

$$Z=\begin{pmatrix}Z_1 & w\\ \bar{w}' & z\end{pmatrix},\qquad T=\begin{pmatrix}T_1 & \frac{1}{2}t\\ \frac{1}{2}\bar{t}' & m\end{pmatrix},$$

where $z\in H(1;\boldsymbol{F})$, $m\in\boldsymbol{N}_0$, $w=u+iv\in\boldsymbol{F}_{\boldsymbol{C}}^{n-1}$, $\bar{w}'=\bar{u}'+i\bar{v}'$ and $\boldsymbol{F}_{\boldsymbol{C}}=\boldsymbol{F}\otimes_{\boldsymbol{R}}\boldsymbol{C}$. A rearrangement of (3) yields the Fourier–Jacobi expansion of f

$$f(Z)=\sum_{m=0}^{\infty}f_m(Z_1,w)e^{2\pi imz},\tag{4}$$

where

$$f_m(Z_1,w)=\sum_{\substack{T_1\in\mathrm{Sym}^{\tau}(n-1;\mathcal{O}),\,t\in\mathcal{O}^{\#n-1}\\ \text{such that } T\geq 0}}\alpha_f(T)e^{2\pi i(\tau(T_1,Z_1)+\tau(t,w))}.$$

Then $f_m(Z_1,w)$ is a *Jacobi form of degree* $n-1$, *weight* k *and index* m. It is characterized by the identities

$$f_m|_{k,m}[M]=f_m|_m[\lambda,\mu]=f\qquad\text{for all } M\in\Gamma_{n-1}\text{ and }\lambda,\mu\in\mathcal{O}^{n-1}.$$

Given $\varphi\colon H(n-1;\, \boldsymbol{F})\times \boldsymbol{F}_{\boldsymbol{C}}^{n-1} \to \boldsymbol{C}$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n-1;\, \boldsymbol{F})$ and $\lambda, \mu \in \boldsymbol{F}^{n-1}$ we use the abbreviations

(5) $$\varphi|_{k,m}[M](Z_1, w)$$
$$:= \varphi\big(M\langle Z_1\rangle, \overline{(CZ_1+D)}'^{-1}w\big)\det(CZ_1+D)^{-k}e^{-2\pi i m((CZ_1+D)^{-1}C)[w]},$$
$$\varphi|_m[\lambda, \mu](Z_1, w) = \varphi(Z_1, w+Z_1\lambda+\mu)e^{2\pi i m Z_1[\lambda]+4\pi i m\tau(\lambda, w)}.$$

The results on Jacobi forms are immediately verified along the same lines as in [4], [5], [16], [19] for arbitrary Siegel modular forms, resp. in [1] and [13] in the case of degree 2.

According to (5) we consider the group $\Gamma_n^* = \Gamma^*(n;\, \boldsymbol{F})$, which acts discontinuously on $H(n-1;\, \boldsymbol{F})\times \boldsymbol{F}_{\boldsymbol{C}}^{n-1}$ and is generated by

$$(Z_1, w) \mapsto \big(M\langle Z_1\rangle, \overline{(CZ_1+D)}'^{-1}w\big), \quad M\in\Gamma_{n-1},$$
$$(Z_1, w) \mapsto (Z_1, w+Z_1\lambda+\mu), \quad \lambda, \mu\in\mathcal{O}^{n-1}.$$

A fundamental domain with respect to the action of $\Gamma^*(n;\, \boldsymbol{F})$ is for instance given by $\mathscr{F}_n^* := \mathscr{F}^*(n;\, \boldsymbol{F})$,

$$\mathscr{F}_n^* := \{(Z_1, w) \mid Z_1 \in \mathscr{F}_{n-1},\ w = u+iv,\ u\colon \boldsymbol{F}^{n-1}/\mathcal{O}^{n-1},\ v\colon \boldsymbol{F}^{n-1}/Y_1\mathcal{O}^{n-1}\},$$

where $\mathscr{F}_n = \mathscr{F}(n;\, \boldsymbol{F})$ is a fundamental domain of $H(n;\, \boldsymbol{F})$ with respect to the action of $\Gamma(n;\, \boldsymbol{F})$ (cf. [7]). Moreover, $dv = (\det Y_1)^{-2-r(n-1)}dX_1\,dY_1\,du\,dv$ is a volume element, which is invariant under the action of $\Gamma^*(n;\, \boldsymbol{F})$.

Given two cusp forms $f, g\in[\Gamma(n;\, \boldsymbol{F}), k]_0$, $n>1$ and $m\in\boldsymbol{N}$, we define the *Petersson scalar product* of the attached Jacobi forms (cf. [4]) by

$$\langle f_m, g_m\rangle := \int_{\mathscr{F}_n^*} f_m(Z_1, w)\overline{g_m(Z_1, w)}(\det Y_1)^k e^{-4\pi m Y_1^{-1}[v]}dv.$$

Lemma 1. *Given $f, g\in[\Gamma(n;\, \boldsymbol{F}), k]_0$ one has*

$$\langle f_m, g_m\rangle = O(m^k),$$

where the O-constant only depends on f and g.

Proof. If $(Z_1, w)\in H(n-1;\, \boldsymbol{F})\times \boldsymbol{F}_{\boldsymbol{C}}^{n-1}$ is fixed, we get

$$f_m(Z_1, w) = \int_{ic}^{ic+1} f(Z)e^{-2\pi i m z}dz,$$

where c is any real constant greater than $Y_1^{-1}[v]$. Since $(\det Y)^{k/2}|f(Z)|$ is bounded on $H(n;\, \boldsymbol{F})$ due to [7, III.2.4], we can choose $c = Y_1^{-1}[v]+1/m$ and get

$$|f_m(Z_1, w)| = O\left(\left(\frac{1}{m}\det Y_1\right)^{-k/2} e^{2\pi m Y_1^{-1}[v]}\right).$$

Thus the Petersson scalar product is majorized by

$$m^k \int_{\mathscr{F}_n^*} (\det Y_1)^{-2-r(n-1)}dX_1\,dY_1\,du\,dv = m^k(\operatorname{vol}\mathcal{O})^{2n-2}\operatorname{vol}(\mathscr{F}_{n-1}). \ \blacksquare$$

Given $f, g\in[\Gamma(n;\, \boldsymbol{F}), k]_0$, $n>1$, we can therefore attach the Dirichlet series

$$D(f, g;\, s) := \sum_{m=1}^{\infty} \langle f_m, g_m\rangle m^{-s},$$

which converges absolutely for $\mathrm{Re}(s) > k+1$ due to Lemma 1.

The analytic continuation of this Dirichlet series is derived by an integral representation. Therefore we have to introduce an Eisenstein series of Klingen type (cf. [3]).

Let $C_n := C(n;\, \boldsymbol{F})$ denote the subgroup of $\Gamma_n = \Gamma(n;\, \boldsymbol{F})$ consisting of all those matrices with $(0, \ldots, 0, 1)$ as its last row. The subscript 1 denotes the upper left $(n-1)\times(n-1)$ corner of an $n\times n$ matrix. Then we define the Eisenstein series of Klingen type

(6) $$E(Z;\, s) := \sum_{M:C_n\backslash\Gamma_n} \left(\frac{\det Y_M}{\det(Y_M)_1}\right)^s, \quad Z\in H(n;\, \boldsymbol{F}).$$

In analogy with [7, V.2.8], we conclude that the series converges absolutely for $\mathrm{Re}(s) > 1+r(n-1)$.

Given $f, g\in[\Gamma(n;\, \boldsymbol{F}), k]$, where at least one is a cusp form, the Petersson scalar product is defined by

$$\{f, g\} := \int_{\mathscr{F}_n} f(Z)\overline{g(Z)}(\det Y)^k d\mu,$$

where $d\mu = (\det Y)^{-2-r(n-1)}dX\,dY$ is the invariant volume element. If $\Gamma(s)$ denotes the gamma function, we get

Lemma 2. *Given $f, g\in[\Gamma(n;\, \boldsymbol{F}), k]_0$, then*

$$\{fE(\cdot;\, s), g\} = 2\cdot(4\pi)^{r(n-1)+1-k-s}\Gamma\big(s+k-1-r(n-1)\big)D\big(f, g;\, s+k-1-r(n-1)\big)$$

holds for $\mathrm{Re}(s) > 2+r(n-1)$.

Proof. We apply the usual unfolding trick. Therefore we calculate

$$\{fE(\cdot;\, s), g\} = \int_{\mathscr{F}_n} f(Z)\overline{g(Z)}(\det Y)^k E(Z, s)d\mu$$
$$= \sum_{M:C_n\backslash\Gamma_n} \int_{\mathscr{F}_n} f(M\langle Z\rangle)\overline{g(M\langle Z\rangle)}(\det Y_M)^{k+s}(\det(Y_M)_1)^{-s}d\mu$$
$$= 2\int_{\mathscr{C}_n} f(Z)\overline{g(Z)}(\det Y)^{k+s}(\det Y_1)^{-s}d\mu,$$

where $\mathscr{C}_n := \mathscr{C}(n; \boldsymbol{F})$ is a fundamental domain of $H(n; \boldsymbol{F})$ with respect to the action of C_n. We can choose

$$\mathscr{C}_n = \{Z \mid (Z_1, w) \in \mathscr{F}_n^*,\ y > Y_1^{-1}[v],\ 0 \leqslant x \leqslant 1\},$$

since $M \in C_n$ has the form

$$M = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{U}' & 0 \\ 0 & U^{-1} \end{pmatrix} \left(N \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

$$S \in \operatorname{Sym}(n; \mathcal{O}), \quad S_1 = 0, \quad U = \begin{pmatrix} I & \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \in \mathcal{O}^{n-1}, \quad N \in \Gamma_{n-1}.$$

Hence the Petersson scalar product equals

$$2 \int_{\mathscr{F}_n^*} \Big(\int_{y > Y_1^{-1}[v], 0 \leqslant x \leqslant 1} \sum_{l,m \geqslant 1} f_l(Z_1, w) \overline{g_m(Z_1, w)} e^{-2\pi(l+m)y} e^{2\pi i(l-m)x}$$
$$\times (\det Y_1)^{k-2-r(n-1)} (y - Y_1^{-1}[v])^{s+k-2-r(n-1)} dx\, dy \Big) dX_1\, dY_1\, du\, dv$$

in view of $\det Y = (\det Y_1)(y - Y_1^{-1}[v])$. Carrying out the integration over x and substituting $q = y - Y_1^{-1}[v]$, we get

$$\{f E(\cdot; s), g\} = 2 \int_{\mathscr{F}_n^*} \sum_{m=1}^{\infty} f_m(Z_1, w) \overline{g_m(Z_1, w)} e^{-4\pi m Y_1^{-1}[v]} (\det Y_1)^k dv$$
$$\times \int_0^\infty q^{s+k-2-r(n-1)} e^{-4\pi m q} dq$$
$$= 2 \sum_{m=1}^{\infty} \langle f_m, g_m \rangle \Gamma(s+k-1-r(n-1)) (4\pi m)^{r(n-1)+1-k-s}. \blacksquare$$

Analytic continuation and functional equation of $D(f, g; s)$ now follow from properties of $E(Z; s)$. The quotient $\det(Y_M)_1$ over $\det(Y_M)$ in (6) equals the (n, n)-entry of Y_M^{-1}. Using (2) we obtain

(7) $$E(Z; s) = \sum_{\lambda} (P_Z[\lambda])^{-s},$$

where $\bar{\lambda}'$ runs through the last rows of all the matrices in $\Gamma(n; \boldsymbol{F})$.

4. Siegel modular forms. In this section let $\boldsymbol{F} = \boldsymbol{R}$. Then it follows from [12, § 11] that λ in (7) runs through all coprime vectors in $\boldsymbol{Z}^{2n}$. If $\zeta(s)$ denotes the Riemann zeta function, we obtain an Epstein zeta function by

$$\zeta(2s) E(Z; s) = \sum_{0 \neq \lambda \in \boldsymbol{Z}^{2n}} (P_Z[\lambda])^{-s}.$$

In view of $P_Z^{-1} = P_Z[J]$ now well-known properties of Epstein zeta functions (cf. [15, p. 59]) yield that

$$\boldsymbol{E}(Z; s) := \pi^{-s} \Gamma(s) \zeta(2s) E(Z; s)$$

is a holomorphic function of s except for two simple poles at $s = 0$ resp. $s = n$ with residue -1 resp. 1 and satisfies the functional equation $\boldsymbol{E}(Z; n-s) = \boldsymbol{E}(Z; s)$.

THEOREM 1 ([17], [18]). *Let $f, g \in [\Gamma(n; \boldsymbol{R}), k]_0$ be two Siegel cusp forms of weight k and degree $n > 1$. Then*

$$\boldsymbol{D}(f, g; s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s+n-k) \zeta(2s+2n-2k) D(f, g; s)$$

is a holomorphic function of $s \in \boldsymbol{C}$ except for two possible simple poles at $s = k$ resp. $s = k-n$ with residue $\frac{1}{2}\pi^{n-k}\{f, g\}$ resp. $-\frac{1}{2}\pi^{n-k}\{f, g\}$ and satisfies the functional equation

$$\boldsymbol{D}(f, g; 2k-n-s) = \boldsymbol{D}(f, g; s).$$

Proof. The claim follows from the considerations above and Lemma 2 due to

$$\{f \boldsymbol{E}(\cdot; s+n-k), g\} = 2\pi^{k-n} \boldsymbol{D}(f, g; s). \blacksquare$$

Clearly $D(f, g; s)$ possesses a meromorphic continuation to the whole complex s-plane. It follows from the well-known properties of $\zeta(s)$ and $\Gamma(s)$ that

$$D(f, g; s) = 0 \quad \text{for } s = k + \tfrac{1}{2} - n \text{ and } s = 0, -1, -2, \ldots$$

as well as

$$D(f, g; k-n) = \frac{(4\pi)^{k-n}}{(k-n-1)!} \{f, g\} \quad \text{provided that } k > n.$$

5. Hermitian modular forms. In this section let $\boldsymbol{F} = \boldsymbol{C}$, $\mathcal{O} = \mathcal{O}(\boldsymbol{C})$. In analogy with [12, § 11], we conclude that λ in (7) runs through all coprime vectors in $\mathcal{O}^{2n}$ with $J[\lambda] = 0$. Hence we obtain

(8) $$E^*(Z; s) := \zeta_{\mathcal{O}}(s) E(Z; s) = \sum_{0 \neq \lambda \in \mathcal{O}^{2n}, J[\lambda] = 0} (P_Z[\lambda])^{-s},$$

where

$$\zeta_{\mathcal{O}}(s) := \frac{1}{4} \sum_{0 \neq a \in \mathcal{O}} N(a)^{-s} = \zeta(s) L(s, (\tfrac{-4}{\cdot})), \qquad \operatorname{Re}(s) > 1,$$

and $L(s, (\frac{-4}{\cdot}))$ is the Dirichlet L-series.

LEMMA 3. *Given $n > 1$ and $Z \in H(n; \boldsymbol{C})$; then*

$$\boldsymbol{E}(Z; s) := \pi^{-2s} \Gamma(s) \Gamma(s+1-n) \zeta(2s+2-2n) E^*(Z; s)$$

becomes a holomorphic function of $s \in \boldsymbol{C}$ except for possible simple poles at $s = 0$, $n-1$, n, $2n-1$ and satisfies the functional equation

$$\boldsymbol{E}(Z; 2n-1-s) = \boldsymbol{E}(Z; s).$$

Proof. We consider the theta series

$$\vartheta(Z, w) = v^n \sum_{\lambda\in\mathcal{O}^{2n}} e^{2\pi i(uJ+ivP_Z)[\lambda]}, \quad Z\in H(n; \boldsymbol{C}),\ w = u+iv\in H = H(1; \boldsymbol{R}).$$

As one easily checks for generators, $\vartheta(Z,w)$ is invariant under the action of $\Gamma(n; \boldsymbol{C})\times\Gamma$, $\Gamma = \mathrm{SL}(2; \boldsymbol{Z})$. Then

$$\Theta(Z, w) := \Delta\vartheta(Z, w) - n(n-1)\vartheta(Z, w),$$

where $\Delta = v^2(\partial^2/\partial u^2 + \partial^2/\partial v^2)$ is the hyperbolic Laplacian on H, is also invariant under the action of $\Gamma(n; \boldsymbol{C})\times\Gamma$. If Z belongs to a compact subset of $H(n; \boldsymbol{C})$ one has

$$\Theta(Z, w) = O(e^{-v}) \quad \text{for all } w\in\mathcal{F}_1.$$

We denote the non-analytic Eisenstein series on the upper half-plane in $\boldsymbol{C}$ by

$$\tilde{E}(w, s) := \sum_{M:C_1\backslash\Gamma} v_M^s, \quad w\in H(1; \boldsymbol{R}),\ \mathrm{Re}(s) > 1.$$

Now the integral

$$\mathcal{I} := \int_{\mathcal{F}_1} \tilde{E}(w, s+1-n)\Theta(Z, w)d\mu(w)$$

converges absolutely for $\mathrm{Re}(s) > n$. The usual unfolding trick (cf. [13]) yields

$$\mathcal{I} = 2s(s+1-2n)\pi^{-s}\Gamma(s)E^*(Z; s).$$

Now the claim follows by virtue of the results on the non-analytic Eisenstein series in [15, §3.5]. ■

A more detailed description of similar arguments will be given in Section 6.

A combination of Lemma 3 and Lemma 2 leads to

THEOREM 2. *Let $f, g\in[\Gamma(n; \boldsymbol{C}), k]_0$ be two Hermitian cusp forms of weight k and degree $n > 1$. Then*

$$\begin{aligned}\boldsymbol{D}(f, g; s) &:= (4\pi^3)^{-s}\Gamma(s)\Gamma(s-k-1+2n)\Gamma(s-k+n)\zeta(2s-2k+2n)\zeta(s-k-1+2n)\\ &\quad\times L\left(s-k-1+2n, \left(\tfrac{-4}{\cdot}\right)\right)D(f, g; s)\end{aligned}$$

becomes a holomorphic function of $s\in\boldsymbol{C}$ except for possible simple poles at $s = k$, $k+1-n$, $k-n$, $k+1-2n$ and satisfies the functional equation

$$\boldsymbol{D}(f, g; 2k+1-2n-s) = \boldsymbol{D}(f, g; s).$$

Again it is clear that $\boldsymbol{D}(f, g; s)$ possesses a meromorphic continuation to the whole s-plane and has trivial zeros at $s = 0, -1, -2, \ldots$

The case $n = 2$ of this theorem coincides with Theorem 1 in [13]. It is evident from the proof that the cases of Hermitian modular forms of an arbitrary imaginary quadratic number field of class number 1 can be dealt with in the same way.

6. Modular forms of quaternions. In this section let $\boldsymbol{F} = \boldsymbol{H}$, $\mathcal{O} = \mathcal{O}(\boldsymbol{H})$, and $\Gamma_n = \Gamma(n; \boldsymbol{H})$. In analogy with (8) we obtain

$$(9) \qquad E^*(Z; s) := \zeta_{\mathcal{O}}(s)E(Z; s) = \sum_{0\neq\lambda\in\mathcal{O}^{2n}, J[\lambda]=0} (P_Z[\lambda])^{-s},$$

where

$$\zeta_{\mathcal{O}}(s) := \tfrac{1}{24}\sum_{0\neq a\in\mathcal{O}} N(a)^{-s} = (1-2^{1-s})\zeta(s)\zeta(s-1), \quad \mathrm{Re}(s) > 2.$$

In order to derive the analytic continuation of (9) we have to express the function as an integral of a modified theta series against an Eisenstein series on the four-dimensional hyperbolic space.

Let $\mathcal{H}$ be the half-space model of the four-dimensional hyperbolic space (cf. [9]), i.e.,

$$\mathcal{H} = \left\{w = u+v = \sum_{j=1}^4 w_j e_j \in \boldsymbol{H} \mid w_1 = v > 0\right\}.$$

The modified symplectic group of degree 1

$$\mathrm{MSp}(1; \boldsymbol{H}) := \{M\in\mathrm{Mat}(2; \boldsymbol{H}) \mid Q[M] = Q\}, \quad Q = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

acts on $\mathcal{H}$ by fractional linear transformations, i.e.,

$$(10) \qquad w \mapsto M\langle w\rangle := (aw+b)(cw+d)^{-1}, \quad M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{MSp}(1; \boldsymbol{H}).$$

The attached hyperbolic Laplacian

$$\Delta = w_1^2\sum_{j=1}^4 \frac{\partial^2}{\partial w_j^2} - 2w_1\frac{\partial}{\partial w_1}$$

is invariant under all the transformations (10). Moreover, consider the modified modular group

$$\tilde{\Gamma} := \mathrm{MSp}(1; \boldsymbol{H})\cap\mathrm{Mat}(2; \mathcal{O})$$

and set

$$\mathrm{Im}\,\boldsymbol{H} = \{a\in\boldsymbol{H} \mid \mathrm{Re}(a) = 0\} = \boldsymbol{R}e_2+\boldsymbol{R}e_3+\boldsymbol{R}e_4.$$

Let $Z \in H(n; \boldsymbol{H})$, $w = u + v \in \mathscr{H}$, $v = \mathrm{Re}(w)$, and let Λ be a lattice in $\boldsymbol{H}^{2n}$. Then we define the *theta series*

$$\vartheta(Z, w; \Lambda) := v^{2n} \sum_{\lambda \in \Lambda} e^{\pi i \langle u, J[\lambda]\rangle - \pi v P_Z[\lambda]}.$$

We derive the theta transformation formula in

Proposition 1. *Let Λ be a lattice in $\boldsymbol{H}^{2n}$ and let*

$$\Lambda^{\#} := \{\mu \in \boldsymbol{H}^{2n} \mid \mathrm{Re}(\bar{\mu}' \lambda) \in \boldsymbol{Z} \text{ for all } \lambda \in \Lambda\}$$

be the dual lattice. Then one has for all $Z \in H(n; \boldsymbol{H})$ and $w \in \mathscr{H}$

$$\vartheta(-Z^{-1}, w^{-1}; \Lambda^{\#}) = (\mathrm{vol}\, \Lambda)\vartheta(Z, w; \Lambda).$$

Proof. We express the theta series as a specialization of a symplectic theta series on the Siegel half-space $H(8n; \boldsymbol{R})$ of degree $8n$. Given a quaternion $a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$, we therefore define the real 4×4 matrices

$$\hat{a} := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & -a_4 & a_3 \\ -a_3 & a_4 & a_1 & -a_2 \\ -a_4 & -a_3 & a_2 & a_1 \end{bmatrix}, \quad \tilde{a} := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & a_4 & -a_3 \\ -a_3 & -a_4 & a_1 & a_2 \\ -a_4 & a_3 & -a_2 & a_1 \end{bmatrix},$$

The mapping $\boldsymbol{H} \to \mathrm{Mat}(4; \boldsymbol{R})$, $a \mapsto \hat{a}$, is a homomorphism, whereas the map $\boldsymbol{H} \to \mathrm{Mat}(4; \boldsymbol{R})$, $a \mapsto \tilde{a}$, is an anti-homomorphism of the algebras. One has

$$\hat{a} \cdot \tilde{b} = \tilde{b} \cdot \hat{a} \quad \text{for all } a, b \in \boldsymbol{H}.$$

We extend the definition of $\hat{}$ to matrices as in [7, p. 15] and use $\otimes$ for the Kronecker product of matrices. Then we obtain

$$(11) \qquad (\tilde{a} \otimes J)[\hat{M}] = \tilde{a} \otimes J \quad \text{for all } a \in \boldsymbol{H},\ M \in \mathrm{Sp}(n; \boldsymbol{H}).$$

Setting

$$W := W(Z, w) := -\tilde{u} \otimes J + iv\hat{P}_Z \in H(8n; \boldsymbol{R}) \quad \text{for } Z \in H(n; \boldsymbol{H}),\ w \in \mathscr{H},$$

we get

$$\vartheta(Z, w; \Lambda) = v^{2n} \sum_{\lambda \in \Lambda_1} e^{\pi i W[\lambda]},$$

where Λ_1 is the lattice in $\boldsymbol{R}^{8n}$ consisting of the first columns of the matrices $\hat{\lambda} \in \mathrm{Mat}(8n, 4; \boldsymbol{R})$, $\lambda \in \Lambda$. One can use (11) in order to verify

$$-W^{-1} = W(-Z^{-1}, w^{-1}), \quad \det(i^{-1} W) = |w|^{8n}.$$

Then the assertion follows from the ordinary theta transformation formula as for instance in [7, IV.2.2]. ∎

Moreover, we need to know the behavior of our theta series under modular substitutions. Therefore let $\Gamma_n = \Gamma(n; \boldsymbol{H})$ and set

$$\tilde{\Gamma}^{\#} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma} \mid b \in \mathcal{O}^{\#} \right\}.$$

Proposition 2. (a) *$\vartheta(M\langle Z\rangle, N\langle w\rangle; \mathcal{O}^{2n}) = \vartheta(Z, w; \mathcal{O}^{2n})$ holds for all $M \in \Gamma_n$ and $N \in \tilde{\Gamma}^{\#}$.*

(b) *$\Theta(Z, w) := \sum_{N: \tilde{\Gamma}^{\#} \backslash \tilde{\Gamma}} \vartheta(Z, N\langle w\rangle; \mathcal{O}^{2n}) = \vartheta(Z, w; \mathcal{O}^{2n}) + \vartheta(Z, w + e_2; \mathcal{O}^{2n}) + 2^{2n} \vartheta(Z, w; \mathcal{O}^{\# 2n})$ is invariant under the transformations $Z \mapsto M\langle Z\rangle$, $M \in \Gamma_n$, as well as under $w \mapsto N\langle w\rangle$, $N \in \tilde{\Gamma}$.*

Proof. (a) The invariance under $Z \mapsto M\langle Z\rangle$, $M \in \Gamma_n$, is clear due to (1). The invariance under the substitutions of $\tilde{\Gamma}^{\#}$ is demonstrated for generators. By the use of the Euclidean algorithm one checks that $\tilde{\Gamma}^{\#}$ is generated by

$$\varepsilon I,\ \varepsilon \in \mathcal{O},\ N(\varepsilon) = 1, \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},\ b \in \mathcal{O}^{\#} \cap \mathrm{Im}\, \boldsymbol{H}, \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},\ c \in \mathcal{O} \cap \mathrm{Im}\, \boldsymbol{H}.$$

(b) The claim follows from part (a) and Proposition 1, since $\tilde{\Gamma}$ is the disjoint union of the right cosets $\tilde{\Gamma}^{\#}$, $\tilde{\Gamma}^{\#} Q$ and $\tilde{\Gamma}^{\#}\begin{pmatrix} 1 & e_2 \\ 0 & 1 \end{pmatrix}$. ∎

Given $n \in \boldsymbol{N}$ then

$$\Omega_n := \Delta - 2n(2n-3)\mathrm{Id}$$

proves to be an $\mathrm{MSp}(1; \boldsymbol{H})$-invariant differential operator.

Proposition 3. (a) *Given $Z \in H(n; \boldsymbol{H})$, $w \in \mathscr{H}$ and a lattice Λ in $\boldsymbol{H}^{2n}$, one has*

$$\Omega_n \vartheta(Z, w; \Lambda) = v^{2n} \sum_{\lambda \in \Lambda} \big((\pi v)^2 (P_Z[\lambda]^2 - |J[\lambda]|^2) - (4n-2)\pi v P_Z[\lambda]\big) e^{\pi i \langle u, J[\lambda]\rangle - \pi v P_Z[\lambda]}.$$

(b) *Given a compact subset $\mathscr{C} \subset H(n; \boldsymbol{H})$, there exist positive constants C and δ depending only on $\mathscr{C}$ such that*

$$|\Omega_n \Theta(Z, w)| \leqslant C v^{-2n} e^{-\delta v}$$

for all $Z \in \mathscr{C}$ and $w \in \mathscr{H}$. Moreover, $\Omega_n \Theta(Z, w)$ is invariant under the transformations $Z \mapsto M\langle Z\rangle$, $M \in \Gamma_n$, as well as $w \mapsto N\langle w\rangle$, $N \in \tilde{\Gamma}$.

Proof. (a) The calculation is straightforward.

(b) Apply the same arguments as in [9, Proposition 3]. ∎

Let $\tilde{\mathscr{F}}$ denote a fundamental domain of $\mathscr{H}$ with respect to the action of $\tilde{\Gamma}$ (cf. [9]). Just as in [9] resp. [2] consider the Eisenstein series

$$\tilde{E}(w, s) := \sum_{N: \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} (v_N)^s, \quad w \in \mathscr{H},$$

where $\tilde{\Gamma}_\infty$ denotes the subgroup of $\tilde{\Gamma}$ with (0 1) as its second row. Moreover, let $dv = w_1^{-4} dw_1 dw_2 dw_3 dw_4$ denote the invariant volume element. Then we get

LEMMA 4. *Given $n > 1$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 4n-3$, one has for all $Z \in H(n; \mathbf{H})$*

$$(12)\qquad \int_{\tilde{\mathscr{F}}} \tilde{E}(w, s+3-2n)\Omega_n \Theta(Z, w) dv(w)$$

$$= 4s(s+3-4n)(1+2^{2n-1-s})\pi^{-s}\Gamma(s)E^*(Z, s).$$

Proof. There exists a constant C such that $|\tilde{E}(w, s)| \leqslant Cv^{\mathrm{Re}(s)}$ for all $w \in \tilde{\mathscr{F}}$. Hence the integral exists due to Proposition 3 and $\mathrm{vol}(\tilde{\mathscr{F}}) < \infty$. Let

$$\tilde{\mathscr{F}}_\infty := \{w = u+v \in \mathscr{H} \mid 0 \leqslant u_j \leqslant 1, j = 2, 3, 4\}$$

be a fundamental domain with respect to the action of $\tilde{\Gamma}_\infty$. Then the integral (12) has the value

$$\int_{\tilde{\mathscr{F}}} \Big(\sum_{N:\tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} (v_N)^{s+3-2n} \Omega_n \Theta(Z, N\langle w\rangle)\Big) dv(w)$$

$$= 2 \int_{\tilde{\mathscr{F}}_\infty} v^{s+3-2n} \Omega_n \Theta(Z, w) dv(w)$$

$$= 4 \sum_{\lambda \in \mathcal{O}^{2n}, J[\lambda]=0} \int_0^\infty v^{s-1}((\pi v P_Z[\lambda])^2 - (4n-2)\pi v P_Z[\lambda]) e^{-\pi v P_Z[\lambda]} dv$$

$$+ 2^{2n+1} \sum_{\lambda \in \mathcal{O}^{*2n}, J[\lambda]=0} \int_0^\infty v^{s-1}((\pi v P_Z[\lambda])^2 - (4n-2)\pi v P_Z[\lambda]) e^{-\pi v P_Z[\lambda]} dv$$

$$= 4(\Gamma(s+2) - (4n-2)\Gamma(s+1))\pi^{-s}\Gamma(s)$$

$$\times \Big(\sum_{0 \neq \lambda \in \mathcal{O}^{2n}, J[\lambda]=0} (P_Z[\lambda])^{-s} + 2^{2n-1} \sum_{0 \neq \lambda \in \mathcal{O}^{*2n}, J[\lambda]=0} (P_Z[\lambda])^{-s}\Big)$$

$$= 4s(s+3-4n)\pi^{-s}\Gamma(s)(1+2^{2n-1-s})\Big(\sum_{0 \neq \lambda \in \mathcal{O}^{2n}, J[\lambda]=0} (P_Z[\lambda])^{-s}\Big).$$

Then (9) completes the proof. ∎

Now set

$$\xi(2s) = \pi^{-s}\Gamma(s)\zeta(2s), \qquad \xi_{\mathcal{O}}(s) = (\sqrt{2}\pi)^{-s}\Gamma(s)\zeta_{\mathcal{O}}(s),$$

which satisfy

$$\xi(1-s) = \xi(s), \qquad \xi_{\mathcal{O}}(2-s) = \xi_{\mathcal{O}}(s).$$

Then the combination of Lemma 4 and [9, Theorem 3], resp. [2] implies

COROLLARY 1. *Given $n > 1$ the Eisenstein series $E(Z, s)$ possesses a meromorphic continuation to the whole complex s-plane. More precisely,*

$$\mathbf{E}(Z, s) := 2^s(1+2^{2n-2-s})(1+2^{2n-1-s})\xi_{\mathcal{O}}(s+3-2n)\xi_{\mathcal{O}}(s)\xi(2s+4-4n)E(Z, s)$$

becomes a holomorphic function of $s \in \mathbf{C}$ except for possible simple poles at $s = 0$, $2n-3$, $2n-2$, $2n-1$, $2n$, $4n-3$ and satisfies the functional equation

$$\mathbf{E}(Z, 4n-3-s) = \mathbf{E}(Z, s).$$

Setting

$$\mathbf{D}(f, g; s) := (1+2^{k+1-2n-s})(1+2^{k+2-2n-s})\xi_{\mathcal{O}}(s+2n-k)\xi_{\mathcal{O}}(s+4n-3-k)$$
$$\times \xi(2s+4n-2-2k)(2\pi)^{-s}\Gamma(s)D(f, g; s),$$

we obtain

$$\{f\mathbf{E}(\cdot, s+4n-3-k), g\} = 2^{4n-2-k}\mathbf{D}(f, g; s)$$

from Lemma 2. Thus Corollary 1 implies

THEOREM 3. *Let $f, g \in [\Gamma(n; \mathbf{H}), k]_0$ be modular forms of quaternions of even weight k and degree $n > 1$, which are cusp forms. Then the Dirichlet series $D(f, g; s)$ possesses a meromorphic continuation to the whole complex s-plane. More precisely, $\mathbf{D}(f, g; s)$ is a holomorphic function of $s \in \mathbf{C}$ except for possible simple poles at*

$$s = k,\ k+3-2n,\ k+2-2n,\ k+1-2n,\ k-2n,\ k+3-4n$$

and satisfies the functional equation

$$\mathbf{D}(f, g; 2k+3-4n-s) = \mathbf{D}(f, g; s).$$

Again it is clear that $D(f, g; s)$ has trivial zeros at $s = 0, -1, -2, \ldots$ One can also cancel several factors in the definition of $\mathbf{D}(f, g; s)$, but the cancellation increases the possible number of poles. The function

$$D^*(f, g; s) := (s+2n-1-k)(s+4n-4-k)(1-2^{k+1-2n-s})(1-2^{k+4-4n-s})$$
$$\times \xi(s+2n-k)\xi(s+4n-3-k)\xi(s+4n-4-k)$$
$$\times \xi(2s+4n-2-2k)(2\pi)^{-s}\Gamma(s)D(f, g; s)$$

is also invariant under $s \mapsto 2k+3-4n-s$.

7. The Maaß space. In this section we consider modular forms in the Maaß space (cf. [8]). Therefore let $n = 2$, $\mathbf{F} = \mathbf{H}$. We always assume Z, T to be

$$Z = \begin{pmatrix} z & w \\ \bar{w} & z^* \end{pmatrix} \in H(2; \mathbf{H}), \qquad T = \begin{pmatrix} l & \frac{1}{2}t \\ \frac{1}{2}\bar{t} & m \end{pmatrix} \in \mathrm{Sym}^\tau(2; \mathcal{O}).$$

The *Maaß space* $\mathscr{M}(k; \mathbf{H})$ consists of all $f \in [\Gamma(2; \mathbf{H}), k]$ with the Fourier expansion (3) such that a function $\alpha_f^*: \mathbf{N}_0 \to \mathbf{C}$ exists satisfying

$$\alpha_f(T) = \sum_{d \in \mathbf{N}, d^{-1}T \in \mathrm{Sym}^\tau(2;\mathcal{O})} d^{k-1}\alpha_f^*(2d^{-2}\det T)$$

for all $0 \neq T \in \mathrm{Sym}^\tau(2; \mathcal{O})$, $T \geqslant 0$. The subspace $\mathscr{M}(k; \mathbf{H})_0$ of cusp forms is characterized by the condition $\alpha_f^*(0) = 0$.

Given $m \in \boldsymbol{N}$ set

$$\mathscr{T}(m) = \{M \in \mathrm{Mat}(2; \boldsymbol{Z}) \mid \det M = m\}, \qquad \mathscr{T}_0(m) = \{M \in \mathscr{T}(m) \mid \gcd(M) = 1\}.$$

Due to [8, Lemma 2], we have

$$(13) \quad f_m(z, w) = f_1|_k[\mathscr{T}(m)](z, w) := m^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right): \Gamma \backslash \mathscr{T}(m)} d^{-k} f_1\left(\frac{az+b}{d}, \frac{mw}{d}\right),$$

where $\Gamma = \mathrm{SL}(2; \boldsymbol{Z})$. We can determine the adjoint $[\mathscr{T}(m)]^*$ of this operator with respect to the Petersson scalar product. Given $f, g \in \mathscr{M}(k; \boldsymbol{H})_0$, $m \geqslant 1$, we obtain

$$\langle f_m, g_1|_k[\mathscr{T}(m)] \rangle = \langle f_m|_k[\mathscr{T}(m)]^*, g_1 \rangle,$$

where

$$(14) \quad f_m|_k[\mathscr{T}(m)]^*(z, w)$$
$$= m^{k-9} \sum_{\lambda, \mu : \mathcal{O}/m\mathcal{O}} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right): \Gamma \backslash \mathscr{T}(m)} d^{-k} e^{2\pi i (zN(\lambda) + 2\langle \lambda, w \rangle)} f_m\left(\frac{az+b}{d}, \frac{w + \lambda z + \mu}{d}\right)$$

is a Jacobi form of index 1. The proof follows the same lines as for the Proposition in [6] and can therefore be omitted.

In analogy with [1, § 4], we introduce a Hecke operator $\mathscr{T}_m^0$ on the space of Jacobi forms of index 1 via

$$f_1|_k \mathscr{T}_m^0(z, w) := m^{k-10} \sum_{\lambda, \mu : \mathcal{O}/m\mathcal{O}} \sum_{M : \Gamma \backslash \mathscr{T}_0(m^2)} f_1|_{k,1}[m^{-1}M]|_1[\lambda, \mu],$$

where we use the notation (5).

Proposition 4. *Given $f \in \mathscr{M}(k; \boldsymbol{H})$ and $m \geqslant 1$ one has*

$$(15) \qquad f_1|_k[\mathscr{T}(m)]|_k[\mathscr{T}(m)]^* = \sum_{l \mid m} \sigma_1\left(\frac{m}{l}\right)\left(\frac{m}{l}\right)^{k-2} f_1|_k \mathscr{T}_l^0.$$

Proof. Using (13) and (14) the left hand side of (15) is equal to

$$m^{k-10} \sum_M \sum_N \sum_{\lambda, \mu} f_1|_{k,1}[m^{-1}MN]|_1[\lambda, \mu],$$

where M and N run through sets of representatives of $\Gamma \backslash \mathscr{T}(m)$. Now we apply Theorem 3.24 in [14] and see that each right coset ΓL, $L \in \mathscr{T}(m^2)$, appears exactly $\sigma_1(m/l)$ times as ΓMN, where l is the gcd of the entries of L. Then a simple calculation completes the proof. ∎

Given $f \in \mathscr{M}(k; \boldsymbol{H})$ with Fourier–Jacobi coefficients f_m in (4) we consider the mapping

$$\Omega \colon f_1 \mapsto F(z) := \sum_{l=0}^{\infty} \alpha_f^*(l) e^{2\pi i l z}.$$

Ω maps the space of Jacobi forms of index 1 onto a certain subspace $\mathfrak{M}_{k-2}$ of elliptic modular forms for $\Gamma_0 := \Gamma_0(4)$ of weight $k-2$ (cf. [10]). Using the same definition for Hecke operators as in [10] without any normalizing factors, we obtain

Proposition 5. *Let $f \in \mathscr{M}(k; \boldsymbol{H})$.*
(a) *Given relatively prime $m, n \in \boldsymbol{N}$ one has*

$$f_1|_k \mathscr{T}_m^0|_k \mathscr{T}_n^0 = f_1|_k \mathscr{T}_{mn}^0 .$$

(b) *If p is an odd prime, then*

$$f_1|_k \mathscr{T}_{p^l}^0 = \Omega^{-1}\left(p^{2l(k-3)} F \mathop{|}_{k-2} \Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & p^{2l} \end{pmatrix} \Gamma_0\right)$$

and

$$f_1|_k \mathscr{T}_{2^l}^0 = \Omega^{-1}\left(2^{2l(k-3)} F \mathop{|}_{k-2} \left(\Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & 2^{2l} \end{pmatrix} \Gamma_0 + \Gamma_0 \begin{pmatrix} 2^{2l} & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0\right)\right)$$

for all $l \geqslant 1$.

Proof. (a) This part is easily verified (cf. [1, § 4]).
(b) Proceed in the same way as Raghavan and Sengupta [13] in the proof of Lemma 2. The arising Gauss sums are evaluated to be

$$G(\alpha, h, p^l) := \sum_{g : \mathcal{O}/p^l\mathcal{O}} e^{2\pi i \alpha p^{-l} N(g + h/2)}$$
$$= \begin{cases} (-1)^{\alpha N(h)/2} p^{2l} & \text{if } p \text{ is odd}, \\ -2^{2l+1} & \text{if } p = 2,\ h \in 2\mathcal{O}, \\ 0 & \text{if } p = 2,\ h \notin 2\mathcal{O}, \end{cases}$$

where $\alpha \in \boldsymbol{Z}$, $p \nmid \alpha$, $l \geqslant 1$, $h \in \mathcal{O}^{\#}$. ∎

Due to [10] the Maaß space $\mathscr{M}(k; \boldsymbol{H})$ is invariant under all Hecke operators. More precisely, it can be concluded from Theorem 2 in [10] that f_1 is a simultaneous eigenform under all operators $\mathscr{T}_m^0$, $m \geqslant 1$, provided that f is a simultaneous eigenform under all Hecke operators. Thus Proposition 5 leads to

Corollary 2. *Let $f \in \mathscr{M}(k; \boldsymbol{H})$ be a simultaneous eigenform under all Hecke operators with $\alpha_f^*(1) = 1$. Then*

$$f_1|_k \mathscr{T}_{p^l}^0 = (\alpha_f^*(p^{2l}) - p^{k-4}\alpha_f^*(p^{2l-2})) f_1, \quad f_1|_k \mathscr{T}_{2^l}^0 = \alpha_f^*(2^{2l}) f_1$$

for $l \geqslant 1$ and all odd primes p.

Now we apply all the results above in order to evaluate the Dirichlet series $D(f, g; s)$.

THEOREM 4. *Let $f, g \in \mathcal{M}(k; \boldsymbol{H})_0$, where f is a simultaneous eigenform under all Hecke operators with $\alpha_f^*(1) = 1$. Then the Dirichlet series $D(f, g; s)$ possesses an Euler product expansion of the form*

$$D(f, g; s) = \langle f_1, g_1 \rangle \frac{1-2^{k-2-s}}{1-2^{k-4-s}}$$

$$\times \prod_p \frac{(1-p^{k-4-s})(1+p^{k-3-s})}{(1-p^{k-2-s})(1-p^{k-1-s})(1+\alpha_f^*(p)p^{-s/2}+p^{k-3-s})(1-\alpha_f^*(p)p^{-s/2}+p^{k-3-s})}.$$

Proof. Propositions 4, 5 and Corollary 2 imply

$$D(f, g; s) = \langle f_1, g_1 \rangle \frac{\zeta(s+2-k)\zeta(s+1-k)}{\zeta(s+4-k)(1-2^{k-4-s})} \prod_p \sum_{l=1}^{\infty} \alpha_f^*(p^{2l}) p^{-ls}.$$

Then Corollary 1 in [10] completes the proof. ∎

Remark. Under the assumptions of Theorem 4 the Dirichlet series $D(f, g; s)$ is proportional to an Andrianov zeta function attached to f, namely

$$D(f, g; s) = \langle f_1, g_1 \rangle \frac{\zeta(s+2-k)}{\zeta(s+4-k)(1-2^{k-4-s})} \sum_{l=1}^{\infty} \alpha_f(lT_0) l^{-s},$$

where

$$2T_0 = \begin{pmatrix} 2 & e_1+e_2 \\ e_1-e_2 & 2 \end{pmatrix}.$$

If $\alpha_f^*(2) \neq 0$, it follows from [10, Theorem 7] that

$$D(f, g; s) = \frac{\langle f_1, g_1 \rangle}{\alpha_f^*(2)} \frac{1+2^{k-3-s}}{1-2^{k-4-s}} \frac{\zeta(s+2-k)}{\zeta(s+4-k)} A(f, s),$$

where $A(f, s)$ denotes the Andrianov zeta function attached to f in [10].

If additionally $\langle f_1, g_1 \rangle \neq 0$, then $\boldsymbol{D}(f, g; s)$ possesses a pole at $s = s_0$ if and only if $A^*(f, s_0) \neq 0$, whenever $s_0 = k, k-1, k-2, k-3, k-4, k-5$.

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