On the Möbius sum function

by

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1. Introduction. Let $M(x) = \sum_{n \leq x} \mu(n)$, $\mu(n)$ being the Möbius function. The inequality $M(x) = O(x^{1/2 + \varepsilon})$ for every $\varepsilon > 0$ is equivalent to the Riemann hypothesis. A major question in the theory of $M(x)$ is whether or not the stronger bound

$$M(x) = O(x^{1/2})$$

holds. Although (1) is probably false, the best known estimate of large values of $|M(x)|x^{-1/2}$ is

$$\lim_{x \to \infty} |M(x)|x^{-1/2} > 1.06$$

due to Odlyzko and te Riele [5].

For any $x$ let

$$M^*(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n(2\pi x)^2n}{n(2n)! \zeta(2n+1)}.$$

If $x_0 > 0$ then

$$|M(x_0) + 2M^*(x_0^{-1})|x_0^{-1/2} \leq \lim_{x \to \infty} |M(x)|x^{-1/2}.$$

This is a result of Jurkat [4, p. 148], also see Anderson and Stark [1, pp. 99–100]. In particular, (1) implies

$$M^*(x) = O(x^{-1/2}).$$

Let $r(t) = t \sum_{n=1}^{\infty} \mu(n)n^{-1}$. The function $M^*(x)$ is the cosine transform of $r(t^{-1})$; thus,

$$M^*(x) = \int_{0}^{1} r(t^{-1}) \cos 2\pi x t \, dt$$

[4, p. 152]. By definition

$$\tilde{M}^*(x) = \int_{0}^{1} r(t^{-1}) \sin 2\pi x t \, dt.$$
It will be seen that on the Riemann hypothesis
\[ \tilde{M}^*(x) = O(x^{-1/2 + \varepsilon}) \]
for each \( \varepsilon > 0 \). On the other hand, one can show, without any hypothesis, that
\[ \lim_{x \to \infty} x^{1/2} \tilde{M}^*(x) = \infty. \]

Hence it is desirable to relate (1) and (2) to the behavior of \( \tilde{M}^*(x) \). This paper obtains several theorems in this direction. The last two results can be improved by making suitable assumptions about \( M(x) \).

**Theorem 1.** If
\[ \int_1^x (M(u)u^{-1})^2 du = O(\log x) \]
then
\[ \lim_{x \to \infty} \frac{x^{1/2} \tilde{M}^*(x)}{\log \log \log x} = \frac{1}{2\pi}. \]

It follows that (1) would be contradicted if it could be shown to imply
\( \tilde{M}^*(x) = o(x^{-1/2} \log \log \log x) \).

Of course (1) implies (3) but it is not difficult to obtain a better result.

**Theorem 2.** If \( M(x) = O(x^{\varepsilon}) \) then \( \tilde{M}^*(x) = O(x^{-1/2 + \varepsilon} \log x) \).

**Theorem 3.** Assume that (4) is true. Then there is a constant \( C > 0 \) such that for any \( N \) we have
\[ |M^*(x) - M^*(y)| > C(x^{-1/2} - y^{-1/2}) \log \log (x + y) \]
for a pair of numbers \( x, y \) with \( \max(x, y) > N \).

**Corollary.** The inequality
\[ M^*(x) - M^*(y) = O(|x^{-1/2} - y^{-1/2}|) \]
does not hold.

This is true without any hypothesis since letting \( y \to \infty \) gives (2), which is equivalent to (1). Therefore the above inequality implies (4).

2. Preliminary results. Let \( \phi(s) = 2^s\pi^{-s} \cos \frac{\pi s}{2} \Gamma(1-s) \) and consider the integral
\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{(s-1) \zeta(s)} ds \]
where \( 1 < c < 2 \). This is absolutely convergent since \( \phi(s) = O(|t|^{1/2 - \varepsilon}) \). Moving the contour to the right leads to the series
\[ \sum_{n=1}^{\infty} (-1)^n \frac{\phi(2n)}{(2n-1)! \zeta(2n)} = \sum_{n=1}^{\infty} \frac{\mu(k)}{k} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)! (2n-1)!} \frac{(2\pi x)^{2n-1}}{k}. \]

Rewriting this as an integral gives \( \int_1^\infty (t^{-1} \sin 2\pi xt) dt \) and proves that
\[ \tilde{M}^*(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{(s-1) \zeta(s)} ds. \]

If the Riemann hypothesis is true then the integral converges absolutely for \( 1/2 < c < 2 \) and (3) is clear. In what follows it is assumed that (4) holds. The proof of Theorem 1 is adapted from Ingham [3].

Let
\[ \tilde{M}_1^*(y) = \int_0^y \tilde{M}^*(x) x^{-1/2} dx. \]

From (6),
\[ \tilde{M}_1^*(y) = \int_{c-i\infty}^{c+i\infty} \frac{\phi(s)}{s-1 \zeta(s)} ds \]
for \( 1/2 < c < 2 \). An explicit formula for this function is required. As in [6, p. 374], shifting the contour to \( \Re(s) = c' \) where \( -1 < c' < 0 \) gives
\[ \tilde{M}_1^*(y) = \lim_{y \to \infty} \sum_{|\gamma| < T} \frac{\phi(\gamma)}{(\gamma-1)(\gamma-1/2) \zeta(\gamma)} \frac{2}{\zeta(1/2)} \]
\[ + \int_{c-i\infty}^{c+i\infty} \frac{y^{s-1/2}}{(s-1) \zeta(s)} ds \]
where \( \{T_n\} \) is a certain sequence and \( \gamma = 1/2 + iy \) is a zero of \( \zeta(s) \). Inequality (4) implies not only the Riemann hypothesis and that \( \phi \) is simple but that
\[ \sum_{|\gamma| < \infty} \frac{1}{|\zeta'(-\gamma)|^2} < \infty, \]
[6, p. 377]. Since \( \sum_{|\gamma| < \infty} \), it follows that the series in (7) is absolutely convergent.

As for the integral in (7), the substitution \( s = 1 - w \) leads to an integral on the line \( \Re(w) = 1 - c' = c'' \), say, which becomes
\[ \frac{1}{2\pi i} \int_{c''-i\infty}^{c''+i\infty} \frac{y^{1-2w}}{w-1/2} \tan(\pi w/2) \frac{1}{\zeta(w)} dw \]
after using the functional equation \( \zeta(s) = \phi(s) \tan \frac{\pi s}{2} \zeta(1-s) \).
Since $e^r > 1$, $\sum \mu(n)n^{-x} = O(1)$ as $N \to \infty$. The last integral becomes

$$\sum_{1}^{\infty} \mu(n)n^{-1/2} J(ny)$$

after termwise integration which is justified by the dominated convergence theorem. Here

$$J(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{1/2-y}}{w(w-1/2)} \tan \frac{1}{2} \pi w \, dw.$$

If the series (9) is denoted $R(y)$ then the explicit formula is

$$\mathcal{M}^*(y) = \sum \frac{\phi(q)}{(q-1)(q-1/2) \zeta(q)} \frac{2}{\zeta(1/2)} + R(y).$$

**Lemma 1.** We have $R(y) = O(y^{-7/2})$ for $y \geq 2$.

**Proof.** If $x > 1$ then

$$J(x) = \sum_{n \geq 1} \frac{x^{1/2-n}}{n(n-1/2)}$$

where $'$ means $n$ is odd. Now

$$J'(x) = -\frac{2}{\pi} \int_{1/2}^{x} \frac{\gamma}{\zeta(1/2)} = O(x^{-7/2})$$

if $x \geq 2$ say. It follows that

$$\frac{d}{dy} J(ny) = \frac{1}{n} \frac{d}{dy} J(ny) = O(n^{-1/2} y^{-7/2})$$

if $y \geq 2$, $n \geq 1$; so

$$\sum_{n \geq 1} \mu(n)n^{-1/2} \frac{d}{dy} J(ny) = O(y^{-7/2}).$$

This series is $R(y)$ since it is uniformly convergent for $y \geq 2$.

**Lemma 2.** We have

$$\text{Im} \sum_{0 < \gamma < T} \frac{1}{\gamma \zeta(\gamma)} \left( 1 - \frac{\gamma}{T} \right) = \frac{1}{2\pi} \log T + O(\log^{1/2} T).$$

**Proof.** According to Ingham [2, p. 317] the interval $(T, T+1)$ contains an $X$ such that

$$\sum_{0 < \gamma < X} \frac{1}{\gamma \zeta(\gamma)} \left( 1 - \frac{\gamma}{T} \right) \neq 1, \pi \log X + O(1).$$

From (8) and the Cauchy–Schwarz inequality,

$$\sum_{T < \gamma < X} \left| \frac{1}{\gamma \zeta'(\gamma)} \right| = O(\log^{1/2} T)$$

since $(T, T+1)$ contains $O(\log T)$ zeros of $\zeta(s)$. Similarly,

$$\sum_{0 < \gamma < T} \left| \frac{1}{\gamma \zeta'(\gamma)} \right| < T^{-1} \sum_{0 < \gamma < T} \left| \frac{1}{\gamma \zeta'(\gamma)} \right| = O(T^{-1/2} \log^{1/2} T).$$

Since $\log X = \log T + O(1)$, the lemma follows from the last two inequalities.

**3. Proofs.**

**Proof of Theorem 1.** Suppose that $\omega > 2$, $T$ is a positive integer and

$$K(y) = \left( \frac{\sin \pi y}{\pi y} \right)^2.$$

Let $K_T(y) = TK(Ty)$ and consider the integral

$$\int_{\omega - 1}^{\omega + 1} K_T(u-\omega) \mathcal{M}^*(e^u) \, du.$$

An integration by parts shows that this is

$$-\int_{\omega - 1}^{\omega + 1} 2 \mathcal{M}^*(e^u) e^{u/2} K_T(u-\omega) \, du.$$

On the other hand, use of (10) in (12) gives

$$-\sum_{\omega - 1}^{\omega + 1} \phi(q) \int_{\omega - 1}^{\omega + 1} e^{iu} K_T(u-\omega) \, du + \int_{\omega - 1}^{\omega + 1} K_T(u-\omega) R(e^u) \, du$$

after termwise integration and integrating by parts.

In the first integral let $u = \omega + T^{-1} y$ to obtain

$$-\sum_{\omega - 1}^{\omega + 1} e^{iyT} \phi(q) \int_{-1}^{1} e^{iyT} K(y) \, dy.$$

Consider the expression

$$\sum_{|\gamma| < X} e^{i\gamma T} K(y) \, dy + \sum_{|\gamma| > X} e^{i\gamma T} K(y) \, dy$$

where $X$ will be chosen. Now

$$\int_{-1}^{1} e^{iyT} K(y) \, dy = \begin{cases} O(T^{-1}), & y < 0 \\ O(1), & y > 0 \end{cases}$$

for any $0 < T$.
[3, p. 206]. By (18) the infinite series in (16) is convergent. Use of (17) in the first term and of (18) in the second shows that (16) is bounded by a constant times

\[ T^{-1} \sum_{|l| < x} \left| \frac{1}{\varphi'(\omega)} \right| + \sum_{|l| > x} \frac{1}{\varphi'(\omega)}. \]

The first term here is \( O\left( T^{-1} X^{1/2} \log^{1/2} X \right) \). By (8) and the Cauchy–Schwarz inequality the other term is \( O(X^{-1/2} \log^{1/2} X) \) since \( \sum_{|l| > x} |l|^{-2} = O(X^{-1} \log X) \); hence (16) is \( O(1) \) if \( X = T^{1/2} \log T \). The range \((-\infty, -T)\) can be handled similarly. When the integral in (15) is extended to \((\infty, \infty)\) the series becomes

\[ \sum_{|l| < x} \frac{e^{i\omega l}}{\varphi'(\omega)} \left( 1 - \left| \frac{\omega}{T} \right| \right) + O(1). \] (19)

The second integral in (14) is

\[ \sum_{\omega = -1}^{\omega = +1} \int_{-\infty}^{\infty} e^{i\omega R(\mu)} K_\tau(u - \omega) du. \]

Here \( e^{i\omega} > 2 \) since \( \omega > 2 \) so Lemma 1 gives the bound

\[ \sum_{\omega = -1}^{\omega = +1} \int_{-\infty}^{\infty} e^{-2\omega \log T} K_\tau(u - \omega) du \]

times a constant for the absolute value of this last term. This is

\[ \int_{-\infty}^{\infty} K_\tau(u - \omega) du < 1. \] (20)

From (13), (19), and (20),

\[ \sum_{\omega = -1}^{\omega = +1} \int_{-\infty}^{\infty} 2\tilde{M}^*(e^{i\omega}) e^{i\omega / 2} K_\tau(u - \omega) du = \sum_{l < T} e^{i\omega l} \phi(\omega) \left( 1 - \left| \frac{\omega}{T} \right| \right) + O(1). \] (21)

By the functional equation the sum becomes

\[ \sum_{l < T} e^{-i\omega l} \tan \frac{1}{2} \pi \omega \left( 1 - \frac{1}{T} \right) \] (22)

when \( \omega \) is changed to \(-\omega\). Now

\[ \tan \frac{1}{2} \pi \omega = i \text{sgn}(\gamma) + O(e^{-\pi|\gamma|}), \]

and substituting into (22) gives

\[ \int_{l < T} e^{-i\omega l} \text{sgn}(\gamma) \left( 1 - \frac{1}{T} \right) + O(1) \]

\[ = -2 \text{Im} \sum_{0 < \gamma < T} e^{-i\omega \gamma} \phi(\omega) \left( 1 - \frac{\gamma}{T} \right) + O(1) = -2S_\tau(\omega) + O(1), \] (23)

say. Equation (21) takes the form

\[ \int_{\omega = -1}^{\omega = +1} \tilde{M}^*(e^{i\omega}) e^{i\omega / 2} K_\tau(u - \omega) du = -S_\tau(\omega) + O(1). \]

To complete the proof observe that

\[ S_\tau(0) = -\frac{1}{2\pi} \log T + O(\log^{1/2} T) \]

by Lemma 2. By Dirichlet's theorem [6, p. 184], for each \( \varepsilon > 0 \) there is a number \( \omega_0 \) and integers \( n(\gamma) \) such that

\[ e^{-N(T)} < \omega < e^{-N(T)}, \]

and

\[ |\nu\omega - 2\pi n(\gamma)| < 2\pi. \]

for each \( 0 < \gamma < T \). Here \( N(T) \) is the number of \( \gamma \)'s. Since \( |e^{-i\omega \gamma} - 1| < 2\pi e, \)

\[ |S_\tau(\omega) - S_\tau(0)| < 2\pi \sum_{0 < \gamma < T} \left| \frac{1}{\phi'(\omega)} \right| = O(eT^{1/2} \log^{1/2} T). \]

Upon choosing \( \varepsilon = T^{-1/2} \), (23) and (24) imply

\[ \int_{\omega = -1}^{\omega = +1} \tilde{M}^*(e^{i\omega}) e^{i\omega / 2} K_\tau(u - \omega) du = \frac{1}{\pi} \log T + O(\log^{1/2} T) \]

for the \( \omega \) in Dirichlet's theorem.

Given \( 0 < \delta < 1 \), the right side exceeds \( \frac{1}{\pi} (1 - \delta) \log T \) if \( T \) is large enough. Hence

\[ \tilde{M}^*(e^{i\omega}) e^{i\omega / 2} \geq \frac{1}{\pi} (1 - \delta) \log T \]

for some \( u \) in \((\omega - 1, \omega + 1)\). Now log \( \omega < \frac{1}{2} T \log^2 T \) for large \( T \) so

\[ T \log^2 T > \log(\omega + 1) > \log u; \]

therefore,

\[ \left( 1 + \frac{2\log \log T}{\log T} \right) \log T \geq \log \log u. \]

The left side is less than \((1 + \delta) \log T \) for large \( T \) so from (25)

\[ \tilde{M}^*(e^{i\omega}) e^{i\omega / 2} \geq \frac{1}{2\pi} \frac{1 - \delta}{1 + \delta} \log \log u. \]

By varying \( T \) one obtains this inequality for arbitrarily large \( u \) so Theorem 1 is proven.
Proof of Theorem 2. The following estimate is needed.

**Lemma.** We have

\[
\frac{d}{dx} M^*(x) = O(x^{-1} \log x).
\]

**Proof.** From the power series defining \( M^*(x) \),

\[
\frac{d}{dx} M^*(x) = \frac{1}{x} \sum_{r=1}^{\infty} \frac{\mu(r)}{r} \left( \frac{2\pi x}{r} - 1 \right).
\]

The contribution of \( r < x \) to the series is \( O(\log x) \). For the remainder one obtains the bound \( x \sum_{r > x} r^{-2} \) times a constant. Since this is \( O(1) \) the lemma follows.

From the formula

\[
\text{sgn}(t) = \frac{2}{\pi} \int_0^\infty \frac{\sin tu}{u} du
\]

one readily obtains

\[
M^*(x) = \frac{1}{\pi} \int_0^x \frac{M^*(x+u) - M^*(x-u)}{u} du.
\]

(26)

For large \( x \) the lemma shows that the integral over \( 0 \leq u \leq 1 \) is \( O(x^{-1} \log x) \).

Assuming (2) in the form \( |M^*(x)| < Cx^{-1/2} \) yields

\[
|M^*(x)| \leq \frac{C}{\pi} \int_0^\infty u^{-1} |x+u|^{-1/2} du + \frac{C}{\pi} \int_0^\infty u^{-1} |x-u|^{-1/2} du + O(x^{-1} \log x).
\]

The integrals are

\[
x^{-1/2} \int_0^1 u^{-1} (1+u)^{-1/2} du \quad \text{and} \quad x^{-1/2} \int_0^1 u^{-1} (1-u)^{-1/2} du
\]

respectively. These are clearly \( O(x^{-1/2} \log x) \) proving Theorem 2.

**Proof of Theorem 3.** For brevity let \( \log_3 x = \log \log \log x \) and assume that

\[
|M^*(x) - M^*(y)| < C|x^{-1/2} - y^{-1/2}| \log_3 (x+y)
\]

for \( x \geq N, y > 0 \). In (26) the integral for \( 0 \leq u \leq 1 \) has already been considered.

For \( 1 \leq u \leq x \), (27) implies the bound

\[
\frac{C}{\pi} \int_1^x \frac{(u-x)^{-1/2} - (x+u)^{-1/2}}{u} \log_3 2u du
\]

\[
\leq \frac{C}{\pi} x^{-1/2} \log_3 2x \int_1^x \frac{(1-u)^{-1/2} - (1+u)^{-1/2}}{u} du < C_1 x^{-1/2} \log_3 x,
\]

say. In absolute value the integral for \( u > x \) is at most

\[
\frac{C}{\pi} \int_x^\infty \frac{(u-x)^{-1/2} - (u+x)^{-1/2}}{u} \log_3 2u du
\]

\[
\leq \frac{C}{\pi} x^{-1/2} \log_3 2x \int_x^\infty \frac{(u-x)^{-1/2} - (u+x)^{-1/2}}{u^{1/2}} du < C_2 x^{-1/2} \log_3 x;
\]

therefore,

\[
|M^*(x)| < C_3 x^{-1/2} \log_3 x + O(x^{-1} \log x).
\]

This contradicts Theorem 1 if \( C \) is small enough.

**References**


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