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On the Möbius sum function

by

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1. Introduction. Let $M(x) = \sum_{n\leqslant x}\mu(n)$, $\mu(n)$ being the Möbius function. The inequality $M(x) = O(x^{1/2+\varepsilon})$ for every $\varepsilon > 0$ is equivalent to the Riemann hypothesis. A major question in the theory of $M(x)$ is whether or not the stronger bound

$$M(x) = O(x^{1/2}) \tag{1}$$

holds. Although (1) is probably false, the best known estimate of large values of $|M(x)|x^{-1/2}$ is

$$\overline{\lim_{x\to\infty}}\, |M(x)|x^{-1/2} > 1.06$$

due to Odlyzko and te Riele [5].

For any x let

$$M^*(x) = 1+\sum_{n=1}^{\infty}\frac{(-1)^n(2\pi x)^{2n}}{2n(2n)!\zeta(2n+1)}.$$

If $x_0 > 0$ then

$$|M(x_0)+2M^*(x_0^{-1})|x_0^{-1/2} \leqslant \overline{\lim_{x\to\infty}}\, |M(x)|x^{-1/2}.$$

This is a result of Jurkat [4, p. 148], also see Anderson and Stark [1, pp. 99–100]. In particular, (1) implies

$$M^*(x) = O(x^{-1/2}). \tag{2}$$

Let $r(t) = t\sum_{n\leqslant t}\mu(n)n^{-1}$. The function $M^*(x)$ is the cosine transform of $r(t^{-1})$; thus,

$$M^*(x) = \int_0^1 r(t^{-1})\cos 2\pi xt\,dt$$

[4, p. 152]. By definition

$$\tilde{M}^*(x) = \int_0^1 r(t^{-1})\sin 2\pi xt\,dt.$$

It will be seen that on the Riemann hypothesis

$$\tilde{M}^*(x) = O(x^{-1/2+\varepsilon}) \tag{3}$$

for each $\varepsilon > 0$. On the other hand, one can show, without any hypothesis, that

$$\overline{\lim_{x\to\infty}}\, x^{1/2}\tilde{M}^*(x) = \infty.$$

Hence it is desirable to relate (1) and (2) to the behavior of $\tilde{M}^*(x)$. This paper obtains several theorems in this direction. The last two results can be improved by making suitable assumptions about $M(x)$.

THEOREM 1. *If*

$$\int_1^x (M(u)u^{-1})^2\,du = O(\log x) \tag{4}$$

then

$$\overline{\lim_{x\to\infty}}\, \frac{x^{1/2}\tilde{M}^*(x)}{\log\log\log x} \geqslant \frac{1}{2\pi}. \tag{5}$$

It follows that (1) would be contradicted if it could be shown to imply

$$\tilde{M}^*(x) = o(x^{-1/2}\log\log\log x).$$

Of course (1) implies (3) but it is not difficult to obtain a better result.

THEOREM 2. *If* $M(x) = O(x^{1/2})$ *then* $\tilde{M}^*(x) = O(x^{-1/2}\log x)$.

THEOREM 3. *Assume that* (4) *is true. Then there is a constant* $C > 0$ *such that for any* N *we have*

$$|M^*(x) - M^*(y)| > C|x^{-1/2} - y^{-1/2}|\log\log\log(x+y)$$

for a pair of numbers x, y *with* $\max(x, y) > N$.

COROLLARY. *The inequality*

$$M^*(x) - M^*(y) = O(|x^{-1/2} - y^{-1/2}|)$$

does not hold.

This is true without any hypothesis since letting $y \to \infty$ gives (2), which is equivalent to (1). Therefore the above inequality implies (4).

2. Preliminary results. Let $\phi(s) = 2^s\pi^{s-1}\cos\frac{\pi s}{2}\Gamma(1-s)$ and consider the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{2(s-1)}\frac{\phi(s)}{\zeta(s)}\,ds$$

where $1 < c < 2$. This is absolutely convergent since $\phi(s) = O(|t|^{1/2-\sigma})$. Moving the contour to the right leads to the series

$$-\sum_{n=1}^{\infty} \frac{(-1)^n(2\pi x)^{2n-1}}{(2n-1)(2n-1)!\,\zeta(2n)} = -\sum_{k=1}^{\infty}\frac{\mu(k)}{k}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n-1)!}\left(\frac{2\pi x}{k}\right)^{2n-1}$$

Rewriting this as an integral gives $\int_0^1 r(t^{-1})\sin 2\pi xt\,dt$ and proves that

$$\tilde{M}^*(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{2(s-1)}\frac{\phi(s)}{\zeta(s)}\,ds. \tag{6}$$

If the Riemann hypothesis is true then the integral converges absolutely for $1/2 < c < 2$ and (3) is clear. In what follows it is assumed that (4) holds. The proof of Theorem 1 is adapted from Ingham [3].

Let

$$\tilde{M}_1^*(y) = \int_0^y 2\tilde{M}^*(x)x^{-1/2}\,dx.$$

From (6),

$$\tilde{M}_1^*(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^{s-1/2}}{(s-1)(s-1/2)}\frac{\phi(s)}{\zeta(s)}\,ds$$

for $1/2 < c < 2$. An explicit formula for this function is required. As in [6, p. 374], shifting the contour to $\mathrm{Re}(s) = c'$ where $-1 < c' < 0$ gives

$$\tilde{M}_1^*(y) = \lim_{\nu\to\infty}\sum_{|\gamma|<T_\nu} \frac{y^{\varrho-1/2}}{(\varrho-1)(\varrho-1/2)}\frac{\phi(\varrho)}{\zeta'(\varrho)} - \frac{2}{\zeta(1/2)} + \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{y^{s-1/2}}{(s-1)(s-1/2)}\frac{\phi(s)}{\zeta(s)}\,ds \tag{7}$$

where $\{T_\nu\}$ is a certain sequence and $\varrho = 1/2 + i\gamma$ is a zero of $\zeta(s)$. Inequality (4) implies not only the Riemann hypothesis and that ϱ is simple but that

$$\sum\left|\frac{1}{\varrho\zeta'(\varrho)}\right|^2 < \infty, \tag{8}$$

[6, p. 377]. Since $\sum|\varrho|^{-2} < \infty$, it follows that the series in (7) is absolutely convergent.

As for the integral in (7), the substitution $s = 1 - w$ leads to an integral on the line $\mathrm{Re}(w) = 1 - c' = c''$ say, which becomes

$$\frac{1}{2\pi i}\int_{c''-i\infty}^{c''+i\infty} \frac{y^{1/2-w}}{w(w-1/2)}\frac{\tan(\pi w/2)}{\zeta(w)}\,dw$$

after using the functional equation $\zeta(s) = \phi(s)\tan\frac{1}{2}\pi s\,\zeta(1-s)$.

Since $c'' > 1$, $\sum_1^N \mu(n)n^{-w} = O(1)$ as $N \to \infty$. The last integral becomes

$$\sum_1^\infty \mu(n)n^{-1/2}J(ny) \tag{9}$$

after termwise integration which is justified by the dominated convergence theorem. Here

$$J(x) = \frac{1}{2\pi i}\int_{c''-i\infty}^{c''+i\infty} \frac{x^{1/2-w}}{w(w-1/2)}\tan\tfrac{1}{2}\pi w\,dw.$$

If the series (9) is denoted $R(y)$ then the explicit formula is

$$\tilde{M}_1^*(y) = \sum \frac{y^{\varrho-1/2}}{(\varrho-1)(\varrho-1/2)}\frac{\phi(\varrho)}{\zeta'(\varrho)} - \frac{2}{\zeta(1/2)} + R(y). \tag{10}$$

LEMMA 1. *We have* $R'(y) = O(y^{-7/2})$ *for* $y \geq 2$.

Proof. If $x > 1$ then

$$J(x) = \frac{2}{\pi}\sum_{n\geq 3}{}' \frac{x^{1/2-n}}{n(n-1/2)}$$

where $'$ means n is odd. Now

$$J'(x) = -\frac{2}{\pi}x^{-5/2}\sum_{n\geq 3}{}' \frac{x^{2-n}}{n} = O(x^{-7/2})$$

if $x \geq 2$ say. It follows that

$$\frac{d}{dy}J(ny) = O(n^{-5/2}y^{-7/2})$$

if $y \geq 2$, $n \geq 1$; so

$$\sum_1^\infty \mu(n)n^{-1/2}\frac{d}{dy}J(ny) = O(y^{-7/2}).$$

This series is $R'(y)$ since it is uniformly convergent for $y \geq 2$.

LEMMA 2. *We have*

$$\operatorname{Im}\sum_{0<\gamma<T}\frac{1}{\varrho\zeta'(\varrho)}\left(1-\frac{\gamma}{T}\right) = -\frac{1}{2\pi}\log T + O(\log^{1/2} T).$$

Proof. According to Ingham [2, p. 317] the interval $(T, T+1)$ contains an X such that

$$\sum_{0<\gamma<X}\frac{1}{\varrho\zeta'(\varrho)}\left(1-\frac{\gamma}{X}\right) = \frac{1}{2\pi i}\log X + O(1). \tag{11}$$

From (8) and the Cauchy–Schwarz inequality,

$$\sum_{T<\gamma<X}\left|\frac{1}{\varrho\zeta'(\varrho)}\right| = O(\log^{1/2} T)$$

since $(T, T+1)$ contains $O(\log T)$ zeros of $\zeta(s)$. Similarly,

$$\left|\sum_{0<\gamma<T}\frac{1}{\varrho\zeta'(\varrho)}\left(\frac{\gamma}{X}-\frac{\gamma}{T}\right)\right| < T^{-1}\sum_{0<\gamma<T}\left|\frac{1}{\varrho\zeta'(\varrho)}\right| = O(T^{-1/2}\log^{1/2} T).$$

Since $\log X = \log T + O(1)$, the lemma follows from the last two inequalities.

3. Proofs.

Proof of Theorem 1. Suppose that $\omega > 2$, T is a positive integer and

$$K(y) = \left(\frac{\sin \pi y}{\pi y}\right)^2.$$

Let $K_T(y) = TK(Ty)$ and consider the integral

$$\int_{\omega-1}^{\omega+1} K_T'(u-\omega)\tilde{M}_1^*(e^u)\,du. \tag{12}$$

An integration by parts shows that this is

$$-\int_{\omega-1}^{\omega+1} 2\tilde{M}^*(e^u)e^{u/2}K_T(u-\omega)\,du. \tag{13}$$

On the other hand, use of (10) in (12) gives

$$-\sum\frac{\phi(\varrho)}{(\varrho-1)\zeta'(\varrho)}\int_{\omega-1}^{\omega+1} e^{i\gamma u}K_T(u-\omega)\,du + \int_{\omega-1}^{\omega+1} K_T'(u-\omega)R(e^u)\,du \tag{14}$$

after termwise integration and integrating by parts.

In the first integral let $u = \omega + T^{-1}y$ to obtain

$$-\sum\frac{e^{i\gamma\omega}}{\varrho-1}\frac{\phi(\varrho)}{\zeta'(\varrho)}\int_{-T}^{T} e^{i\gamma y/T}K(y)\,dy. \tag{15}$$

Consider the expression

$$\sum_{|\gamma|\leq X}\frac{e^{i\gamma\omega}}{\varrho-1}\frac{\phi(\varrho)}{\zeta'(\varrho)}\int_T^\infty e^{i\gamma y/T}K(y)\,dy + \sum_{|\gamma|>X}\frac{e^{i\gamma\omega}}{\varrho-1}\frac{\phi(\varrho)}{\zeta'(\varrho)}\int_T^\infty e^{i\gamma y/T}K(y)\,dy \tag{16}$$

where X will be chosen. Now

(17)
(18)

$$\int_T^\infty e^{i\gamma y/T}K(y)\,dy = \begin{cases} O(T^{-1}), \\ O(\gamma^{-1}) \end{cases}$$

[3, p. 206]. By (18) the infinite series in (16) is convergent. Use of (17) in the first term and of (18) in the second shows that (16) is bounded by a constant times

$$T^{-1}\sum_{|\gamma|\leqslant X}\left|\frac{1}{\varrho\zeta'(\varrho)}\right|+\sum_{|\gamma|>X}\left|\frac{1}{\varrho^2\zeta'(\varrho)}\right|.$$

The first term here is $O(T^{-1}X^{1/2}\log^{1/2}X)$. By (8) and the Cauchy–Schwarz inequality the other term is $O(X^{-1/2}\log^{1/2}X)$ since $\sum_{|\gamma|>X}|\varrho|^{-2}=O(X^{-1}\log X)$; hence (16) is $O(1)$ if $X=T^2/(\log T)$. The range $(-\infty,-T)$ can be handled similarly. When the integral in (15) is extended to $(-\infty,\infty)$ the series becomes

$$(19)\qquad -\sum_{|\gamma|<T}\frac{e^{i\gamma\omega}}{\varrho-1}\frac{\phi(\varrho)}{\zeta'(\varrho)}\left(1-\frac{|\gamma|}{T}\right)+O(1).$$

The second integral in (14) is

$$-\int_{\omega-1}^{\omega+1}e^uR'(e^u)K_T(u-\omega)du.$$

Here $e^u>2$ since $\omega>2$ so Lemma 1 gives the bound

$$\int_{\omega-1}^{\omega+1}e^{-5u/2}K_T(u-\omega)du$$

times a constant for the absolute value of this last term. This is

$$(20)\qquad <\int_1^\infty K_T(u-\omega)du<1.$$

From (13), (19), and (20),

$$(21)\qquad \int_{\omega-1}^{\omega+1}2\tilde{M}^*(e^u)e^{u/2}K_T(u-\omega)du=\sum_{|\gamma|<T}\frac{e^{i\gamma\omega}}{\varrho-1}\frac{\phi(\varrho)}{\zeta'(\varrho)}\left(1-\frac{|\gamma|}{T}\right)+O(1).$$

By the functional equation the sum becomes

$$(22)\qquad \sum_{|\gamma|<T}\frac{e^{-i\gamma\omega}}{\varrho\zeta'(\varrho)}\tan\tfrac{1}{2}\pi\varrho\left(1-\frac{|\gamma|}{T}\right)$$

when ϱ is changed to $1-\varrho$. Now

$$\tan\tfrac{1}{2}\pi\varrho=i\,\mathrm{sgn}(\gamma)+O(e^{-\pi|\gamma|}),$$

and substituting into (22) gives

$$i\sum_{|\gamma|<T}\frac{e^{-i\gamma\omega}}{\varrho\zeta'(\varrho)}\mathrm{sgn}(\gamma)\left(1-\frac{|\gamma|}{T}\right)+O(1)$$

$$=-2\,\mathrm{Im}\sum_{0<\gamma<T}\frac{e^{-i\gamma\omega}}{\varrho\zeta'(\varrho)}\left(1-\frac{\gamma}{T}\right)+O(1)=-2S_T(\omega)+O(1),$$

say. Equation (21) takes the form

$$(23)\qquad \int_{\omega-1}^{\omega+1}\tilde{M}^*(e^u)e^{u/2}K_T(u-\omega)du=-S_T(\omega)+O(1).$$

To complete the proof observe that

$$(24)\qquad S_T(0)=-\frac{1}{2\pi}\log T+O(\log^{1/2}T)$$

by Lemma 2. By Dirichlet's theorem [6, p. 184], for each $\varepsilon>0$ there is a number ω and integers $n(\gamma)$ such that

$$\varepsilon^{-N(T)}<\omega<\varepsilon^{-2N(T)},$$

and

$$|\gamma\omega-2\pi n(\gamma)|<2\pi\varepsilon$$

for each $0<\gamma<T$. Here $N(T)$ is the number of γ's. Since $|e^{-i\gamma\omega}-1|<2\pi\varepsilon$,

$$|S_T(\omega)-S_T(0)|<2\pi\varepsilon\sum_{0<\gamma<T}\left|\frac{1}{\varrho\zeta'(\varrho)}\right|=O(\varepsilon T^{1/2}\log^{1/2}T).$$

Upon choosing $\varepsilon=T^{-1/2}$, (23) and (24) imply

$$\int_{\omega-1}^{\omega+1}\tilde{M}^*(e^u)e^{u/2}K_T(u-\omega)du=\frac{1}{2\pi}\log T+O(\log^{1/2}T)$$

for the ω in Dirichlet's theorem.

Given $0<\delta<1$, the right side exceeds $\frac{1}{2\pi}(1-\delta)\log T$ if T is large enough. Hence

$$(25)\qquad \tilde{M}^*(e^u)e^{u/2}\geqslant\frac{1}{2\pi}(1-\delta)\log T$$

for some u in $(\omega-1,\omega+1)$. Now $\log\omega<\frac{1}{2}T\log^2T$ for large T so

$$T\log^2T>\log(\omega+1)>\log u;$$

therefore,

$$\left(1+\frac{2\log\log T}{\log T}\right)\log T>\log\log u.$$

The left side is less than $(1+\delta)\log T$ for large T so from (25)

$$\tilde{M}^*(e^u)e^{u/2}>\frac{1}{2\pi}\frac{1-\delta}{1+\delta}\log\log u.$$

By varying T one obtains this inequality for arbitrarily large u so Theorem 1 is proven.

Proof of Theorem 2. The following estimate is needed.

LEMMA. *We have*

$$\frac{d}{dx}M^*(x) = O(x^{-1}\log x).$$

Proof. From the power series defining $M^*(x)$,

$$\frac{d}{dx}M^*(x) = \frac{1}{x}\sum_{r=1}^{\infty}\frac{\mu(r)}{r}\left(\cos\frac{2\pi x}{r} - 1\right).$$

The contribution of $r \leqslant x$ to the series is $O(\log x)$. For the remainder one obtains the bound $x\sum_{r>x} r^{-2}$ times a constant. Since this is $O(1)$ the lemma follows.

From the formula

$$\operatorname{sgn}(t) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin tu}{u}\,du$$

one readily obtains

$$\tilde{M}^*(x) = \frac{-1}{\pi}\int_0^{\infty}\frac{M^*(x+u) - M^*(x-u)}{u}\,du. \tag{26}$$

For large x the lemma shows that the integral over $0 \leqslant u \leqslant 1$ is $O(x^{-1}\log x)$. Assuming (2) in the form $|M^*(x)| < Cx^{-1/2}$ yields

$$|\tilde{M}^*(x)| < \frac{C}{\pi}\int_1^{\infty}u^{-1}(x+u)^{-1/2}\,du + \frac{C}{\pi}\int_1^{\infty}u^{-1}|x-u|^{-1/2}\,du + O(x^{-1}\log x).$$

The integrals are

$$x^{-1/2}\int_{1/x}^{\infty}u^{-1}(1+u)^{-1/2}\,du \quad\text{and}\quad x^{-1/2}\int_{1/x}^{\infty}u^{-1}|1-u|^{-1/2}\,du$$

respectively. These are clearly $O(x^{-1/2}\log x)$ proving Theorem 2.

Proof of Theorem 3. For brevity let $\log_3 x = \log\log\log x$ and assume that

$$|M^*(x) - M^*(y)| < C|x^{-1/2} - y^{-1/2}|\log_3(x+y) \tag{27}$$

for $x \geqslant N$, $y > 0$. In (26) the integral for $0 \leqslant u \leqslant 1$ has already been considered. For $1 \leqslant u \leqslant x$, (27) implies the bound

$$\frac{C}{\pi}\int_1^{x}\frac{(x-u)^{-1/2} - (x+u)^{-1/2}}{u}\log_3 2x\,du$$
$$< \frac{C}{\pi}x^{-1/2}\log_3 2x\int_{1/x}^{1}\frac{(1-u)^{-1/2} - (1+u)^{-1/2}}{u}\,du < C_1 x^{-1/2}\log_3 x,$$

say. In absolute value the integral for $u > x$ is at most

$$\frac{C}{\pi}\int_x^{\infty}\frac{(u-x)^{-1/2} - (u+x)^{-1/2}}{u}\log_3 2u\,du$$
$$< \frac{C}{\pi}x^{-1/2}\log_3 2x\int_x^{\infty}\frac{(u-x)^{-1/2} - (u+x)^{-1/2}}{u^{1/2}}\,du < C_2 x^{-1/2}\log_3 x;$$

therefore,

$$|\tilde{M}^*(x)| < C_3 x^{-1/2}\log_3 x + O(x^{-1}\log x).$$

This contradicts Theorem 1 if C is small enough.

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