ACTA ARITHMETICA
LIX.2 (1991)

Expression of real numbers with the help of infinite series

by

JAROSŁAW HANČL (Ostrava)

Suppose that we have a sequence $\{a_n\}_{n=1}^{\infty}$, where a_n are positive real numbers. There are many papers describing how to express real numbers by means of $\{a_n\}_{n=1}^{\infty}$. J. Galambos (see [2]) deals with Cantor's series and shows that if we have a sequence $\{a_n\}_{n=1}^{\infty}$ such that $1/a_n$ are positive integers, $1/a_n$ is a divisor of $1/a_{n+1}$, $a_n > a_{n+1}$, then for every positive real number x $(0 \leqslant x \leqslant 1)$ there are positive integers q_n $(0 \leqslant q_n < a_n/a_{n+1})$ such that $x = \sum_{n=1}^{\infty} q_n a_n$. Theorems 1, 2 and 3 deal with similar expressions for every $x \in (0, B\rangle$; here, however, the a_n are arbitrary positive numbers and q_n are reciprocals of elements of some fixed unbouded set S.

Erdős in his paper [1] (see also [3]) introduced the notion of irrational sequences of positive integers. He proved, e.g., that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is irrational and also stated the problem whether there is an irrational sequence increasing less quickly. We extend his definition of irrational sequences to sequences of positive real numbers and Corollary 1 of Theorem 2 gives a negative answer to his problem not only in the domain of positive integers but also in the domain of positive real numbers.

Note that even though Theorems 1, 2 and 3 look very different, their proofs are based on the same idea.

THEOREM 1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive real numbers such that* $\sum_{n=1}^{\infty} 1/a_n = K < \infty$. *Let* $S = \{b_1 = 1, b_2, b_3, \ldots\}$, $b_n < b_{n+1}$ $(n = 1, 2, \ldots)$, *be a set of positive real numbers such that* $\lim_{n\to\infty} b_n = \infty$. *Suppose that*

$$(1) \qquad \Big(\max_{k=1,2,\ldots} (1/b_k - 1/b_{k+1})\Big)/a_n \leqslant \sum_{j=n+1}^{\infty} 1/a_j$$

for every $n = 1, 2, \ldots$ *Then for every* A *with*

$$(2) \qquad 0 < A \leqslant K$$

there is a sequence $\{g_n\}_{n=1}^{\infty}$, $g_n \in S$ $(n = 1, 2, \ldots)$, *such that*

$$(3) \qquad A = \sum_{n=1}^{\infty} 1/(a_n g_n).$$

In addition, if $\{a_n\}_{n=1}^{\infty}$ *is a nondecreasing sequence and*

$$\max_{k=1,2,\dots} (1/b_k - 1/b_{k+1}) = 1 - 1/b_2 \tag{4}$$

then (1) *is also a necessary condition.*

If $K = \infty$ *and there is a positive real number* B *such that*

$$1 > B \geqslant 1 - b_n/b_{n+1} \tag{5}$$

for every $n = 1, 2, \dots$ *then for every positive real number* A *there is a sequence* $\{g_n\}_{n=1}^{\infty}$, $g_n \in S$, $n = 1, 2, \dots$, *such that* (3) *holds.*

Proof. 1. Assume $K < \infty$. We first prove that condition (1) is sufficient. Let (2) hold. The coefficients $g_1, g_2, \dots$ will be constructed by induction. For $n = 1$ we define

$$g_1 = \min\{b\colon 1/(a_1 b) < A,\ b \in S\}.$$

Thus $1/(a_1 g_1) < A$. Suppose that we have $g_1, \dots, g_{n-1}$ such that

$$\sum_{j=1}^{n-1} 1/(a_j g_j) < A$$

and g_n is defined in the following way:

$$g_n = \min\Big\{b\colon \sum_{j=1}^{n-1} 1/(a_j g_j) + 1/(a_n b) < A,\ b \in S\Big\}. \tag{6}$$

Thus $\sum_{j=1}^{n} 1/(a_j g_j) < A$. It follows that

$$\sum_{j=1}^{\infty} 1/(a_j g_j) \leqslant A. \tag{7}$$

On the other hand, we will prove that

$$\sum_{j=1}^{\infty} 1/(a_j g_j) \geqslant A.$$

First we prove by induction that

$$\sum_{j=1}^{n} 1/(a_j g_j) + \sum_{j=n+1}^{\infty} 1/a_j \geqslant A \tag{8}$$

for every positive integer n. For $n = 0$, (8) follows from (2). Suppose that (8) holds with n replaced by $n-1$. If $g_n = 1$ then (8) with n replaced by $n-1$ and (8) are identical. If $g_n = b_{k(n)}$ $(k(n) \neq 1)$, then (1) and (6) imply

$$\begin{aligned}\sum_{j=1}^{n} 1/(a_j g_j) + \sum_{j=n+1}^{\infty} 1/a_j &= \sum_{j=1}^{n-1} 1/(a_j g_j) + 1/(a_n b_{k(n)-1}) \\ &\quad + \sum_{j=n+1}^{\infty} 1/a_j - (1/b_{k(n)-1} - 1/b_{k(n)})/a_n \\ &\geqslant \sum_{j=1}^{n-1} 1/(a_j g_j) + 1/(a_n b_{k(n)-1}) \geqslant A;\end{aligned}$$

thus the inductive proof is complete.

Since (7) and (8) imply (3), condition (1) is proved to be sufficient.

We now prove that condition (1) is necessary. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence, (4) holds, $K < \infty$ and there is a natural number n such that

$$(1 - 1/b_2)/a_n > \sum_{j=n+1}^{\infty} 1/a_j. \tag{9}$$

Put

$$A = \sum_{i=1}^{n} 1/a_i - ((1 - 1/b_2)/a_n - \sum_{j=n+1}^{\infty} 1/a_i)/2.$$

Then (9) implies $0 < A < K$. Now we suppose that A can be expressed as in (3) and we proceed to find a contradiction. (3) implies

$$0 = \sum_{i=1}^{n} 1/a_i - ((1 - 1/b_2)/a_n - \sum_{i=n+1}^{\infty} 1/a_i)/2 - \sum_{i=1}^{\infty} 1/(a_i g_i). \tag{10}$$

If there is a $j \in \{1, 2, \dots, n\}$ such that $g_j \neq 1$ then (10) implies

$$\begin{aligned}0 = &\sum_{\substack{i=1 \\ i \neq j}}^{\infty} (1 - 1/g_i)/a_i + (1 - 1/b_2)(1/a_j - 1/a_n) \\ &+ ((1 - 1/b_2)/a_n - \sum_{i=n+1}^{\infty} 1/a_i)/2 + (1/b_2 - 1/g_j)/a_j > 0.\end{aligned}$$

Thus $g_1 = g_2 = \dots = g_n = 1$. This and (10) imply

$$0 = -((1 - 1/b_2)/a_n - \sum_{i=n+1}^{\infty} 1/a_i)/2 - \sum_{i=n+1}^{\infty} 1/(a_i g_i) < 0.$$

It follows that the number A cannot be expressed as in (3), and condition (1) is proved to be necessary.

2. Assume that $K = \infty$, (5) holds and $A > 0$. We will construct simultaneously $k(n)$ and $g_{k(n-1)+1}, \dots, g_{k(n)}$ as follows: $k(0) = 0$. Suppose that we have $g_1, \dots, g_{k(n-1)} \in S$ and $\sum_{i=1}^{k(n-1)} 1/(a_i g_i) < A$. Then $k(n)$ is the least positive integer such that

$$b_2\Big(A - \sum_{i=1}^{k(n-1)} 1/(a_i g_i)\Big) \leqslant \sum_{i=k(n-1)+1}^{k(n)} 1/a_i = S_n, \quad g_{k(n-1)+1} = \dots = g_{k(n)} = b_{H(n)}$$

where $H(n)$ is the greatest positive integer such that

$$S_n/b_{H(n)} = S_n/g_{k(n)} < A - \sum_{i=1}^{k(n-1)} 1/(a_i g_i) \leqslant S_n/b_{H(n)-1}.$$

It follows that

$$\sum_{i=1}^{k(n)} 1/(a_i g_i) < A,$$

and

$$A-\sum_{i=1}^{k(n)} 1/(a_i g_i) \leqslant A-\sum_{i=1}^{k(n-1)} 1/(a_i g_i)-\Big(A-\sum_{i=1}^{k(n-1)} 1/(a_i g_i)\Big)(S_n/b_{H(n)})(b_{H(n)-1}/S_n)$$
$$=\Big(A-\sum_{i=1}^{k(n-1)} 1/(a_i g_i)\Big)(1-b_{H(n)-1}/b_{H(n)}) \leqslant B\Big(A-\sum_{i=1}^{k(n-1)} 1/(a_i g_i)\Big).$$

Since $1>B$, (3) follows. The proof of Theorem 1 is complete.

Remark 1. Note that sequences $\{g_n\}_{n=1}^{\infty}$ are in general not uniquely determined.

THEOREM 2. *Let* $S=\{b_1, b_2, \ldots\}$, $b_n<b_{n+1}$, $\lim_{n\to\infty} b_n=\infty$, *be a set of positive real numbers such that there is a positive integer* D *with* $D>b_{n-1}-b_n$ *for every positive integer* n.

Let $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive real numbers containing a subsequence* $\{a_{p(n)}\}_{n=1}^{\infty}=\{c_n\}_{n=1}^{\infty}$ *with the following property:*

There is a positive real function $F(n)<n$ *on the set of positive integers and a positive integer* K *such that*

$$2^{-2^{n-F(n)}}<K/c_n \tag{11}$$

and

$$\sum_{n=1}^{\infty} 2^{-F(n)}<\infty \tag{12}$$

for every positive integer n. *Then there is a positive real number* B *such that for every* B_1, $0<B_1\leqslant B$, *there is a sequence* $\{g_n\}_{n=1}^{\infty}$, $g_n\in S$, *satisfying*

$$B_1=\sum_{n=1}^{\infty} 1/(a_n g_n).$$

Proof. It is convenient to define

$$H(n)=\log_2\Big(1/\sum_{i=n}^{\infty} 2^{-F(i)}\Big).$$

We have

$$H(n)\leqslant \log_2(1/2^{-F(n)})=F(n) \tag{13}$$

and

$$2^{n-H(n)+1}-2^{n-F(n)}-2^{n+1-H(n+1)} \tag{14}$$
$$=2^{n+1}\Big(\sum_{i=n}^{\infty} 2^{-F(i)}\Big)-2^n\cdot 2^{-F(n)}-2^{n+1}\Big(\sum_{i=n+1}^{\infty} 2^{-F(i)}\Big)=2^{n-F(n)}\geqslant 0.$$

Assume $0<\varepsilon\leqslant 1$ and put

$$d_n=2[c_1+1]\cdot D\cdot K\cdot[2^{2^{2-H(2)}}+1][b_1+1]/c_n. \tag{15}$$

Now we prove that there are $b_{k(n)}\in S$, $n=1, 2, \ldots$, such that $\varepsilon=\sum_{n=1}^{\infty} d_n/b_{k(n)}$ and

$$0<\varepsilon-\sum_{i=1}^{n} d_i/b_{k(i)}\leqslant 2^{-2^{n+1-H(n+1)}} \tag{16}$$

for every positive integer n. The proof is by induction. For $n=1$ we have $d_1\geqslant 2b_1$ and thus there is a positive integer $b_{k(1)}$ such that

$$d_1/b_{k(1)}<\varepsilon\leqslant d_1/b_{k(1)-1}.$$

It follows that

$$0<\varepsilon-d_1/b_{k(1)}\leqslant \varepsilon-\varepsilon d_1/b_{k(1)}\cdot b_{k(1)-1}/d_1 \tag{17}$$
$$=\varepsilon(b_{k(1)}-b_{k(1)-1})/b_{k(1)}<D\varepsilon^2/d_1.$$

(15) and (17) imply (16) for $n=1$.

Now suppose (16) holds for $n=N-1$; we will prove (16) for $n=N$. Because of (11), (13), (15) (for $n=N$) and (16) (for $n=N-1$) we have

$$\varepsilon-\sum_{i=1}^{N-1} d_i/b_{k(i)}\leqslant 2^{-2^{N-H(N)}}\leqslant 2^{-2^{N-F(N)}}<K/c_N\leqslant d_N/b_1.$$

It follows that there is $b_{k(N)}\in S$ such that

$$d_N/b_{k(n)}<\varepsilon-\sum_{i=1}^{N-1} d_i/b_{k(i)}\leqslant d_N/b_{k(N)-1}$$

and it follows that

$$0<\varepsilon-\sum_{i=1}^{N} d_i/b_{k(i)}\leqslant\Big(\varepsilon-\sum_{i=1}^{N-1} d_i/b_{k(i)}\Big)-\Big(\varepsilon-\sum_{i=1}^{N-1} d_i/b_{k(i)}\Big)(d_N/b_{k(N)}) \tag{18}$$
$$\times(b_{k(N)-1}/d_N)$$
$$=\Big(\varepsilon-\sum_{i=1}^{N-1} d_i/b_{k(i)}\Big)(b_{k(N)}-b_{k(N)-1})/b_{k(N)}$$
$$<\Big(\varepsilon-\sum_{i=1}^{N-1} d_i/b_{k(i)}\Big)^2(D/d_N).$$

(11), (13), (15) and (18) imply

$$0<\varepsilon-\sum_{i=1}^{N} d_i/b_{k(i)}\leqslant 2^{-(2^{N-H(N)+1}-2^{N-F(N)})}. \tag{19}$$

(14) and (19) imply (16) for $n=N$. We have proved that (16) holds for every positive integer n. Thus

$$\varepsilon=\sum_{n=1}^{\infty} d_n/b_{k(n)}. \tag{20}$$

(15) and (20) imply

$$\sum_{n=1}^{\infty} 1/(c_n b_{k(n)}) = \varepsilon(2[c_1+1]\cdot D\cdot K\cdot[2^{2^{2-H(2)}}+1][b_1+1])^{-1}. \tag{21}$$

We have found for every ε $(0<\varepsilon\leqslant 1)$ a sequence $\{b_{k(n)}\}_{n=1}^{\infty}$, $b_{k(n)}\in S$, such that (21) holds. Now we put

$$B=(2[c_1+1]\cdot D\cdot K\cdot[2^{2^{2-H(2)}}+1][b_1+1])^{-1}.$$

If $0<B_1\leqslant B$, then there is a sequence $\{g_j\}_{\substack{j=1\\ j\neq P(n)}}^{\infty}$, $g_j\in S$, such that

$$D_1=\sum_{\substack{n=1\\ n\neq P(j)}}^{\infty} 1/(a_n g_n)<B_1.$$

Put $\varepsilon=(B_1-D_1)/B$ and find $\{b_{k(P(n))}\}_{n=1}^{\infty}$ satisfying (21). If $g_{P(n)}=b_{k(P(n))}$ $(n=1,2,\ldots)$ then

$$B_1=\sum_{n=1}^{\infty} 1/(g_n a_n)$$

and the proof of Theorem 2 is complete.

DEFINITION 1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If there is a sequence of positive integers $\{b_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} 1/(a_n b_n)$ is rational then we call $\{a_n\}_{n=1}^{\infty}$ a *rational sequence*; otherwise we call $\{a_n\}_{n=1}^{\infty}$ an *irrational sequence.*

COROLLARY 1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying all the assumptions of Theorem 2 and let $S=\{1,2,\ldots\}$. Then $\{a_n\}_{n=1}^{\infty}$ is a rational sequence.*

COROLLARY 2. *Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that* $\limsup(\log_2\log_2 c_n)/n<1$. *Then $\{c_n\}_{n=1}^{\infty}$ is a rational sequence.*

EXAMPLE. $\{2^{2^{(1-\varepsilon)n}}\}_{n=1}^{\infty}$, $\{(n!)^k\}_{n=1}^{\infty}$ (k is a real number), $\{n^n\}_{n=1}^{\infty}$, $\{S^n\}_{n=1}^{\infty}$ (S is a positive real number) are rational sequences.

Remark 1. The problem remains open whether $\{2^{2^n/n}\}_{n=1}^{\infty}$ is a rational sequence.

THEOREM 3. *Let $S=\{b_1,b_2,\ldots\}$ be a set of positive real numbers $b_1<b_2<\ldots$, $\lim_{n\to\infty} b_n=\infty$, such that*

$$1>K\geqslant 1-b_{n-1}/b_n \tag{22}$$

for every $n\geqslant n_0$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers containing a subsequence $\{a_{k(n)}\}_{n=1}^{\infty}=\{C_n\}_{n=1}^{\infty}$ such that

$$\liminf 1/(C_n K^n)>0. \tag{23}$$

Then there is a positive real number B such that for every $0<B_1\leqslant B$ there is a sequence $\{g_n\}_{n=1}^{\infty}$ with $g_n\in S$ and

$$B_1=\sum_{n=1}^{\infty} 1/(a_n g_n). \tag{24}$$

Proof. Set $S_1=\{b_{n_0},b_{n_0+1},\ldots\}$. In view of (23) there is a positive integer A with $A/C_n>K^n$ for every positive integer n. Put

$$d_n=2A\cdot[b_{n_0}+1]/C_n. \tag{25}$$

Now the proof is similar to the proof of Theorem 2.

By induction we prove that for every $0<\varepsilon\leqslant 1$ there are $h_1,h_2,\ldots\in S_1$ such that

$$\varepsilon=\sum_{n=1}^{\infty} d_n/h_n$$

and

$$0<\varepsilon-\sum_{i=1}^{n} d_i/h_i\leqslant K^n \tag{26}$$

for every nonnegative integer n. Since $0<\varepsilon\leqslant 1$, (26) holds for $n=0$. Now assume that (26) holds with n replaced by $n-1$. Because of (25) and of the inductive assumption there is a positive integer n_1 such that

$$d_n/b_{n_1}<\varepsilon-\sum_{i=1}^{n-1} d_i/h_i\leqslant d_n/b_{n_1-1}$$

where $n_1\neq n_0$. Put $h_n=b_{n_1}$. It follows that

$$0<\varepsilon-\sum_{i=1}^{n} d_i/h_i\leqslant\varepsilon-\sum_{i=1}^{n-1} d_i/h_i-(\varepsilon-\sum_{i=1}^{n-1} d_i/h_i)(d_n/b_{n_1})(b_{n_1-1}/d_n) \tag{27}$$
$$=(\varepsilon-\sum_{i=1}^{n-1} d_i/h_i)(1-b_{n_1-1}/b_{n_1}).$$

(22), the inductive hypothesis and (27) imply (26). Thus the proof is complete and

$$\varepsilon=\sum_{n=1}^{\infty} d_n/h_n.$$

It follows that

$$\varepsilon(2A\cdot[b_{n_0+1}+1])^{-1}=\sum_{n=1}^{\infty} 1/(C_n h_n). \tag{28}$$

Put $B=1/(2A\cdot[b_{n_0+1}+1])$. If $0<B_1\leqslant B$, then there are $g_n\in S_1$, where n is a positive integer, $n\neq k(i)$, such that

$$B_1>R=\sum_{\substack{n=1\\ n\neq k(i)}}^{\infty} 1/(a_n g_n).$$

If $\varepsilon = (B_1 - R)/B$, we find $h_n \in S_1$, $n = 1, 2, \ldots$, such that (28) holds. It suffices to put $g_{k(n)} = h_n$ $(n = 1, 2, \ldots)$ and (24) is satisfied. The proof of Theorem 3 is complete.

Acknowledgments. I would like to thank Professor A. Schinzel and Professor B. Novák for their encouragement.

References

[1] P. Erdős, *Some problems and results on the irrationality of the sum of infinite series*, J. Math. Sci. 10 (1975), 1–7.
[2] J. Galambos, *Representations of Real Numbers by Infinite Series*, Lecture Notes in Math. 502, Springer, 1976.
[3] R. K. Guy, *Unsolved Problems in Number Theory*, Springer, 1981.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSTRAVA
DVOŘÁKOVA 7
701 03 OSTRAVA 1, CZECHOSLOVAKIA

Received on 12.1.1990
and in revised form on 24.9.1990 (2003)

ACTA ARITHMETICA
LIX.2 (1991)

The classification of pairs of binary quadratic forms

by

JORGE MORALES (Baton Rouge, La.)

Introduction. We consider ordered pairs (Q_1, Q_2) of binary quadratic forms with coefficients in $\boldsymbol{Z}$. In the present paper we classify such pairs up to equivalence, where two pairs of forms (Q_1, Q_2) and (Q'_1, Q'_2) are said to be *equivalent* if there is a transformation U in $\mathrm{SL}_2(\boldsymbol{Z})$ such that $Q_i(Ux) = Q'_i(x)$ for $i = 1$, or 2. If Q_1 and Q_2 are linearly dependent then the problem is obviously equivalent to the classification of single forms, which goes back to Gauss' *Disquisitiones Arithmeticae*.

It can be shown (see Appendices I and II) that the number of equivalence classes of pairs with given discriminants δ_1, δ_2 and codiscriminant Δ is finite if and only if $\Delta^2 \neq 4\delta_1\delta_2$. Moreover, the classification of pairs with $\Delta^2 = 4\delta_1\delta_2$ turns out to be elementary (see Appendix II).

Thus the interesting case is when $\Delta^2 \neq 4\delta_1\delta_2$. The classification we will give uses a new invariant, called the *index* and denoted by μ. Our main result is that there is a natural finite group $\mathfrak{G}$ that acts transitively and freely on the set of equivalence classes of pairs with prescribed set of invariants $(\delta_1, \delta_2, \Delta, \mu)$ (see Theorem 1.3 and Corollary 1.5). This approach to classification is illustrated by a numerical example in Appendix IV.

The group $\mathfrak{G}$ turns out to depend solely on the Sylow 2-subgroup of the Picard group of a certain quadratic order. As a consequence, the evaluation of the order of $\mathfrak{G}$ gives an explicit formula for the number of classes of pairs with given invariants $(\delta_1, \delta_2, \Delta, \mu)$. We also obtain the formula for the number of pairs with prescribed $(\delta_1, \delta_2, \Delta)$ found by Hardy and Williams (see [3]) for positive-definite forms with fundamental discriminant.

1. The index of a pair of symmetric forms. Recall that quadratic forms correspond bijectively to even symmetric bilinear forms. In this section we study triples (M, b_1, b_2) where M is an *oriented* free $\boldsymbol{Z}$-module of rank two and $b_i\colon M \to M^* = \mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{Z})$ $(i = 1, 2)$ are nondegenerate (i.e. injective) symmetric homomorphisms. We shall say that (M, b_1, b_2) and (N, c_1, c_2) are *equivalent* if there exists an orientation-preserving isomorphism $f\colon M \to N$ such that $f^* c_i f = b_i$ for $i = 1, 2$, where as usual f^* stands for the dual map of f.

1980 Mathematics Subject Classification: 11D09, 11E41.